Solving Composite Fixed Point Problems with Block Updates*

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Abstract

Various strategies are available to construct iteratively a common fixed point of nonexpansive operators by activating only a block of operators at each iteration. In the more challenging class of composite fixed point problems involving operators that do not share common fixed points, current methods require the activation of all the operators at each iteration, and the question of maintaining convergence while updating only blocks of operators is open. We propose a method that achieves this goal and analyze its asymptotic behavior. Weak, strong, and linear convergence results are established by exploiting a connection with the theory of concentrating arrays. Applications to several nonlinear and nonsmooth analysis problems are presented, ranging from monotone inclusions and inconsistent feasibility problems, to variational inequalities and minimization problems arising in data science.

Key words. averaged operator; constrained minimization; forward-backward splitting; fixed point iterations; monotone operator; nonexpansive operator; variational inequality.

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1 Introduction

Throughout, $\mathcal{H}$ is a real Hilbert space with power set $2^{\mathcal{H}}$, identity operator Id, scalar product $\langle\cdot\mid\cdot\rangle$, and associated norm $\|\cdot\|$. Recall that an operator $T\colon\mathcal{H}\to\mathcal{H}$ is nonexpansive if it is 1-Lipschitzian, and α-averaged for some $\alpha\in\,]0,1[$ if $\operatorname{Id}+\alpha^{-1}(T-\operatorname{Id})$ is nonexpansive [4]. We consider the broad class of nonlinear analysis problems which can be cast in the following format.

Problem 1.1 Let m be a strictly positive integer and let $(\omega_i)_{1\leqslant i\leqslant m}\in\,]0,1]^m$ be such that $\sum_{i=1}^m\omega_i=1$. For every $i\in\{0,\ldots,m\}$, let $T_i\colon\mathcal{H}\to\mathcal{H}$ be α_i-averaged for some $\alpha_i\in\,]0,1[$. The task is to find a fixed point of $T_0\circ\sum_{i=1}^m\omega_iT_i$.

A classical instantiation of Problem 1.1 is found in the area of best approximation [8, 38]: given two nonempty closed convex subsets C and D of $\mathcal{H}$, with projection operators proj_C and proj_D, find a fixed point of the composition $\operatorname{proj}_C\circ\operatorname{proj}_D$. Geometrically, such points are those in C at minimum distance from D, and they can be constructed via the method of alternating projections [8, 26]

$$(\forall n\in\mathbb{N})\quad x_{n+1}=\operatorname{proj}_C(\operatorname{proj}_Dx_n). \tag{1.1}$$

This problem was extended in [1] to that of finding a fixed point of the composition $\operatorname{prox}_f\circ\operatorname{prox}_g$ of the proximity operators of proper lower semicontinuous convex functions $f\colon\mathcal{H}\to\,]-\infty,+\infty]$ and $g\colon\mathcal{H}\to\,]-\infty,+\infty]$. Recall that, given $x\in\mathcal{H}$, prox_fx is the unique minimizer of the function $y\mapsto f(y)+\|x-y\|^2/2$ or, equivalently, $\operatorname{prox}_fx=(\operatorname{Id}+\partial f)^{-1}$ where ∂f is the subdifferential of f, which is maximally monotone [6]. A further generalization of this formalism was proposed in [7] where, given two maximally monotone operators $A\colon\mathcal{H}\to2^{\mathcal{H}}$ and $B\colon\mathcal{H}\to2^{\mathcal{H}}$, with associated resolvents $J_A=(\operatorname{Id}+A)^{-1}$ and $J_B=(\operatorname{Id}+B)^{-1}$, the asymptotic behavior of the iterations

$$(\forall n\in\mathbb{N})\quad x_{n+1}=J_A(J_Bx_n) \tag{1.2}$$

for constructing a fixed point of $J_A\circ J_B$ was investigated. We recall that J_A and J_B are 1/2-averaged operators [6]. Now let A_0 and $(B_i)_{1\leqslant i\leqslant m}$ be maximally monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$ and, for every $i\in\{1,\ldots,m\}$, let ${}^{\rho_i}B_i=(\operatorname{Id}-J_{\rho_iB_i})/\rho_i$ be the Yosida approximation of B_i of index $\rho_i\in\,]0,+\infty[$. Set $\beta=1/(\sum_{i=1}^m1/\rho_i)$ and $(\forall i\in\{1,\ldots,m\})\ \omega_i=\beta/\rho_i$. In connection with the inclusion problem

$$\text{find}\quad x\in\mathcal{H}\quad\text{such that}\quad 0\in A_0x+\sum_{i=1}^m\big({}^{\rho_i}B_i\big)x, \tag{1.3}$$

the iterative process

$$(\forall n\in\mathbb{N})\quad x_{n+1}=J_{\beta\gamma_nA_0}\left(x_n+\gamma_n\left(\sum_{i=1}^m\omega_iJ_{\rho_iB_i}x_n-x_n\right)\right),\quad\text{where}\quad 0<\gamma_n<2, \tag{1.4}$$

was studied in [11]. This algorithm captures (1.2) as well as methods such as those proposed in [31, 32]; see also [45] for related problems. To make its structure more apparent, let us set

$$(\forall n\in\mathbb{N})\quad T_{0,n}=J_{\beta\gamma_nA_0}\quad\text{and}\quad(\forall i\in\{1,\ldots,m\})\quad T_{i,n}=(1-\gamma_n)\operatorname{Id}+\gamma_nJ_{\rho_iB_i}. \tag{1.5}$$

Then we observe that, for every $n\in\mathbb{N}$, the following hold:

- Problem (1.3) is the special case of Problem 1.1 in which $T_0=J_{\beta A_0}$, $T_1=J_{\rho_1B_1}$, ..., and $T_m=J_{\rho_mB_m}$. Its set of solutions is

$$\operatorname{Fix}\left(J_{\beta A_0}\circ\sum_{i=1}^m\omega_iJ_{\rho_iB_i}\right)=\operatorname{Fix}\left(T_{0,n}\circ\sum_{i=1}^m\omega_iT_{i,n}\right). \tag{1.6}$$

- For every $i \in \{0,\ldots,m\}$, $T_{i,n}$ is an averaged nonexpansive operator.
- The updating rule in (1.4) can be written as

$$x_{n+1} = T_{0,n}\Bigg(\sum_{i=1}^{m} \omega_i t_{i,n}\Bigg), \quad \text{where} \quad (\forall i \in \{1,\ldots,m\}) \quad t_{i,n} = T_{i,n}x_n. \tag{1.7}$$

The implementation of (1.7) requires the activation of $T_{0,n}$ and the m operators $(T_{i,n})_{1\leqslant i\leqslant m}$. If the operators $(T_i)_{0\leqslant i\leqslant m}$ have common fixed points, then Problem 1.1 amounts to finding such a point, and this can be achieved via block-iterative methods that require activating only subgroups of operators over the iterations; see, for instance, [2, 5, 10, 24]. In the absence of common fixed points, whether Problem 1.1 can be solved by updating only subgroups of operators is an open question. In the present paper, we address it by showing that it is possible to lighten the computational burden of iteration n of (1.7) by activating only a subgroup $(T_{i,n})_{i\in I_n\subset\{1,\ldots,m\}}$ of the operators and by recycling older evaluations of the remaining operators. This leads to the iteration template

$$\left\lfloor \begin{array}{l} \text{for every } i \in I_n \\ \lfloor\; t_{i,n} = T_{i,n}x_n \\ \text{for every } i \in \{1,\ldots,m\} \smallsetminus I_n \\ \lfloor\; t_{i,n} = t_{i,n-1} \\ x_{n+1} = T_{0,n}\Bigg(\displaystyle\sum_{i=1}^{m} \omega_i t_{i,n}\Bigg). \end{array}\right. \tag{1.8}$$

The proposed framework will feature a flexible deterministic rule for selecting the blocks of indices $(I_n)_{n\in\mathbb{N}}$, as well as tolerances in the evaluation of the operators in (1.8). Somewhat unexpectedly, our analysis will rely on the theory of concentrating arrays, which appears predominantly in the area of mean iteration methods [13, 15, 29, 33, 34, 40, 41]. In Section 2, we propose a new type of concentrating array that will be employed in Section 3 to investigate the asymptotic behavior of the method. Finally, various applications to nonlinear analysis problems are presented in Section 4.

Notation. Let $M\colon \mathcal{H} \to 2^{\mathcal{H}}$. Then $\operatorname{gra} M = \big\{(x,u) \in \mathcal{H}\times\mathcal{H} \;\big|\; u \in Mx\big\}$ is the graph of M, $\operatorname{zer} M = \big\{x \in \mathcal{H} \;\big|\; 0 \in Mx\big\}$ the set of zeros of M, $\operatorname{dom} M = \big\{x \in \mathcal{H} \;\big|\; Mx \neq \varnothing\big\}$ the domain of M, $\operatorname{ran} M = \big\{u \in \mathcal{H} \;\big|\; (\exists\, x \in \mathcal{H})\; u \in Mx\big\}$ the range of M, M^{-1} the inverse of M, which has graph $\big\{(u,x) \in \mathcal{H}\times\mathcal{H} \;\big|\; u \in Mx\big\}$, and $J_M = (\mathrm{Id} + M)^{-1}$ the resolvent of M. The parallel sum of M and $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is $M \,\square\, A = (M^{-1} + A^{-1})^{-1}$. Further, M is monotone if

$$\big(\forall (x,u) \in \operatorname{gra} M\big)\big(\forall (y,v) \in \operatorname{gra} M\big) \quad \langle x - y \mid u - v\rangle \geqslant 0, \tag{1.9}$$

and maximally monotone if, in addition, there exists no monotone operator $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ such that $\operatorname{gra} M \subset \operatorname{gra} A \neq \operatorname{gra} M$. If $M - \rho\,\mathrm{Id}$ is monotone for some $\rho \in \left]0,+\infty\right[$, then M is strongly monotone. We denote by $\Gamma_0(\mathcal{H})$ the class of lower semicontinuous convex functions $f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ such that $\operatorname{dom} f = \big\{x \in \mathcal{H} \;\big|\; f(x) < +\infty\big\} \neq \varnothing$. Let $f \in \Gamma_0(\mathcal{H})$. The subdifferential of f is the maximally monotone operator $\partial f\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{u \in \mathcal{H} \;\big|\; (\forall y \in \mathcal{H})\; \langle y - x \mid u\rangle + f(x) \leqslant f(y)\big\}$. For every $x \in \mathcal{H}$, the unique minimizer of the function $f + (1/2)\|\cdot - x\|^2$ is denoted by $\operatorname{prox}_f x$. We have $\operatorname{prox}_f = J_{\partial f}$. Let C be a nonempty closed convex subset of $\mathcal{H}$. Then proj_C is the projector onto C, d_C the distance function to C, and ι_C is the indicator function of C, which takes the value 0 on C and $+\infty$ on its complement.

2 Concentrating arrays

Mann's mean value iteration method seeks a fixed point of an operator $T\colon \mathcal{H} \to \mathcal{H}$ via the iterative process $x_{n+1} = T\overline{x}_n$, where $\overline{x}_n$ is a convex combination of the points $(x_j)_{0\leqslant j\leqslant n}$ [33, 34]. The notion of a concentrating array was introduced in [15] to study the asymptotic behavior of such methods. Interestingly, it will turn out to be also quite useful in our investigation of the asymptotic behavior of (1.8).

Definition 2.1 [15, Definition 2.1] A triangular array $(\mu_{n,j})_{n\in\mathbb{N},0\leqslant j\leqslant n}$ in $[0,+\infty[$ is concentrating if the following hold:

[a] $(\forall n \in \mathbb{N})\ \sum_{j=0}^{n} \mu_{n,j} = 1$.

[b] $(\forall j \in \mathbb{N})\ \lim_{n\to+\infty} \mu_{n,j} = 0$.

[c] Every sequence $(\xi_n)_{n\in\mathbb{N}}$ in $[0,+\infty[$ that satisfies

$$\big(\forall n \in \mathbb{N}\big) \quad \xi_{n+1} \leqslant \sum_{j=0}^{n} \mu_{n,j}\xi_j + \varepsilon_n, \tag{2.1}$$

for some summable sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $[0,+\infty[$, converges.

We shall require the following convergence principle, which extends that of quasi-Fejér monotonicity [10].

Lemma 2.2 *Let C be a nonempty subset of $\mathcal{H}$, let $\phi\colon [0,+\infty[\to [0,+\infty[$ be strictly increasing and such that $\lim_{t\to+\infty}\phi(t) = +\infty$, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$, let $(\mu_{n,j})_{n\in\mathbb{N},0\leqslant j\leqslant n}$ be a concentrating array in $[0,+\infty[$, let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence in $[0,+\infty[$, and let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a summable sequence in $[0,+\infty[$ such that*

$$(\forall x \in C)(\forall n \in \mathbb{N}) \quad \phi(\|x_{n+1} - x\|) \leqslant \sum_{j=0}^{n} \mu_{n,j}\phi(\|x_j - x\|) - \beta_n + \varepsilon_n. \tag{2.2}$$

Then the following hold:

(i) *$(x_n)_{n\in\mathbb{N}}$ is bounded.*

(ii) *$\beta_n \to 0$.*

(iii) *Suppose that every weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$ belongs to C. Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point in C.*

(iv) *Suppose that $(x_n)_{n\in\mathbb{N}}$ has a strong sequential cluster point in C. Then $(x_n)_{n\in\mathbb{N}}$ converges strongly to a point in C.*

Proof. Let $x \in C$. Let us first show that

$$(\|x_n - x\|)_{n\in\mathbb{N}} \text{ converges.} \tag{2.3}$$

It follows from (2.2) and Definition 2.1 that $(\phi(\|x_n - x\|))_{n\in\mathbb{N}}$ converges, say $\phi(\|x_n - x\|) \to \lambda$. However, since $\lim_{t\to+\infty}\phi(t) = +\infty$, $(\|x_n - x\|)_{n\in\mathbb{N}}$ is bounded and, to establish (2.3), it suffices to show that it does not have two distinct cluster points. Suppose to the contrary that there exist subsequences $(\|x_{k_n} - x\|)_{n\in\mathbb{N}}$ and $(\|x_{l_n} - x\|)_{n\in\mathbb{N}}$ such that $\|x_{k_n} - x\| \to \eta$ and $\|x_{l_n} - x\| \to \zeta > \eta$, and fix $\varepsilon \in]0, (\zeta-\eta)/2[$. Then, for n sufficiently large, $\|x_{k_n}-x\| \leqslant \eta+\varepsilon < \zeta-\varepsilon \leqslant \|x_{l_n}-x\|$ and, since ϕ is strictly increasing, $\phi(\|x_{k_n}-x\|) \leqslant \phi(\eta+\varepsilon) < \phi(\zeta-\varepsilon) \leqslant \phi(\|x_{l_n}-x\|)$. Taking the limit as $n \to +\infty$ yields $\lambda \leqslant \phi(\eta+\varepsilon) < \phi(\zeta-\varepsilon) \leqslant \lambda$, which is impossible.

(i) and (iv): Clear in view of (2.3).

(ii): As shown above, there exists $\lambda \in [0,+\infty[$ such that $\phi(\|x_n-x\|) \to \lambda$. In turn, [28, Theorem 3.5.4] implies that $\sum_{j=0}^{n} \mu_{n,j}\phi(\|x_j-x\|) \to \lambda$. We thus derive from (2.2) that $0 \leqslant \beta_n \leqslant \sum_{j=0}^{n} \mu_{n,j}\phi(\|x_j-x\|) - \phi(\|x_{n+1}-x\|) + \varepsilon_n \to 0$.

(iii): This follows from (2.3) and [6, Lemma 2.47]. □

Several examples of concentrating arrays are provided in [15]. Here is a novel construction which is not only of interest to mean iteration processes in fixed point theory [13, 15, 29, 33, 34, 41] but will also play a pivotal role in establishing our main result, Theorem 3.2.

Proposition 2.3 *Let K be a strictly positive integer and let $(\mu_{n,j})_{n\in\mathbb{N},0\leqslant j\leqslant n}$ be a triangular array in $[0,+\infty[$ such that the following hold:*

(i) $(\forall n \in \mathbb{N})\ \sum_{j=0}^{n} \mu_{n,j} = 1$.

(ii) $(\forall n \in \mathbb{N})(\forall j \in \mathbb{N})\ n-j \geqslant K \Rightarrow \mu_{n,j} = 0$.

(iii) $\inf_{n\in\mathbb{N}} \mu_{n,n} > 0$.

Then $(\mu_{n,j})_{n\in\mathbb{N},0\leqslant j\leqslant n}$ is a concentrating array.

Proof. Properties [a] and [b] in Definition 2.1 clearly hold. To verify [c], let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence in $[0,+\infty[$ and let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a summable sequence in $[0,+\infty[$ such that

$$(\forall n \in \mathbb{N}) \quad \xi_{n+1} \leqslant \sum_{j=0}^{n} \mu_{n,j}\xi_j + \varepsilon_n. \tag{2.4}$$

Then, in view of (ii), for every integer $n \geqslant K-1$,

$$\xi_{n+1} \leqslant \sum_{k=0}^{K-1} \mu_{n,n-k}\xi_{n-k} + \varepsilon_n. \tag{2.5}$$

Set $\mu = \inf_{n\in\mathbb{N}} \mu_{n,n}$. If $\mu = 1$, then (i) and (2.5) imply that, for every integer $n \geqslant K-1$,

$$0 \leqslant \xi_{n+1} \leqslant \xi_n + \varepsilon_n, \tag{2.6}$$

and the convergence of $(\xi_n)_{n\in\mathbb{N}}$ therefore follows from [6, Lemma 5.31]. We henceforth assume that $\mu < 1$ and, without loss of generality, that $K > 1$. For every integer $n \geqslant K-1$, define $\widehat{\xi}_n = \max_{0\leqslant k\leqslant K-1} \xi_{n-k}$, and observe that (i) and (2.5) yield $\xi_{n+1} \leqslant \widehat{\xi}_n + \varepsilon_n$. Hence,

$$(\forall n \in \{K-1, K, \ldots\}) \quad 0 \leqslant \widehat{\xi}_{n+1} \leqslant \widehat{\xi}_n + \varepsilon_n \tag{2.7}$$

and we deduce from [6, Lemma 5.31] that $(\widehat{\xi}_n)_{n\in\mathbb{N}}$ converges to some number $\eta \in [0,+\infty[$. Therefore, if $(\xi_n)_{n\in\mathbb{N}}$ converges, then its limit is η as well. Let us argue by contradiction by assuming that $\xi_n \not\to \eta$. Then there exists $\nu \in]0,+\infty[$ such that

$$(\forall N \in \mathbb{N})(\exists\, n_0 \in \{N, N+1, \ldots\}) \quad |\xi_{n_0} - \eta| > \nu. \tag{2.8}$$

Set

$$\delta = \min\left\{\frac{\mu^{K-1}}{1-\mu^{K-1}}, 1\right\} \quad \text{and} \quad \nu' = \frac{\delta\nu}{4}. \tag{2.9}$$

Since $\widehat{\xi}_n \to \eta$ and $\sum_{n\in\mathbb{N}} \varepsilon_n < +\infty$, let us fix an integer $N \geqslant K-1$ such that

$$(\forall n \in \{N, N+1, \ldots\}) \quad \eta - \mu^{K-1}\frac{\nu}{4} \leqslant \widehat{\xi}_n \leqslant \eta + \nu' \quad \text{and} \quad \sum_{j\geqslant n} \varepsilon_j \leqslant (1-\mu^{K-1})\nu'. \tag{2.10}$$

Then

$$(\forall k \in \{1, 2, \ldots\})(\forall n \in \{N, N+1, \ldots\}) \quad \sum_{j=1}^{k} \mu^{j-1}\varepsilon_{n+k-j} \leqslant \sum_{j\geqslant n} \varepsilon_j \leqslant (1-\mu^{K-1})\nu', \tag{2.11}$$

while (2.5) and (i) imply that

$$\begin{aligned}(\forall n \in \{N, N+1, \ldots\}) \quad \xi_{n+1} &\leqslant \mu_{n,n}\xi_n + \sum_{k=1}^{K-1} \mu_{n,n-k}\xi_{n-k} + \varepsilon_n \\ &\leqslant \mu_{n,n}\xi_n + (1-\mu_{n,n})\widehat{\xi}_n + \varepsilon_n \\ &= \mu\xi_n + (1-\mu)\widehat{\xi}_n + (\mu_{n,n}-\mu)(\xi_n - \widehat{\xi}_n) + \varepsilon_n \\ &\leqslant \mu\xi_n + (1-\mu)\widehat{\xi}_n + \varepsilon_n \\ &\leqslant \mu\xi_n + (1-\mu)(\eta+\nu') + \varepsilon_n. \end{aligned} \tag{2.12}$$

It follows from (2.8) that there exists an integer $n_0 \geqslant N$ such that $|\xi_{n_0} - \eta| > \nu$, i.e.,

$$\xi_{n_0} > \eta + \nu \quad \text{or} \quad 0 \leqslant \xi_{n_0} < \eta - \nu. \tag{2.13}$$

Suppose that $\xi_{n_0} > \eta + \nu$. Then (2.9) and (2.10) imply that $\nu < \xi_{n_0} - \eta \leqslant \widehat{\xi}_{n_0} - \eta \leqslant \nu' \leqslant \nu/4$, which is impossible. Therefore, $0 \leqslant \xi_{n_0} < \eta - \nu$ and it follows from (2.12) that

$$\xi_{n_0+1} \leqslant \mu(\eta-\nu) + (1-\mu)(\eta+\nu') + \varepsilon_{n_0} = \eta + (1-\mu)\nu' - \mu\nu + \varepsilon_{n_0}. \tag{2.14}$$

Let us show by induction that, for every integer $k \geqslant 1$,

$$\xi_{n_0+k} \leqslant \eta + (1-\mu^k)\nu' - \mu^k\nu + \sum_{j=1}^{k} \mu^{j-1}\varepsilon_{n_0+k-j}. \tag{2.15}$$

In view of (2.14), this inequality holds for $k = 1$. Now suppose that it holds for some integer $k \geqslant 1$. Then we deduce from (2.12) and (2.15) that

$$\begin{aligned}\xi_{n_0+k+1} &\leqslant \mu\xi_{n_0+k} + \varepsilon_{n_0+k} + (1-\mu)(\eta+\nu') \\ &\leqslant \mu\eta + \mu(1-\mu^k)\nu' - \mu^{k+1}\nu + \sum_{j=0}^{k} \mu^j\varepsilon_{n_0+k-j} + (1-\mu)(\eta+\nu') \\ &= \eta + (1-\mu^{k+1})\nu' - \mu^{k+1}\nu + \sum_{j=1}^{k+1} \mu^{j-1}\varepsilon_{n_0+k+1-j}, \end{aligned} \tag{2.16}$$

which completes the induction argument. Since $\mu \in]0,1[$, we derive from (2.15), (2.11), and (2.9) that

$$(\forall k \in \{1,\ldots,K-1\}) \quad \xi_{n_0+k} \leqslant \eta + (1-\mu^k)\nu' - \mu^k\nu + (1-\mu^{K-1})\nu'$$
$$\leqslant \eta + 2(1-\mu^{K-1})\nu' - \mu^{K-1}\nu$$
$$= \eta + (1-\mu^{K-1})\frac{\delta\nu}{2} - \mu^{K-1}\nu$$
$$\leqslant \eta - \mu^{K-1}\frac{\nu}{2}. \tag{2.17}$$

Therefore, by (2.10),

$$\eta - \mu^{K-1}\frac{\nu}{4} \leqslant \widehat{\xi}_{n_0+K-1} \leqslant \eta - \mu^{K-1}\frac{\nu}{2}. \tag{2.18}$$

We thus reach a contradiction and conclude that $(\xi_n)_{n\in\mathbb{N}}$ converges. ☐

We derive from Proposition 2.3 a new instance of a concentrating array on which the main result of Section 3 will hinge.

Example 2.4 Let I be a nonempty finite set, let $(\omega_i)_{i\in I}$ be a family in $]0,1]$ such that $\sum_{i\in I}\omega_i = 1$, let $(I_n)_{n\in\mathbb{N}}$ be a sequence of nonempty subsets of I, and let K be a strictly positive integer such that $(\forall n \in \mathbb{N}) \bigcup_{0\leqslant k\leqslant K-1} I_{n+k} = I$. Set

$$(\forall n \in \mathbb{N})(\forall j \in \{0,\ldots,n\}) \quad \mu_{n,j} = \begin{cases} 1, & \text{if } n = j < K;\\ \displaystyle\sum_{i\in I_j\smallsetminus\bigcup_{k=j+1}^n I_k} \omega_i, & \text{if } 0 \leqslant n-K < j;\\ 0, & \text{otherwise.}\end{cases} \tag{2.19}$$

Then the following hold:

(i) $(\mu_{n,j})_{n\in\mathbb{N},0\leqslant j\leqslant n}$ is a concentrating array.

(ii) Let $\mathbb{N} \ni n \geqslant K-1$, let $(\xi_j)_{0\leqslant j\leqslant n}$ be in $[0,+\infty[$, and, for every $i \in I$, define $\ell_i(n) = \max\big\{k \in \{n-K+1,\ldots,n\} \bigm| i \in I_k\big\}$. Then

$$\sum_{j=0}^n \mu_{n,j}\xi_j = \sum_{i\in I}\omega_i\xi_{\ell_i(n)}. \tag{2.20}$$

Proof. Let $n \in \mathbb{N}$. If $n \geqslant K-1$, we have $\bigcup_{0\leqslant k\leqslant K-1} I_{n-k} = I$ and therefore

I is the union of the disjoint sets

$$\left(I_n, I_{n-1}\smallsetminus I_n, I_{n-2}\smallsetminus(I_n\cup I_{n-1}),\ldots, I_{n-K+2}\smallsetminus\bigcup_{k=n-K+3}^n I_k, I_{n-K+1}\smallsetminus\bigcup_{k=n-K+2}^n I_k\right). \tag{2.21}$$

(i): It is clear from (2.19) that, for every integer $j \in [0, n-K]$, $\mu_{n,j} = 0$. In turn, we derive from (2.19) and (2.21) that

$$\begin{cases} \displaystyle\sum_{j=0}^n \mu_{n,j} = \mu_{n,n} = 1, & \text{if } n < K;\\ \displaystyle\sum_{j=0}^n \mu_{n,j} = \sum_{j=n-K+1}^n \mu_{n,j} = \sum_{j=n-K+1}^n \sum_{i\in I_j\smallsetminus\bigcup_{k=j+1}^n I_k}\omega_i = \sum_{i\in I}\omega_i = 1, & \text{if } n \geqslant K.\end{cases} \tag{2.22}$$

Finally, $\inf_{n\in\mathbb{N}}\mu_{n,n} = \inf_{n\in\mathbb{N}}\sum_{i\in I_n}\omega_i \geqslant \min_{i\in I}\omega_i > 0$. All the properties of Proposition 2.3 are therefore satisfied.

(ii): We have

$$(\forall j\in\{n-K+1,\ldots,n\})\left(\forall i\in I_j\smallsetminus\bigcup_{k=j+1}^{n} I_k\right)\quad \ell_i(n)=j. \tag{2.23}$$

Hence, in view of (2.19),

$$(\forall j\in\{n-K+1,\ldots,n\})\qquad \sum_{i\in I_j\smallsetminus\bigcup_{k=j+1}^{n} I_k}\omega_i\xi_{\ell_i(n)} = \sum_{i\in I_j\smallsetminus\bigcup_{k=j+1}^{n} I_k}\omega_i\xi_j = \mu_{n,j}\xi_j. \tag{2.24}$$

Consequently, (2.21) yields

$$\sum_{j=0}^{n}\mu_{n,j}\xi_j = \sum_{j=n-K+1}^{n}\;\sum_{i\in I_j\smallsetminus\bigcup_{k=j+1}^{n} I_k}\omega_i\xi_{\ell_i(n)} = \sum_{i\in I}\omega_i\xi_{\ell_i(n)}, \tag{2.25}$$

which concludes the proof. □

3 Solving Problem 1.1 with block updates

We formalize the ideas underlying (1.8) by proposing a method in which variable subgroups of operators are updated over the course of the iterations, and establish its convergence properties. At iteration n, the block of operators to be updated is $(T_{i,n})_{i\in I_n}$. For added flexibility, an error $e_{i,n}$ is tolerated in the application of the operator $T_{i,n}$. We operate under the following assumption, where m is as in Problem 1.1.

Assumption 3.1 *K is a strictly positive integer and $(I_n)_{n\in\mathbb{N}}$ is a sequence of nonempty subsets of $\{1,\ldots,m\}$ such that*

$$(\forall n\in\mathbb{N})\quad \bigcup_{k=0}^{K-1} I_{n+k}=\{1,\ldots,m\}. \tag{3.1}$$

For every integer $n\geqslant K-1$, define

$$(\forall i\in\{1,\ldots,m\})\quad \ell_i(n)=\max\big\{k\in\{n-K+1,\ldots,n\}\;\big|\;i\in I_k\big\}. \tag{3.2}$$

The sequences $(e_{0,n})_{n\in\mathbb{N}}$, $(e_{1,n})_{n\in\mathbb{N}}$, ..., $(e_{m,n})_{n\in\mathbb{N}}$ are in $\mathcal{H}$ and satisfy

$$\sum_{n\geqslant K-1}\|e_{0,n}\|<+\infty\quad\text{and}\quad(\forall i\in\{1,\ldots,m\})\quad\sum_{n\geqslant K-1}\|e_{i,\ell_i(n)}\|<+\infty. \tag{3.3}$$

Theorem 3.2 *Consider the setting of Problem 1.1 together with Assumption 3.1. Let $\varepsilon\in\,]0,1[$ and, for every $n\in\mathbb{N}$ and every $i\in\{0\}\cup I_n$, let $\alpha_{i,n}\in\,]0,1/(1+\varepsilon)[$ and let $T_{i,n}\colon\mathcal{H}\to\mathcal{H}$ be $\alpha_{i,n}$-averaged. Suppose that, for every integer $n\geqslant K-1$,*

$$\varnothing\neq\operatorname{Fix}\left(T_0\circ\sum_{i=1}^{m}\omega_iT_i\right)\subset\operatorname{Fix}\left(T_{0,n}\circ\sum_{i=1}^{m}\omega_iT_{i,\ell_i(n)}\right). \tag{3.4}$$

Let $x_0 \in \mathcal{H}$, let $(t_{i,-1})_{1\leqslant i\leqslant m} \in \mathcal{H}^m$, and iterate

$$
\begin{array}{l}
\text{for } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\text{for every } i \in I_n\\
\left\lfloor\; t_{i,n} = T_{i,n}x_n + e_{i,n}\right.\\
\text{for every } i \in \{1,\ldots,m\} \smallsetminus I_n\\
\left\lfloor\; t_{i,n} = t_{i,n-1}\right.\\
x_{n+1} = T_{0,n}\left(\displaystyle\sum_{i=1}^m \omega_i t_{i,n}\right) + e_{0,n}.
\end{array}
\right.
\end{array}
\tag{3.5}
$$

Let x be a solution to Problem 1.1. Then the following hold:

(i) $(x_n)_{n\in\mathbb{N}}$ *is bounded.*

(ii) *Let $i \in \{1,\ldots,m\}$. Then* $x_{\ell_i(n)} - T_{i,\ell_i(n)}x_{\ell_i(n)} + T_{i,\ell_i(n)}x - x \to 0$.

(iii) *Let $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,m\}$. Then* $T_{i,\ell_i(n)}x_{\ell_i(n)} - T_{j,\ell_j(n)}x_{\ell_j(n)} - T_{i,\ell_i(n)}x + T_{j,\ell_j(n)}x \to 0$.

(iv) *Let $i \in \{1,\ldots,m\}$. Then* $x_{\ell_i(n)} - x_n \to 0$.

(v) $x_n - T_{0,n}(\sum_{i=1}^m \omega_i T_{i,\ell_i(n)}x_n) \to 0$.

(vi) *Suppose that every weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$ solves Problem 1.1. Then the following hold:*

  (a) $(x_n)_{n\in\mathbb{N}}$ *converges weakly to a solution to Problem 1.1.*

  (b) *Suppose that $(x_n)_{n\in\mathbb{N}}$ has a strong sequential cluster point. Then $(x_n)_{n\in\mathbb{N}}$ converges strongly to a solution to Problem 1.1.*

(vii) *For every $n \geqslant K-1$ and every $i \in \{0\} \cup I_n$, let $\rho_i \in \left]0,1\right]$ be a Lipschitz constant of $T_{i,n}$. Suppose that (3.5) is implemented without errors and that, for some $i \in \{0,\ldots,m\}$, $\rho_i < 1$. Then $(x_n)_{n\in\mathbb{N}}$ converges linearly to the unique solution to Problem 1.1.*

Proof. Let us fix temporarily an integer $n \geqslant K-1$. We first observe that, by nonexpansiveness of the operators $T_{0,n}$ and $(T_{i,\ell_i(n)})_{1\leqslant i\leqslant m}$,

$$
\begin{aligned}
(\forall (y,e_0,\ldots,e_m) \in \mathcal{H}^{m+2})\quad & \left\| T_{0,n}\left(\sum_{i=1}^m \omega_i\big(T_{i,\ell_i(n)}x_{\ell_i(n)} + e_i\big)\right) + e_0 - T_{0,n}\left(\sum_{i=1}^m \omega_i T_{i,\ell_i(n)}y\right)\right\|\\
&\leqslant \left\| \sum_{i=1}^m \omega_i T_{i,\ell_i(n)}x_{\ell_i(n)} - \sum_{i=1}^m \omega_i T_{i,\ell_i(n)}y + \sum_{i=1}^m \omega_i e_i\right\| + \|e_0\|\\
&\leqslant \sum_{i=1}^m \omega_i \big\|T_{i,\ell_i(n)}x_{\ell_i(n)} - T_{i,\ell_i(n)}y\big\| + \|e_0\| + \sum_{i=1}^m \omega_i\|e_i\|\\
&\leqslant \sum_{i=1}^m \omega_i \|x_{\ell_i(n)} - y\| + \|e_0\| + \sum_{i=1}^m \|e_i\|.
\end{aligned}
\tag{3.6}
$$

We also note that (3.2) and (3.5) yield

$$
(\forall i \in \{1,\ldots,m\})\quad t_{i,n} = T_{i,\ell_i(n)}x_{\ell_i(n)} + e_{i,\ell_i(n)}.
\tag{3.7}
$$

It follows from (3.5), (3.7), (3.4), and (3.6) that

$$\begin{aligned}\|x_{n+1}-x\| &= \left\|T_{0,n}\left(\sum_{i=1}^m \omega_i\big(T_{i,\ell_i(n)}x_{\ell_i(n)}+e_{i,\ell_i(n)}\big)\right)+e_{0,n}-T_{0,n}\left(\sum_{i=1}^m \omega_i T_{i,\ell_i(n)}x\right)\right\|\\ &\leqslant \sum_{i=1}^m \omega_i\|x_{\ell_i(n)}-x\|+\|e_{0,n}\|+\sum_{i=1}^m\|e_{i,\ell_i(n)}\|. \end{aligned}\tag{3.8}$$

Now define $(\mu_{k,j})_{k\in\mathbb{N},0\leqslant j\leqslant k}$ as in (2.19), with $I=\{1,\ldots,m\}$, and set $\varepsilon_n=\|e_{0,n}\|+\sum_{i=1}^m\|e_{i,\ell_i(n)}\|$. Then we derive from Example 2.4(ii) that

$$\sum_{i=1}^m \omega_i\|x_{\ell_i(n)}-x\|=\sum_{j=0}^n \mu_{n,j}\|x_j-x\|,\tag{3.9}$$

and it follows from (3.8) and (3.3) that

$$\|x_{n+1}-x\|\leqslant\sum_{j=0}^n \mu_{n,j}\|x_j-x\|+\varepsilon_n,\quad\text{where}\quad\sum_{k\geqslant K-1}\varepsilon_k<+\infty.\tag{3.10}$$

Hence, Lemma 2.2(i) guarantees that

$$(x_k)_{k\in\mathbb{N}}\ \text{is bounded.}\tag{3.11}$$

Consequently, using (3.3) and (3.6), we obtain

$$\nu_0=\sup_{k\geqslant K-1}\left(2\left\|T_{0,k}\left(\sum_{i=1}^m\omega_i\big(T_{i,\ell_i(k)}x_{\ell_i(k)}+e_{i,\ell_i(k)}\big)\right)-T_{0,k}\left(\sum_{i=1}^m\omega_iT_{i,\ell_i(k)}x\right)\right\|+\|e_{0,k}\|\right)<+\infty\tag{3.12}$$

and

$$\nu=\sup_{k\geqslant K-1}\left(\sum_{i=1}^m\omega_i\|e_{i,\ell_i(k)}\|+2\left\|\sum_{i=1}^m\omega_i\big(T_{i,\ell_i(k)}x_{\ell_i(k)}-T_{i,\ell_i(k)}x\big)\right\|\right)<+\infty.\tag{3.13}$$

In addition, for every $y\in\mathcal{H}$ and every $z\in\mathcal{H}$, it follows from [6, Proposition 4.35] that

$$\begin{aligned}\|T_{0,n}y-T_{0,n}z\|^2&\leqslant\|y-z\|^2-\frac{1-\alpha_{0,n}}{\alpha_{0,n}}\|(\mathrm{Id}-T_{0,n})y-(\mathrm{Id}-T_{0,n})z\|^2\\ &\leqslant\|y-z\|^2-\varepsilon\|(\mathrm{Id}-T_{0,n})y-(\mathrm{Id}-T_{0,n})z\|^2\end{aligned}\tag{3.14}$$

and, likewise, that

$$(\forall i\in\{1,\ldots,m\})\quad\|T_{i,\ell_i(n)}y-T_{i,\ell_i(n)}z\|^2\leqslant\|y-z\|^2-\varepsilon\|(\mathrm{Id}-T_{i,\ell_i(n)})y-(\mathrm{Id}-T_{i,\ell_i(n)})z\|^2.\tag{3.15}$$

Hence, we deduce from (3.5), (3.7), (3.4), and [6, Lemma 2.14(ii)] that

$$
\begin{aligned}
\|x_{n+1}-x\|^2 &= \bigg\|T_{0,n}\bigg(\sum_{i=1}^m \omega_i(T_{i,\ell_i(n)}x_{\ell_i(n)}+e_{i,\ell_i(n)})\bigg)-T_{0,n}\bigg(\sum_{i=1}^m \omega_i T_{i,\ell_i(n)}x\bigg)+e_{0,n}\bigg\|^2\\
&\leqslant \bigg\|T_{0,n}\bigg(\sum_{i=1}^m \omega_i(T_{i,\ell_i(n)}x_{\ell_i(n)}+e_{i,\ell_i(n)})\bigg)-T_{0,n}\bigg(\sum_{i=1}^m \omega_i T_{i,\ell_i(n)}x\bigg)\bigg\|^2+\nu_0\|e_{0,n}\|\\
&\leqslant \bigg\|\sum_{i=1}^m \omega_i\big(T_{i,\ell_i(n)}x_{\ell_i(n)}-T_{i,\ell_i(n)}x\big)\bigg\|^2\\
&\quad-\varepsilon\bigg\|(\mathrm{Id}-T_{0,n})\bigg(\sum_{i=1}^m \omega_i(T_{i,\ell_i(n)}x_{\ell_i(n)}+e_{i,\ell_i(n)})\bigg)-(\mathrm{Id}-T_{0,n})\bigg(\sum_{i=1}^m \omega_i T_{i,\ell_i(n)}x\bigg)\bigg\|^2\\
&\quad+\nu_0\|e_{0,n}\|+\nu\sum_{i=1}^m \omega_i\|e_{i,\ell_i(n)}\|\\
&\leqslant \sum_{i=1}^m \omega_i\|T_{i,\ell_i(n)}x_{\ell_i(n)}-T_{i,\ell_i(n)}x\|^2\\
&\quad-\frac{1}{2}\sum_{i=1}^m\sum_{j=1}^m \omega_i\omega_j\big\|T_{i,\ell_i(n)}x_{\ell_i(n)}-T_{i,\ell_i(n)}x-T_{j,\ell_j(n)}x_{\ell_j(n)}+T_{j,\ell_j(n)}x\big\|^2\\
&\quad-\varepsilon\bigg\|(\mathrm{Id}-T_{0,n})\bigg(\sum_{i=1}^m \omega_i(T_{i,\ell_i(n)}x_{\ell_i(n)}+e_{i,\ell_i(n)})\bigg)+x-\sum_{i=1}^m \omega_i T_{i,\ell_i(n)}x\bigg\|^2\\
&\quad+\nu_0\|e_{0,n}\|+\nu\sum_{i=1}^m \omega_i\|e_{i,\ell_i(n)}\|\\
&\leqslant \sum_{i=1}^m \omega_i\|x_{\ell_i(n)}-x\|^2-\varepsilon\sum_{i=1}^m \omega_i\big\|(\mathrm{Id}-T_{i,\ell_i(n)})x_{\ell_i(n)}-(\mathrm{Id}-T_{i,\ell_i(n)})x\big\|^2\\
&\quad-\frac{1}{2}\sum_{i=1}^m\sum_{j=1}^m \omega_i\omega_j\big\|T_{i,\ell_i(n)}x_{\ell_i(n)}-T_{i,\ell_i(n)}x-T_{j,\ell_j(n)}x_{\ell_j(n)}+T_{j,\ell_j(n)}x\big\|^2\\
&\quad-\varepsilon\bigg\|(\mathrm{Id}-T_{0,n})\bigg(\sum_{i=1}^m \omega_i(T_{i,\ell_i(n)}x_{\ell_i(n)}+e_{i,\ell_i(n)})\bigg)+x-\sum_{i=1}^m \omega_i T_{i,\ell_i(n)}x\bigg\|^2\\
&\quad+\nu_0\|e_{0,n}\|+\nu\sum_{i=1}^m \omega_i\|e_{i,\ell_i(n)}\|. \qquad (3.16)
\end{aligned}
$$

It therefore follows from (3.9) that

$$
\begin{aligned}
&\|x_{n+1}-x\|^2\\
&\leqslant \sum_{j=0}^{n}\mu_{n,j}\|x_j-x\|^2-\varepsilon\sum_{i=1}^{m}\omega_i\big\|x_{\ell_i(n)}-T_{i,\ell_i(n)}x_{\ell_i(n)}+T_{i,\ell_i(n)}x-x\big\|^2\\
&\quad-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\omega_i\omega_j\big\|T_{i,\ell_i(n)}x_{\ell_i(n)}-T_{i,\ell_i(n)}x-T_{j,\ell_j(n)}x_{\ell_j(n)}+T_{j,\ell_j(n)}x\big\|^2\\
&\quad-\varepsilon\Bigg\|\sum_{i=1}^{m}\omega_i(T_{i,\ell_i(n)}x_{\ell_i(n)}+e_{i,\ell_i(n)})-T_{0,n}\bigg(\sum_{i=1}^{m}\omega_i(T_{i,\ell_i(n)}x_{\ell_i(n)}+e_{i,\ell_i(n)})\bigg)+x-\sum_{i=1}^{m}\omega_iT_{i,\ell_i(n)}x\Bigg\|^2\\
&\quad+\nu_0\|e_{0,n}\|+\nu\sum_{i=1}^{m}\omega_i\|e_{i,\ell_i(n)}\|.
\end{aligned}
\tag{3.17}
$$

Hence, Example 2.4(i), (3.3), and Lemma 2.2(ii) imply that

$$
\begin{cases}
\max\limits_{1\leqslant i\leqslant m}\big\|x_{\ell_i(n)}-T_{i,\ell_i(n)}x_{\ell_i(n)}+T_{i,\ell_i(n)}x-x\big\|\to 0\\
\max\limits_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant m}}\big\|T_{i,\ell_i(n)}x_{\ell_i(n)}-T_{j,\ell_j(n)}x_{\ell_j(n)}-T_{i,\ell_i(n)}x+T_{j,\ell_j(n)}x\big\|\to 0,
\end{cases}
\tag{3.18}
$$

and that

$$
\Bigg\|\sum_{i=1}^{m}\omega_i(T_{i,\ell_i(n)}x_{\ell_i(n)}+e_{i,\ell_i(n)})-T_{0,n}\bigg(\sum_{i=1}^{m}\omega_i(T_{i,\ell_i(n)}x_{\ell_i(n)}+e_{i,\ell_i(n)})\bigg)+x-\sum_{i=1}^{m}\omega_iT_{i,\ell_i(n)}x\Bigg\|\to 0. \tag{3.19}
$$

(i): See (3.11).

(ii)–(iii): See (3.18).

(iv)–(v): It follows from (ii) that

$$
\sum_{i=1}^{m}\omega_i x_{\ell_i(n)}-\sum_{i=1}^{m}\omega_iT_{i,\ell_i(n)}x_{\ell_i(n)}+\sum_{i=1}^{m}\omega_iT_{i,\ell_i(n)}x-x\to 0. \tag{3.20}
$$

We also derive from (ii) that, for every i and every j in $\{1,\ldots,m\}$,

$$
x_{\ell_i(n)}-T_{i,\ell_i(n)}x_{\ell_i(n)}-x_{\ell_j(n)}+T_{j,\ell_j(n)}x_{\ell_j(n)}+T_{i,\ell_i(n)}x-T_{j,\ell_j(n)}x\to 0. \tag{3.21}
$$

Combining (iii) and (3.21), we obtain

$$
(\forall i\in\{1,\ldots,m\})(\forall j\in\{1,\ldots,m\})\quad x_{\ell_i(n)}-x_{\ell_j(n)}\to 0. \tag{3.22}
$$

Now, let $\bar{\imath}\in\{1,\ldots,m\}$ and $\delta\in\left]0,+\infty\right[$. Then (3.22) implies that, for every $j\in\{1,\ldots,m\}$, there exists an integer $\overline{N}_{\delta,j}\geqslant K-1$ such that

$$
\big(\forall n\in\{\overline{N}_{\delta,j},\overline{N}_{\delta,j}+1,\ldots\}\big)\quad \|x_{\ell_{\bar{\imath}}(n)}-x_{\ell_j(n)}\|\leqslant\delta. \tag{3.23}
$$

Set $\overline{N}_\delta=\max_{1\leqslant j\leqslant m}\overline{N}_{\delta,j}$. Then

$$
(\forall j\in\{1,\ldots,m\})\big(\forall n\in\{\overline{N}_\delta,\overline{N}_\delta+1,\ldots\}\big)\quad \|x_{\ell_{\bar{\imath}}(n)}-x_{\ell_j(n)}\|\leqslant\delta. \tag{3.24}
$$

Thus, in view of (3.2), for every integer $n \geqslant \overline{N}_\delta$, taking $j_n \in I_n$ yields $\ell_{j_n}(n) = n$ and hence $\|x_{\ell_{\bar{\imath}}(n)} - x_n\| \leqslant \delta$. This shows that

$$(\forall i \in \{1,\ldots,m\}) \quad x_{\ell_i(n)} - x_n \to 0. \tag{3.25}$$

Consequently, it follows from (3.6) that

$$\begin{aligned}
&\left\| T_{0,n}\left(\sum_{i=1}^m \big(\omega_i T_{i,\ell_i(n)} x_{\ell_i(n)} + e_{i,\ell_i(n)}\big)\right) - T_{0,n}\left(\sum_{i=1}^m \omega_i T_{i,\ell_i(n)} x_n\right)\right\| \\
&\leqslant \sum_{i=1}^m \omega_i \|x_{\ell_i(n)} - x_n\| + \sum_{i=1}^m \|e_{i,\ell_i(n)}\| \\
&\to 0.
\end{aligned} \tag{3.26}$$

In turn, we derive from (3.19), (3.20), (3.25), and (3.3) that

$$\begin{aligned}
&x_n - T_{0,n}\left(\sum_{i=1}^m \omega_i T_{i,\ell_i(n)} x_n\right) \\
&= T_{0,n}\left(\sum_{i=1}^m \omega_i (T_{i,\ell_i(n)} x_{\ell_i(n)} + e_{i,\ell_i(n)})\right) - T_{0,n}\left(\sum_{i=1}^m \omega_i T_{i,\ell_i(n)} x_n\right) \\
&\quad + \sum_{i=1}^m \omega_i (T_{i,\ell_i(n)} x_{\ell_i(n)} + e_{i,\ell_i(n)}) - T_{0,n}\left(\sum_{i=1}^m \omega_i (T_{i,\ell_i(n)} x_{\ell_i(n)} + e_{i,\ell_i(n)})\right) + x - \sum_{i=1}^m \omega_i T_{i,\ell_i(n)} x \\
&\quad + \sum_{i=1}^m \omega_i x_{\ell_i(n)} - \sum_{i=1}^m \omega_i T_{i,\ell_i(n)} x_{\ell_i(n)} + \sum_{i=1}^m \omega_i T_{i,\ell_i(n)} x - x + \sum_{i=1}^m \omega_i (x_n - x_{\ell_i(n)}) - \sum_{i=1}^m \omega_i e_{i,\ell_i(n)} \\
&\to 0.
\end{aligned} \tag{3.27}$$

(vi)(a): This follows from (3.10) and Lemma 2.2(iii).

(vi)(b): By (vi)(a), there exists a solution z to Problem 1.1 such that $x_n \rightharpoonup z$. Therefore, z must be the strong cluster point in question, say $x_{k_n} \to z$. In view of (3.10) and Lemma 2.2(iv), we conclude that $x_n \to z$.

(vii): Set $\rho = \rho_0 \sum_{i=1}^m \omega_i \rho_i$ and note that $\rho \in \,]0,1[$. For every integer $n \geqslant K-1$ and every $(y_i)_{1\leqslant i\leqslant m} \in \mathcal{H}^m$, (3.4) yields

$$\begin{aligned}
\left\| T_{0,n}\left(\sum_{i=1}^m \omega_i T_{i,\ell_i(n)} y_i\right) - x\right\| &= \left\| T_{0,n}\left(\sum_{i=1}^m \omega_i T_{i,\ell_i(n)} y_i\right) - T_{0,n}\left(\sum_{i=1}^m \omega_i T_{i,\ell_i(n)} x\right)\right\| \\
&\leqslant \rho_0 \left\| \sum_{i=1}^m \omega_i T_{i,\ell_i(n)} y_i - \sum_{i=1}^m \omega_i T_{i,\ell_i(n)} x\right\| \\
&\leqslant \rho_0 \sum_{i=1}^m \omega_i \|T_{i,\ell_i(n)} y_i - T_{i,\ell_i(n)} x\| \\
&\leqslant \rho_0 \sum_{i=1}^m \omega_i \rho_i \|y_i - x\|.
\end{aligned} \tag{3.28}$$

Now let $y \in \operatorname{Fix}(T_0 \circ \sum_{i=1}^m \omega_i T_i)$. Since (3.28) implies that

$$\|y - x\| = \left\| T_{0,K-1}\left(\sum_{i=1}^m \omega_i T_{i,\ell_i(K-1)} y\right) - x\right\| \leqslant \rho \|y - x\|, \tag{3.29}$$

we infer that $y = x$, which shows uniqueness. For every integer $n \geqslant K-1$, (3.28) also yields

$$\|x_{n+1} - x\| = \left\| T_{0,n}\left(\sum_{i=1}^m \omega_i T_{i,\ell_i(n)} x_{\ell_i(n)}\right) - x \right\| \leqslant \rho_0 \sum_{i=1}^m \omega_i \rho_i \|x_{\ell_i(n)} - x\|. \tag{3.30}$$

Now set

$$(\forall n \in \mathbb{N}) \quad \xi_n = \|x_n - x\|. \tag{3.31}$$

It follows from (3.30) that

$$(\forall n \in \{K-1, K, \ldots\}) \quad \xi_{n+1} \leqslant \rho_0 \sum_{i=1}^m \omega_i \rho_i \xi_{\ell_i(n)} \leqslant \rho \widehat{\xi}_n, \quad \text{where} \quad \widehat{\xi}_n = \max_{1 \leqslant i \leqslant m} \xi_{\ell_i(n)}. \tag{3.32}$$

Let us show that

$$(\forall n \in \mathbb{N}) \quad \xi_n \leqslant \rho^{\frac{n-K+1}{K}} \widehat{\xi}_{K-1}. \tag{3.33}$$

We proceed by strong induction. We have

$$(\forall k \in \{0, \ldots, K-1\}) \quad \xi_k \leqslant \widehat{\xi}_{K-1} \leqslant \rho^{\frac{k-K+1}{K}} \widehat{\xi}_{K-1}. \tag{3.34}$$

Next, let $\mathbb{N} \ni n \geqslant K-1$ and suppose that

$$(\forall k \in \{0, \ldots, n\}) \quad \xi_k \leqslant \rho^{\frac{k-K+1}{K}} \widehat{\xi}_{K-1}. \tag{3.35}$$

Since $\{\ell_i(n)\}_{1 \leqslant i \leqslant m} \subset \{n-K+1, \ldots, n\}$, there exists $k_n \in \{n-K+1, \ldots, n\}$ such that $\widehat{\xi}_n = \xi_{k_n}$. Therefore, we derive from (3.32) and (3.35) that

$$\xi_{n+1} \leqslant \rho \widehat{\xi}_n = \rho \xi_{k_n} \leqslant \rho \rho^{\frac{k_n-K+1}{K}} \widehat{\xi}_{K-1} = \rho^{\frac{k_n+1}{K}} \widehat{\xi}_{K-1} \leqslant \rho^{\frac{n-K+2}{K}} \widehat{\xi}_{K-1}. \tag{3.36}$$

We have thus shown that

$$(\forall n \in \mathbb{N}) \quad \|x_n - x\| \leqslant \rho^{\frac{1-K}{K}} \widehat{\xi}_{K-1} \big(\rho^{\frac{1}{K}}\big)^n, \tag{3.37}$$

which establishes the linear convergence of $(x_n)_{n \in \mathbb{N}}$ to x. □

Remark 3.3 In applications, the cardinality of I_n may be small compared to m. In such scenarios, it is advantageous to set $z_{-1} = \sum_{i=1}^m \omega_i t_{i,-1}$ and write (3.5) as

$$\begin{array}{l} \text{for } n = 0, 1, \ldots \\ \left\lfloor \begin{array}{l} y_n = z_{n-1} - \sum_{i \in I_n} \omega_i t_{i,n-1} \\ \text{for every } i \in I_n \\ \left\lfloor t_{i,n} = T_{i,n} x_n + e_{i,n} \right. \\ \text{for every } i \in \{1, \ldots, m\} \smallsetminus I_n \\ \left\lfloor t_{i,n} = t_{i,n-1} \right. \\ z_n = y_n + \sum_{i \in I_n} \omega_i t_{i,n} \\ x_{n+1} = T_{0,n} z_n + e_{0,n}, \end{array} \right. \end{array} \tag{3.38}$$

which provides a more economical update equation.

Next, we specialize our results to the autonomous case, wherein the operators $(T_i)_{0\leqslant i\leqslant m}$ of Problem 1.1 are used directly.

Corollary 3.4 *Consider the setting of Problem 1.1 under Assumption 3.1 and the assumption that it has a solution. Let $x_0 \in \mathcal{H}$, let $(t_{i,-1})_{1\leqslant i\leqslant m} \in \mathcal{H}^m$, and iterate*

$$
\begin{array}{l}
\text{for } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\text{for every } i \in I_n\\
\left\lfloor\; t_{i,n} = T_i x_n + e_{i,n}\right.\\
\text{for every } i \in \{1,\ldots,m\}\smallsetminus I_n\\
\left\lfloor\; t_{i,n} = t_{i,n-1}\right.\\
x_{n+1} = T_0\Bigg(\displaystyle\sum_{i=1}^{m}\omega_i t_{i,n}\Bigg) + e_{0,n}.
\end{array}
\right.
\end{array}
\tag{3.39}
$$

Then the following hold:

(i) *Let x be a solution to Problem 1.1 and let $i \in \{1,\ldots,m\}$. Then $x_n - T_i x_n \to x - T_i x$.*

(ii) *$(x_n)_{n\in\mathbb{N}}$ converges weakly to a solution to Problem 1.1.*

(iii) *Suppose that, for some $i \in \{0,\ldots,m\}$, T_i is* demicompact *[39], i.e., every bounded sequence $(y_n)_{n\in\mathbb{N}}$ such that $(y_n - T_i y_n)_{n\in\mathbb{N}}$ converges has a strong sequential cluster point. Then $(x_n)_{n\in\mathbb{N}}$ converges strongly to a solution to Problem 1.1.*

(iv) *Suppose that (3.5) is implemented without errors and that, for some $i \in \{0,\ldots,m\}$, T_i is a Banach contraction. Then $(x_n)_{n\in\mathbb{N}}$ converges linearly to the unique solution to Problem 1.1.*

Proof. We operate in the special case of Theorem 3.2 for which $(\forall n\in\mathbb{N})(\forall i\in\{0\}\cup I_n)\; T_{i,n} = T_i$. Set $T = T_0\circ(\sum_{i=1}^m \omega_i T_i)$. Then the set of solutions to Problem 1.1 is $\operatorname{Fix} T$ and T is nonexpansive since the operators $(T_i)_{0\leqslant i\leqslant m}$ are likewise. In addition, we derive from Theorem 3.2(v) that

$$
x_n - Tx_n \to 0. \tag{3.40}
$$

Altogether, [6, Corollary 4.28] asserts that, if $z\in\mathcal{H}$ is a weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$, then $z \in \operatorname{Fix} T$. Thus,

$$
\text{every weak sequential cluster point of } (x_n)_{n\in\mathbb{N}} \text{ solves Problem 1.1.} \tag{3.41}
$$

Recall from Theorem 3.2(ii) that

$$
(\forall x\in\operatorname{Fix}T)(\forall i\in\{1,\ldots,m\})\quad x_{\ell_i(n)} - T_i x_{\ell_i(n)} \to x - T_i x \tag{3.42}
$$

and from Theorem 3.2(iv) that

$$
(\forall i\in\{1,\ldots,m\})\quad x_{\ell_i(n)} - x_n \to 0. \tag{3.43}
$$

(i): We derive from the nonexpansiveness of T_i, (3.42), and (3.43) that

$$
\begin{aligned}
\|(\operatorname{Id}-T_i)x_n - (\operatorname{Id}-T_i)x\| &\leqslant \|(\operatorname{Id}-T_i)x_n - (\operatorname{Id}-T_i)x_{\ell_i(n)}\| + \|(\operatorname{Id}-T_i)x_{\ell_i(n)} - (\operatorname{Id}-T_i)x\|\\
&\leqslant 2\|x_n - x_{\ell_i(n)}\| + \|(\operatorname{Id}-T_i)x_{\ell_i(n)} - (\operatorname{Id}-T_i)x\|\\
&\to 0.
\end{aligned}
\tag{3.44}
$$

(ii): This is a consequence of (3.41) and Theorem 3.2(vi)(a).

(iii): In view of (3.41) and Theorem 3.2(vi)(b), it is enough to show that $(x_n)_{n\in\mathbb{N}}$ has a strong sequential cluster point. It follows from (ii) and [6, Lemma 2.46] that $(x_n)_{n\in\mathbb{N}}$ is bounded. Hence, if $1 \leqslant i \leqslant m$, we infer from (i) and the demicompactness of T_i that $(x_n)_{n\in\mathbb{N}}$ has a strong sequential cluster point. Now suppose that $i = 0$ and let $x \in \operatorname{Fix} T$. Arguing as in (3.19), we obtain

$$(\mathrm{Id} - T_0)\Bigg(\sum_{i=1}^{m} \omega_i T_i x_{\ell_i(n)}\Bigg) = \sum_{i=1}^{m} \omega_i T_i x_{\ell_i(n)} - T_0\Bigg(\sum_{i=1}^{m} \omega_i T_i x_{\ell_i(n)}\Bigg) \to \sum_{i=1}^{m} \omega_i T_i x - x. \tag{3.45}$$

However, we derive from the nonexpansiveness of the operators $(T_i)_{0\leqslant i\leqslant m}$ and (3.43) that

$$\begin{aligned}\Bigg\|(\mathrm{Id} - T_0)\Bigg(\sum_{i=1}^{m} \omega_i T_i x_n\Bigg) - (\mathrm{Id} - T_0)\Bigg(\sum_{i=1}^{m} \omega_i T_i x_{\ell_i(n)}\Bigg)\Bigg\| &\leqslant 2\Bigg\|\sum_{i=1}^{m} \omega_i T_i x_n - \sum_{i=1}^{m} \omega_i T_i x_{\ell_i(n)}\Bigg\| \\ &\leqslant 2\sum_{i=1}^{m} \omega_i \Big\|T_i x_n - T_i x_{\ell_i(n)}\Big\| \\ &\leqslant 2\|x_n - x_{\ell_i(n)}\| \\ &\to 0.\end{aligned} \tag{3.46}$$

Combining (3.45) and (3.46) yields

$$(\mathrm{Id} - T_0)\Bigg(\sum_{i=1}^{m} \omega_i T_i x_n\Bigg) \to \sum_{i=1}^{m} \omega_i T_i x - x. \tag{3.47}$$

Therefore, by demicompactness of T_0, the bounded sequence $(\sum_{i=1}^{m} \omega_i T_i x_n)_{n\in\mathbb{N}}$ has a strong sequential cluster point and so does $(Tx_n)_{n\in\mathbb{N}} = (T_0(\sum_{i=1}^{m} \omega_i T_i x_n))_{n\in\mathbb{N}}$ since T_0 is nonexpansive. Consequently, (3.40) entails that $(x_n)_{n\in\mathbb{N}}$ has a strong sequential cluster point.

(iv): This is a consequence of Theorem 3.2(vii). ☐

In connection with Corollary 3.4(iii), here are examples of demicompact operators.

Example 3.5 Le $T\colon \mathcal{H} \to \mathcal{H}$ be a nonexpansive operator. Then T is demicompact if one of the following holds:

(i) $\operatorname{ran} T$ is boundedly relatively compact (the intersection of its closure with every closed ball in $\mathcal{H}$ is compact).

(ii) $\operatorname{ran} T$ lies in a finite-dimensional subspace.

(iii) $T = J_A$, where $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone and one of the following is satisfied:

(a) A is *demiregular* [3], i.e., for every sequence $(x_n, u_n)_{n\in\mathbb{N}}$ in $\operatorname{gra} A$ and for every $(x, u) \in \operatorname{gra} A$, $[x_n \rightharpoonup x$ and $u_n \to u] \Rightarrow x_n \to x$.

(b) A is uniformly monotone, i.e., there exists an increasing function $\phi\colon\ [0, +\infty[\ \to [0, +\infty]$ vanishing only at 0 such that $(\forall (x, u) \in \operatorname{gra} A)(\forall (y, v) \in \operatorname{gra} A)\ \langle x - y \mid u - v\rangle \geqslant \phi(\|x-y\|)$.

(c) $A = \partial f$, where $f \in \Gamma_0(\mathcal{H})$ is uniformly convex, i.e., there exists an increasing function $\phi\colon\ [0, +\infty[\ \to [0, +\infty]$ vanishing only at 0 such that

$$\begin{aligned}&(\forall \alpha \in\,]0, 1[)(\forall x \in \operatorname{dom} f)(\forall y \in \operatorname{dom} f) \\ &\qquad f\big(\alpha x + (1 - \alpha)y\big) + \alpha(1 - \alpha)\phi(\|x - y\|) \leqslant \alpha f(x) + (1 - \alpha)f(y).\end{aligned} \tag{3.48}$$

(d) $A = \partial f$, where $f \in \Gamma_0(\mathcal{H})$ and the lower level sets of f are boundedly compact.

(e) $\operatorname{dom} A$ is boundedly relatively compact.

(f) $A\colon \mathcal{H} \to \mathcal{H}$ is single-valued with a single-valued continuous inverse.

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{H}$ such that $y_n - Ty_n \to u$, for some $u \in \mathcal{H}$. Set $(\forall n \in \mathbb{N})$ $x_n = Ty_n$.

(i): By construction, $(x_n)_{n\in\mathbb{N}}$ lies in $\operatorname{ran} T$ and it is bounded since $(\forall n \in \mathbb{N})$ $\|x_n\| \leqslant \|Ty_n - Ty_0\| + \|Ty_0\| \leqslant \|y_n - y_0\| + \|Ty_0\|$. Thus, $(x_n)_{n\in\mathbb{N}}$ lies in a compact set and it therefore possesses a strongly convergent subsequence, say $x_{k_n} \to x \in \mathcal{H}$. In turn $y_{k_n} = y_{k_n} - Ty_{k_n} + x_{k_n} \to u + x$.

(ii)⇒(i): Clear.

(iii)(a): Set $(\forall n \in \mathbb{N})$ $u_n = y_n - x_n$. Then $u_n \to u$. In addition, $(\forall n \in \mathbb{N})$ $(x_n, u_n) \in \operatorname{gra} A$. On the other hand, since $(y_n)_{n\in\mathbb{N}}$ is bounded, we can extract from it a weakly convergent subsequence, say $y_{k_n} \rightharpoonup y$. Then $x_{k_n} = y_{k_n} - u_{k_n} \rightharpoonup y - u$ and $u_{k_n} \to u$. By demiregularity, we get $x_{k_n} \to y - u$ and therefore $y_{k_n} = x_{k_n} + u_{k_n} \to y$.

(iii)(b)–(iii)(f): These are special cases of (iii)(a) [3, Proposition 2.4]. □

4 Applications

We present several applications of Theorem 3.2 to classical nonlinear analysis problems which will be seen to reduce to instantiations of Problem 1.1. These range from common fixed point and inconsistent feasibility problems to composite monotone inclusion and minimization problems. In each scenario, the main benefit of the proposed framework will lie in its ability to achieve convergence while updating only subgroups of the pool of operators involved.

4.1 Finding common fixed point of firmly nonexpansive operators

Firmly nonexpansive operators are operators which are $1/2$-averaged [6, 25]. This application concerns the following ubiquitous fixed point problem [5, 9, 23, 24, 43].

Problem 4.1 Let m be a strictly positive integer and, for every $i \in \{1, \ldots, m\}$, let $T_i\colon \mathcal{H} \to \mathcal{H}$ be firmly nonexpansive. The task is to find a point in $\bigcap_{i=1}^m \operatorname{Fix} T_i$.

Corollary 4.2 *Consider the setting of Problem 4.1 under Assumption 3.1 and the assumption that* $\bigcap_{i=1}^m \operatorname{Fix} T_i \neq \varnothing$. *Let* $(\omega_i)_{1\leqslant i\leqslant m} \in \left]0,1\right]^m$ *be such that* $\sum_{i=1}^m \omega_i = 1$. *For every* $n \in \mathbb{N}$ *and every* $i \in I_n$, *let* $T_{i,n}\colon \mathcal{H} \to \mathcal{H}$ *be a firmly nonexpansive operator such that* $\operatorname{Fix} T_i \subset \operatorname{Fix} T_{i,n}$. *Let* $x_0 \in \mathcal{H}$, *let* $(t_{i,-1})_{1\leqslant i\leqslant m} \in \mathcal{H}^m$, *and iterate*

$$
\begin{array}{l}
\text{for } n = 0, 1, \ldots \\
\left\lfloor
\begin{array}{l}
\text{for every } i \in I_n \\
\left\lfloor\ t_{i,n} = T_{i,n} x_n + e_{i,n} \right. \\
\text{for every } i \in \{1, \ldots, m\} \smallsetminus I_n \\
\left\lfloor\ t_{i,n} = t_{i,n-1} \right. \\
x_{n+1} = \displaystyle\sum_{i=1}^m \omega_i t_{i,n}.
\end{array}
\right.
\end{array}
\tag{4.1}
$$

Then the following hold:

(i) *Let $i \in \{1,\ldots,m\}$. Then $(T_{i,\ell_i(n)}x_{\ell_i(n)})_{n\in\mathbb{N}}$ is bounded.*

(ii) *Suppose that, for every $z \in \mathcal{H}$, every $i \in \{1\ldots,m\}$, and every strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of integers greater than K,*

$$\begin{cases} x_{\ell_i(k_n)} \rightharpoonup z \\ x_{\ell_i(k_n)} - T_{i,\ell_i(k_n)}x_{\ell_i(k_n)} \to 0 \end{cases} \quad \Rightarrow \quad z \in \operatorname{Fix} T_i. \tag{4.2}$$

Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a solution to Problem 4.1.

(iii) *Suppose that, for some $i \in \{1,\ldots,m\}$, $(T_{i,\ell_i(n)}x_{\ell_i(n)})_{n\in\mathbb{N}}$ has a strong sequential cluster point. Then $(x_n)_{n\in\mathbb{N}}$ converges strongly to a solution to Problem 4.1.*

Proof. Set $T_0 = \operatorname{Id}$ and $(\forall i \in \{1,\ldots,m\})$ $\alpha_i = 1/2$. In addition, set $(\forall n \in \mathbb{N})$ $T_{0,n} = \operatorname{Id}$. By assumption, for every $i \in \{1,\ldots,m\}$ and every integer $n \geqslant K-1$, $\operatorname{Fix} T_i \subset \operatorname{Fix} T_{i,\ell_i(n)}$. Therefore, it follows from [6, Proposition 4.47] that, for every integer $n \geqslant K-1$,

$$\operatorname{Fix}\left(T_0 \circ \sum_{i=1}^m \omega_i T_i\right) = \bigcap_{i=1}^m \operatorname{Fix} T_i \subset \bigcap_{i=1}^m \operatorname{Fix} T_{i,\ell_i(n)} = \operatorname{Fix}\left(T_{0,n} \circ \sum_{i=1}^m \omega_i T_{i,\ell_i(n)}\right). \tag{4.3}$$

This shows that (3.4) holds, that Problem 4.1 is a special case of Problem 1.1, and that (4.1) is a special case of (3.5). Let us derive the claims from Theorem 3.2. First, let $x \in \bigcap_{i=1}^m \operatorname{Fix} T_i$. Then, for every $i \in \{1,\ldots,m\}$ and every integer $n \geqslant K-1$, $x \in \operatorname{Fix} T_i \subset \operatorname{Fix} T_{i,\ell_i(n)}$. This allows us to deduce from Theorem 3.2(ii) that

$$(\forall i \in \{1,\ldots,m\}) \quad x_{\ell_i(n)} - T_{i,\ell_i(n)}x_{\ell_i(n)} = x_{\ell_i(n)} - T_{i,\ell_i(n)}x_{\ell_i(n)} + T_{i,\ell_i(n)}x - x \to 0. \tag{4.4}$$

We also recall from Theorem 3.2(iv) that

$$(\forall i \in \{1,\ldots,m\}) \quad x_{\ell_i(n)} - x_n \to 0. \tag{4.5}$$

(i): This follows from Theorem 3.2(i), (4.4), and (4.5).

(ii): Let $i \in \{1,\ldots,m\}$ and let $z \in \mathcal{H}$ be a weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$, say $x_{k_n} \rightharpoonup z$. In view of Theorem 3.2(vi)(a), it is enough to show that $z \in \operatorname{Fix} T_i$. We derive from (4.4) that $x_{\ell_i(k_n)} - T_{i,\ell_i(k_n)}x_{\ell_i(k_n)} \to 0$. On the other hand, (4.5) yields $x_{\ell_i(k_n)} = (x_{\ell_i(k_n)} - x_{k_n}) + x_{k_n} \rightharpoonup z$. Using (4.2), we obtain $z \in \operatorname{Fix} T_i$.

(iii): Let $z \in \mathcal{H}$ be a strong sequential cluster point of $(T_{i,\ell_i(n)}x_{\ell_i(n)})_{n\in\mathbb{N}}$, say $T_{i,\ell_i(k_n)}x_{\ell_i(k_n)} \to z$. Then (4.4) yields $x_{\ell_i(k_n)} \to z$. In turn, (4.5) implies that $x_{k_n} \to z$ and the conclusion follows from Theorem 3.2(vi)(b). □

Example 4.3 We revisit a problem investigated in [16]. Let m be a strictly positive integer, let $(\omega_i)_{1\leqslant i\leqslant m} \in \left]0,1\right]^m$ be such that $\sum_{i=1}^m \omega_i = 1$, and, for every $i \in \{1,\ldots,m\}$, let $\rho_i \in \left[0,+\infty\right[$ and let $A_i \colon \mathcal{H} \to 2^{\mathcal{H}}$ be maximally ρ_i-cohypomonotone in the sense that $A_i^{-1} + \rho_i \operatorname{Id}$ is maximally monotone. The task is to

$$\text{find } x \in \mathcal{H} \quad \text{such that} \quad (\forall i \in \{1,\ldots,m\}) \quad 0 \in A_i x, \tag{4.6}$$

under the assumption that such a point exists. Suppose that Assumption 3.1 is satisfied, let $\varepsilon \in]0,1[$, let $x_0 \in \mathcal{H}$, let $(t_{i,-1})_{1\leqslant i\leqslant m} \in \mathcal{H}^m$, and let $(\forall n \in \mathbb{N})(\forall i \in I_n)\ \gamma_{i,n} \in [\rho_i+\varepsilon,+\infty[$. Iterate

$$
\begin{array}{l}
\text{for } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\text{for every } i \in I_n\\
\left\lfloor\; t_{i,n} = x_n + \big(1-\rho_i/\gamma_{i,n}\big)\big(J_{\gamma_{i,n}A_i}x_n + e_{i,n} - x_n\big)\right.\\
\text{for every } i \in \{1,\ldots,m\} \smallsetminus I_n\\
\left\lfloor\; t_{i,n} = t_{i,n-1}\right.\\
x_{n+1} = \displaystyle\sum_{i=1}^{m} \omega_i t_{i,n}.
\end{array}
\right.
\end{array}
\tag{4.7}
$$

Then the following hold:

(i) $(x_n)_{n\in\mathbb{N}}$ converges weakly to a solution to (4.6).

(ii) Suppose that, for some $i \in \{1,\ldots,m\}$, $\operatorname{dom} A_i$ is boundedly relatively compact. Then $(x_n)_{n\in\mathbb{N}}$ converges strongly to a solution to (4.6).

Proof. Set

$$
(\forall i \in \{1,\ldots,m\}) \quad
\begin{cases}
T_i = \operatorname{Id} + \left(1-\dfrac{\rho_i}{\gamma_i}\right)\big(J_{\gamma_i A_i} - \operatorname{Id}\big), \quad \text{where} \quad \gamma_i \in]\rho_i,+\infty[\\
M_i = (A_i^{-1} + \rho_i \operatorname{Id})^{-1}.
\end{cases}
\tag{4.8}
$$

Then it follows from [6, Proposition 20.22] that the operators $(M_i)_{1\leqslant i\leqslant m}$ are maximally monotone and therefore from [16, Lemma 2.4] and [6, Corollary 23.9] that

$$
(\forall i \in \{1,\ldots,m\}) \quad T_i = J_{(\gamma_i-\rho_i)M_i} \text{ is firmly nonexpansive and } \operatorname{Fix} T_i = \operatorname{zer} M_i = \operatorname{zer} A_i, \tag{4.9}
$$

which makes (4.6) an instantiation of Problem 4.1. Now set

$$
(\forall n \in \mathbb{N})(\forall i \in I_n) \quad T_{i,n} = \operatorname{Id} + \left(1-\frac{\rho_i}{\gamma_{i,n}}\right)\big(J_{\gamma_{i,n}A_i} - \operatorname{Id}\big) \quad \text{and} \quad e'_{i,n} = \left(1-\frac{\rho_i}{\gamma_{i,n}}\right)e_{i,n}. \tag{4.10}
$$

Then $(\forall i \in \{1,\ldots,m\})\ \sum_{n\geqslant K-1}\|e'_{i,\ell_i(n)}\| \leqslant \sum_{n\geqslant K-1}\|e_{i,\ell_i(n)}\| < +\infty$. In addition, $(\forall n \in \mathbb{N})(\forall i \in I_n)$ $t_{i,n} = T_{i,n}x_n + e'_{i,n}$. This places (4.7) in the same operating conditions as (4.1). We also derive from [16, Lemma 2.4] that

$$
(\forall n \in \mathbb{N})(\forall i \in I_n) \quad T_{i,n} = J_{(\gamma_{i,n}-\rho_i)M_i} \text{ is firmly nonexpansive and } \operatorname{Fix} T_{i,n} = \operatorname{zer} M_i = \operatorname{zer} A_i. \tag{4.11}
$$

(i): In view of Corollary 4.2(ii), it suffices to check that condition (4.2) holds. Let us take $z \in \mathcal{H}$, $i \in \{1,\ldots,m\}$, and a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of integers greater than K such that

$$
x_{\ell_i(k_n)} \rightharpoonup z \quad \text{and} \quad x_{\ell_i(k_n)} - T_{i,\ell_i(k_n)}x_{\ell_i(k_n)} \to 0. \tag{4.12}
$$

Then we must show that $0 \in A_i z$. Note that

$$
T_{i,\ell_i(k_n)}x_{\ell_i(k_n)} \rightharpoonup z. \tag{4.13}
$$

Now set

$$
(\forall n \in \mathbb{N}) \quad u_{\ell_i(k_n)} = \big(\gamma_{i,\ell_i(k_n)} - \rho_i\big)^{-1}\big(x_{\ell_i(k_n)} - T_{i,\ell_i(k_n)}x_{\ell_i(k_n)}\big). \tag{4.14}
$$

Then

$$\|u_{\ell_i(k_n)}\| = \frac{\|x_{\ell_i(k_n)} - T_{i,\ell_i(k_n)}x_{\ell_i(k_n)}\|}{\gamma_{i,\ell_i(k_n)} - \rho_i} \leqslant \frac{\|x_{\ell_i(k_n)} - T_{i,\ell_i(k_n)}x_{\ell_i(k_n)}\|}{\varepsilon} \to 0. \tag{4.15}$$

On the other hand, we derive from (4.11) that $(\forall n \in \mathbb{N})\ T_{i,\ell_i(k_n)} = J_{(\gamma_{i,\ell_i(k_n)}-\rho_i)M_i}$. Therefore, (4.14) yields

$$(\forall n \in \mathbb{N}) \quad \big(T_{i,\ell_i(k_n)}x_{\ell_i(k_n)}, u_{\ell_i(k_n)}\big) \in \operatorname{gra} M_i. \tag{4.16}$$

However, since M_i is maximally monotone, gra M_i is sequentially closed in $\mathcal{H}^{\text{weak}} \times \mathcal{H}^{\text{strong}}$ [6, Proposition 20.38(ii)]. Hence, (4.13), (4.15), and (4.16) imply that $z \in \operatorname{zer} M_i = \operatorname{zer} A_i$.

(ii): By (4.11), for every $n \geqslant K-1$, $T_{i,\ell_i(n)}x_{\ell_i(n)} \in \operatorname{ran} T_{i,\ell_i(n)} = \operatorname{dom}(\operatorname{Id} + (\gamma_{i,\ell_i(n)} - \rho_i)M_i) = \operatorname{dom} M_i$. However, Corollary 4.2(i) asserts that $(T_{i,\ell_i(n)}x_{\ell_i(n)})_{n\in\mathbb{N}}$ lies in a closed ball. Altogether, it possesses a strong sequential cluster point and the conclusion follows from Corollary 4.2(iii). ☐

Remark 4.4 Suppose that, in Example 4.3, the operators $(A_i)_{1\leqslant i\leqslant m}$ are maximally monotone, i.e., $(\forall i \in \{1,\ldots,m\})\ \rho_i = 0$. Suppose that, in addition, all the operators are used at each iteration, i.e., $(\forall n \in \mathbb{N})\ I_n = \{1,\ldots,m\}$. Then the implementation of (4.7) with no errors reduces to the barycentric proximal method of [31].

Example 4.5 As shown in [19], many problems in data science and harmonic analysis can be cast as follows. Let m be a strictly positive integer and, for every $i \in \{1,\ldots,m\}$, let $R_i\colon \mathcal{H} \to \mathcal{H}$ be firmly nonexpansive and let $r_i \in \mathcal{H}$. The task is to

$$\text{find } x \in \mathcal{H} \quad \text{such that} \quad (\forall i \in \{1,\ldots,m\}) \quad r_i = R_i x, \tag{4.17}$$

under the assumption that such a point exists. Let $(\omega_i)_{1\leqslant i\leqslant m} \in \left]0,1\right]^m$ be such that $\sum_{i=1}^m \omega_i = 1$, suppose that Assumption 3.1 is satisfied, let $x_0 \in \mathcal{H}$, and let $(t_{i,-1})_{1\leqslant i\leqslant m} \in \mathcal{H}^m$. Iterate

$$\begin{array}{l} \text{for } n = 0,1,\ldots \\ \left\lfloor \begin{array}{l} \text{for every } i \in I_n \\ \left\lfloor t_{i,n} = r_i + x_n - R_i x_n + e_{i,n} \right. \\ \text{for every } i \in \{1,\ldots,m\} \smallsetminus I_n \\ \left\lfloor t_{i,n} = t_{i,n-1} \right. \\ x_{n+1} = \displaystyle\sum_{i=1}^m \omega_i t_{i,n}. \end{array} \right. \end{array} \tag{4.18}$$

Then the following hold:

(i) $(x_n)_{n\in\mathbb{N}}$ converges weakly to a solution to (4.17).

(ii) Suppose that, for some $i \in \{1,\ldots,m\}$, $\operatorname{Id} - R_i$ is demicompact. Then $(x_n)_{n\in\mathbb{N}}$ converges strongly to a solution to (4.17).

Proof. Following [19], (4.17) can be formulated as an instance of Problem 4.1, by choosing $(\forall i \in \{1,\ldots,m\})\ T_i = r_i + \operatorname{Id} - R_i$. A straightforward implementation of (4.1) consists of setting $(\forall n \in \mathbb{N})(\forall i \in I_n)\ T_{i,n} = T_i$, which reduces (4.1) to (4.18).

(i): Since the operators $(T_i)_{1\leqslant i\leqslant m}$ are nonexpansive, [6, Theorem 4.27] asserts that the operators $(\operatorname{Id} - T_i)_{1\leqslant i\leqslant m}$ are demiclosed, which implies that condition (4.2) holds. Thus, the claim follows from Corollary 4.2(ii).

(ii): We deduce from (4.4) that $x_{\ell_i(n)} - T_i x_{\ell_i(n)} \to 0$, and from (4.5) and (i) that $(x_{\ell_i(n)})_{n\in\mathbb{N}}$ is bounded. Hence, since T_i is demicompact, $(x_{\ell_i(n)})_{n\in\mathbb{N}}$ has a strong sequential cluster point and so does $(T_i x_{\ell_i(n)})_{n\in\mathbb{N}}$. We conclude with Corollary 4.2(iii). □

Remark 4.6 If (4.17) has no solution, (4.18) will produce a fixed point of the operator $\sum_{i=1}^m \omega_i T_i = \mathrm{Id} + \sum_{i=1}^m \omega_i (r_i - R_i)$, provided one exists. As discussed in [19], this is a valid relaxation of (4.17).

4.2 Forward-backward operator splitting

We consider the following monotone inclusion problem.

Problem 4.7 Let m be a strictly positive integer and let $(\omega_i)_{1\leqslant i\leqslant m} \in \left]0,1\right]^m$ be such that $\sum_{i=1}^m \omega_i = 1$. Let $A_0 \colon \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone and, for every $i \in \{1,\ldots,m\}$, let $\beta_i \in \left]0,+\infty\right[$ and let $A_i \colon \mathcal{H} \to \mathcal{H}$ be β_i-cocoercive, i.e.,

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \langle x - y \mid A_i x - A_i y\rangle \geqslant \beta_i \|A_i x - A_i y\|^2. \tag{4.19}$$

The task is to find $x \in \mathcal{H}$ such that $0 \in A_0 x + \sum_{i=1}^m \omega_i A_i x$.

Remark 4.8 In Problem 4.7, suppose that A_0 is the normal cone operator of a nonempty closed convex set C, i.e., $A_0 = \partial\iota_C$. Then the problem is to solve the variational inequality

$$\text{find} \;\; x \in C \quad \text{such that} \quad (\forall y \in \mathcal{H}) \quad \left\langle x - y \,\middle|\, \sum_{i=1}^m \omega_i A_i x \right\rangle \leqslant 0. \tag{4.20}$$

If $m = 1$, a standard method for solving Problem 4.7 is the forward-backward splitting algorithm [11, 42, 44]. We propose below a multi-operator version of it with block-updates.

Proposition 4.9 *Consider the setting of Problem 4.7 under Assumption 3.1 and the assumption that it has a solution. Let $\gamma \in \left]0, 2\min_{1\leqslant i\leqslant m} \beta_i\right[$, let $x_0 \in \mathcal{H}$, let $(t_{i,-1})_{1\leqslant i\leqslant m} \in \mathcal{H}^m$, and iterate*

$$\begin{array}{l}
\text{for } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\text{for every } i \in I_n\\
\left\lfloor\; t_{i,n} = x_n - \gamma(A_i x_n + e_{i,n})\right.\\
\text{for every } i \in \{1,\ldots,m\} \smallsetminus I_n\\
\left\lfloor\; t_{i,n} = t_{i,n-1}\right.\\
x_{n+1} = J_{\gamma A_0}\left(\displaystyle\sum_{i=1}^m \omega_i t_{i,n}\right) + e_{0,n}.
\end{array}
\right.
\end{array} \tag{4.21}$$

Then the following hold:

(i) *Let x be a solution to Problem 4.7 and let $i \in \{1,\ldots,m\}$. Then $A_i x_n \to A_i x$.*

(ii) *$(x_n)_{n\in\mathbb{N}}$ converges weakly to a solution to Problem 4.7.*

(iii) *Suppose that, for some $i \in \{0,\ldots,m\}$, A_i is demiregular. Then $(x_n)_{n\in\mathbb{N}}$ converges strongly to a solution to Problem 4.7.*

(iv) *Suppose that, for some $i \in \{0,\ldots,m\}$, A_i is strongly monotone. Then $(x_n)_{n\in\mathbb{N}}$ converges linearly to the unique solution to Problem 4.7.*

Proof. We apply Corollary 3.4 with $T_0 = J_{\gamma A_0}$ and $(\forall i \in \{1,\ldots,m\})$ $T_i = \mathrm{Id} - \gamma A_i$. It follows from [6, Proposition 4.39 and Corollary 23.9] that the operators $(T_i)_{0\leqslant i\leqslant m}$ are averaged, and hence from [6, Proposition 26.1(iv)(a)] that Problem 4.7 coincides with Problem 1.1. In addition, (4.21) is an instance of (3.39).

(i): See Corollary 3.4(i).

(ii): See Corollary 3.4(ii).

(iii): This follows from Corollary 3.4(iii). Indeed, if $i = 0$, the demicompactness of T_i follows from Example 3.5(iii)(a). On the other hand, if $i \neq 0$, take a bounded sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that $(y_n - T_i y_n)_{n\in\mathbb{N}}$ converges, say $y_n - T_i y_n \to u$. Then $A_i y_n \to u/\gamma$. On the other hand, $(y_n)_{n\in\mathbb{N}}$ has a weak sequential cluster point, say $y_{k_n} \rightharpoonup y$. So by demiregularity of A_i, $y_{k_n} \to y$, which shows that T_i is demicompact.

(iv): If $i = 0$, we derive from [6, Proposition 23.13] that $T_0 = J_{\gamma A_0}$ is a Banach contraction. If $i \neq 0$, as in the proof of [6, Proposition 26.16], we obtain that $T_i = \mathrm{Id} - \gamma A_i$ is a Banach contraction. The conclusion follows from Corollary 3.4(iv). □

Example 4.10 Consider maximally operators $A_0 : \mathcal{H} \to 2^{\mathcal{H}}$ and, for every $i \in \{1,\ldots,m\}$, $B_i : \mathcal{H} \to 2^{\mathcal{H}}$. The associated common zero problem is [10, 31, 46]

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in A_0 x \cap \bigcap_{i=1}^{m} B_i x. \tag{4.22}$$

As shown in [12], when (4.22) has no solution, a suitable relaxation is

$$\text{find } x \in \mathcal{H} \quad \text{such that} \quad 0 \in A_0 x + \sum_{i=1}^{m} \omega_i (B_i \,\square\, C_i) x \tag{4.23}$$

where, for every $i \in \{1,\ldots,m\}$, $C_i : \mathcal{H} \to 2^{\mathcal{H}}$ is such that C_i^{-1} is at most single-valued and strictly monotone, with $C_i^{-1} 0 = \{0\}$. In this setting, if (4.22) happens to have solutions, they coincide with those of (4.23) [12]. Let us consider the particular instance in which, for every $i \in \{1,\ldots,m\}$, C_i is cocoercive, and set $A_i = B_i \,\square\, C_i$. Then the operators $(C_i^{-1})_{1\leqslant i\leqslant m}$ are strongly monotone and, therefore, the operators $(A_i)_{1\leqslant i\leqslant m}$ are cocoercive. In addition, (4.23) is a special case of Problem 4.7, which can be solved via Proposition 4.9. Let us observe that if we further specialize by setting, for every $i \in \{1,\ldots,m\}$, $C_i = \rho_i^{-1}\,\mathrm{Id}$ for some $\rho_i \in \left]0,+\infty\right[$, then (4.23) reduces to (1.3).

We now focus on minimization problems.

Problem 4.11 Let m be a strictly positive integer and let $(\omega_i)_{1\leqslant i\leqslant m} \in \left]0,1\right]^m$ be such that $\sum_{i=1}^m \omega_i = 1$. Let $f_0 \in \Gamma_0(\mathcal{H})$ and, for every $i \in \{1,\ldots,m\}$, let $\beta_i \in \left]0,+\infty\right[$ and let $f_i : \mathcal{H} \to \mathbb{R}$ be a differentiable convex function with a $1/\beta_i$-Lipschitzian gradient. The task is to

$$\underset{x\in\mathcal{H}}{\text{minimize}} \;\; f_0(x) + \sum_{i=1}^{m} \omega_i f_i(x). \tag{4.24}$$

Proposition 4.12 *Consider the setting of Problem 4.11 under Assumption 3.1 and assume that*

$$\lim_{\substack{x\in\mathcal{H}\\ \|x\|\to+\infty}} \left(f_0(x) + \sum_{i=1}^{m} \omega_i f_i(x) \right) = +\infty. \tag{4.25}$$

Let $\gamma\in\,]0,2\min_{1\leqslant i\leqslant m}\beta_i[$*, let* $x_0\in\mathcal{H}$*, let* $(t_{i,-1})_{1\leqslant i\leqslant m}\in\mathcal{H}^m$*, and iterate*

$$
\begin{array}{l}
\text{for } n=0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\text{for every } i\in I_n\\
\left\lfloor\; t_{i,n}=x_n-\gamma(\nabla f_i(x_n)+e_{i,n})\right.\\
\text{for every } i\in\{1,\ldots,m\}\smallsetminus I_n\\
\left\lfloor\; t_{i,n}=t_{i,n-1}\right.\\
x_{n+1}=\operatorname{prox}_{\gamma f_0}\left(\displaystyle\sum_{i=1}^m\omega_i t_{i,n}\right)+e_{0,n}.
\end{array}
\right.
\end{array}
\tag{4.26}
$$

Then the following hold:

(i) *Let* x *be a solution to Problem 4.11 and let* $i\in\{1,\ldots,m\}$*. Then* $\nabla f_i(x_n)\to\nabla f_i(x)$*.*

(ii) $(x_n)_{n\in\mathbb{N}}$ *converges weakly to a solution to Problem 4.11.*

(iii) *Suppose that, for some* $i\in\{0,\ldots,m\}$*, one of the following holds:*

  (a) f_i *is uniformly convex.*

  (b) *The lower level sets of* f_i *are boundedly compact.*

  Then $(x_n)_{n\in\mathbb{N}}$ *converges strongly to a solution to Problem 4.11.*

(iv) *Suppose that, for some* $i\in\{0,\ldots,m\}$*,* f_i *is strongly convex, i.e., it satisfies (3.48) with* $\phi=|\cdot|^2/2$*. Then* $(x_n)_{n\in\mathbb{N}}$ *converges linearly to the unique solution to Problem 4.11.*

Proof. We derive from [6, Theorem 20.25] that $A_0=\partial f_0$ is maximally monotone and from [6, Corollary 18.17] that, for every $i\in\{1,\ldots,m\}$, $A_i=\nabla f_i$ is β_i-cocoercive. In this setting, it follows from [6, Corollary 27.3(i)] that Problem 4.7 reduces to Problem 4.11. On the other hand, since the assumptions imply that $f_0+\sum_{i=1}^m\omega_i f_i$ is proper, lower semicontinuous, convex, and coercive, it follows from [6, Corollary 11.16(ii)] that Problem 4.11 has a solution. The claims therefore follow from Proposition 4.9, Example 3.5(iii)(c)&(iii)(d), and [6, Example 22.4(iv)]. □

An algorithm related to (4.26) has recently been proposed in [35] in a finite-dimensional setting; see also [36] for a special case.

We illustrate an application of Proposition 4.12 in the context of a variational model that captures various formulations found in data analysis.

Example 4.13 Suppose that $\mathcal{H}$ is separable, let $(e_k)_{k\in\mathbb{K}\subset\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$, and, for every $k\in\mathbb{K}$, let $\psi_k\in\Gamma_0(\mathbb{R})$ be such that $\psi_k\geqslant 0=\psi_k(0)$. For every $i\in\{1,\ldots,m\}$, let $0\neq a_i\in\mathcal{H}$, let $\mu_i\in\,]0,+\infty[$, and let $\phi_i\colon\mathbb{R}\to[0,+\infty[$ be a differentiable convex function such that ϕ_i' is μ_i-Lipschitzian. The task is to

$$
\underset{x\in\mathcal{H}}{\text{minimize}}\;\;\sum_{k\in\mathbb{K}}\psi_k(\langle x\mid e_k\rangle)+\frac{1}{m}\sum_{i=1}^m\phi_i(\langle x\mid a_i\rangle).
\tag{4.27}
$$

Let us note that (4.27) is an instantiation of (4.24) with $f_0=\sum_{k\in\mathbb{K}}\psi_k\circ\langle\cdot\mid e_k\rangle$ and, for every $i\in\{1,\ldots,m\}$, $f_i=\phi_i\circ\langle\cdot\mid a_i\rangle$ and $\omega_i=1/m$. The fact that $f_0\in\Gamma_0(\mathcal{H})$ is established in [18], where it is also shown that, given $\gamma\in\,]0,+\infty[$,

$$
\operatorname{prox}_{\gamma f_0}\colon x\mapsto\sum_{k\in\mathbb{K}}\big(\operatorname{prox}_{\gamma\psi_k}\langle x\mid e_k\rangle\big)e_k.
\tag{4.28}
$$

On the other hand, for every $i \in \{1, \ldots, m\}$, f_i is a differentiable convex function and its gradient

$$\nabla f_i \colon x \mapsto \phi_i'(\langle x \mid a_i \rangle) a_i \tag{4.29}$$

has Lipschitz constant $\mu_i \|a_i\|^2$. Let $\gamma \in \left]0, 2/(\max_{1 \leqslant i \leqslant m} \mu_i \|a_i\|^2)\right[$ and let $(I_n)_{n \in \mathbb{N}}$ be as in Assumption 3.1. In view of (4.26), (4.28), and (4.29), we can solve (4.27) via the algorithm

$$\begin{array}{l}
\text{for } n = 0, 1, \ldots \\
\left\lfloor
\begin{array}{l}
\text{for every } i \in I_n \\
\left\lfloor\; t_{i,n} = x_n - \gamma \phi_i'(\langle x_n \mid a_i \rangle) a_i \right. \\
\text{for every } i \in \{1, \ldots, m\} \smallsetminus I_n \\
\left\lfloor\; t_{i,n} = t_{i,n-1} \right. \\
y_n = \displaystyle\sum_{i=1}^{m} \omega_i t_{i,n} \\
x_{n+1} = \displaystyle\sum_{k \in \mathbb{K}} \big(\mathrm{prox}_{\gamma \psi_k} \langle y_n \mid e_k \rangle\big) e_k.
\end{array}
\right.
\end{array} \tag{4.30}$$

Infinite-dimensional instances of (4.27) are discussed in [17, 18, 20, 21]. A popular finite-dimensional setting is obtained by choosing $\mathcal{H} = \mathbb{R}^N$, $\mathbb{K} = \{1, \ldots, N\}$, $(e_k)_{1 \leqslant k \leqslant N}$ as the canonical basis, $\alpha \in \left]0, +\infty\right[$, and, for every $k \in \mathbb{K}$, $\psi_k = \alpha |\cdot|$. This reduces (4.27) to

$$\underset{x \in \mathbb{R}^N}{\text{minimize}} \;\; \alpha \|x\|_1 + \sum_{i=1}^{m} \phi_i(\langle x \mid a_i \rangle). \tag{4.31}$$

Thus, choosing for every $i \in \{1, \ldots, m\}$ $\phi_i \colon t \mapsto |t - \eta_i|^2$, where $\eta_i \in \mathbb{R}$ models an observation, yields the Lasso formulation, whereas choosing $\phi_i \colon t \mapsto \ln(1 + \exp(t)) - \eta_i t$, where $\eta_i \in \{0, 1\}$ models a label, yields the penalized logistic regression framework [27].

4.3 Hard constrained inconsistent convex feasibility problems

The next application revisits a model proposed in [14] to relax inconsistent feasibility problems.

Problem 4.14 Let m be a strictly positive integer and let $(\omega_i)_{1 \leqslant i \leqslant m} \in \left]0, 1\right]^m$ be such that $\sum_{i=1}^m \omega_i = 1$. Let C_0 be a nonempty closed convex subset of $\mathcal{H}$ and, for every $i \in \{1, \ldots, m\}$, let $\mathcal{G}_i$ be a real Hilbert space, let $L_i \colon \mathcal{H} \to \mathcal{G}_i$ be a nonzero bounded linear operator, let D_i be a nonempty closed convex subset of $\mathcal{G}_i$, let $\mu_i \in \left]0, +\infty\right[$, and let $\phi_i \colon \mathbb{R} \to \left[0, +\infty\right[$ be an even differentiable convex function that vanishes only at 0 and such that ϕ_i' is μ_i-Lipschitzian. The task is to

$$\underset{x \in C_0}{\text{minimize}} \;\; \sum_{i=1}^{m} \omega_i \phi_i\big(d_{D_i}(L_i x)\big). \tag{4.32}$$

The variational formulation (4.32) is a relaxation of the convex feasibility problem

$$\text{find } \; x \in C_0 \quad \text{such that} \quad (\forall i \in \{1, \ldots, m\}) \quad L_i x \in D_i \tag{4.33}$$

in the sense that, if (4.33) is consistent, then its solution set is precisely that of (4.32); see [14, Section 4.4] for details on this formulation and background on inconsistent convex feasibility problems. Here C_0 models a hard constraint. An early instance of (4.33) as a relaxation of (4.32) is Legendre's method of least-squares to deal with an inconsistent system of m linear equations in $\mathcal{H} = \mathbb{R}^N$ [30].

There, $C_0 = \mathbb{R}^N$ and, for every $i \in \{1,\ldots,m\}$, $\mathcal{G}_i = \mathbb{R}$, $D_i = \{\beta_i\}$, $L_i = \langle \cdot \mid a_i \rangle$ for some $a_i \in \mathbb{R}^N$ such that $\|a_i\| = 1$, $\omega_i = 1/m$, and $\phi_i = |\cdot|^2$. The formulation (4.32) can also be regarded as a smooth version of the set-theoretic Fermat-Weber problem [37] arising in location theory, namely,

$$\underset{x\in\mathcal{H}}{\text{minimize}} \;\; \frac{1}{m}\sum_{i=1}^{m} d_{C_i}(x). \tag{4.34}$$

The following version of the Closed Range Theorem will be required.

Lemma 4.15 [22, Theorem 8.18] *Let $\mathcal{G}$ be a real Hilbert space and let $L\colon \mathcal{H} \to \mathcal{G}$ be a nonzero bounded linear operator. Then* ran L *is closed* $\Leftrightarrow$ ran $L^* \circ L$ *is closed* $\Leftrightarrow$ $(\exists\, \rho \in \left]0,+\infty\right[)(\forall x \in (\ker L)^{\perp})\ \|Lx\| \geqslant \rho\|x\|$.

Corollary 4.16 *Consider the setting of Problem 4.14 under one of the following assumptions:*

[a] *There exists $j \in \{1,\ldots,m\}$ such that $\lim_{\|x\|\to+\infty} \big(\iota_{C_0}(x) + \phi_j\big(d_{D_j}(L_j x)\big)\big) = +\infty$.*

[b] *There exists $j \in \{1,\ldots,m\}$ such that* ran L_j *is closed, $C_0 \subset (\ker L_j)^{\perp}$, and D_j is bounded.*

[c] *There exists $j \in \{1,\ldots,m\}$ such that $\mathcal{G}_j = \mathcal{H}$, $L_j = \mathrm{Id}$, and D_j is bounded.*

[d] *C_0 is bounded.*

Set $\beta = 1/(\max_{1\leqslant i\leqslant m} \mu_i\|L_i\|^2)$, let $\gamma \in \left]0,2\beta\right[$, let $(I_n)_{n\in\mathbb{N}}$ be as in Assumption 3.1, let $x_0 \in C_0$, let $(t_{i,-1})_{1\leqslant i\leqslant m} \in \mathcal{H}^m$, and iterate

$$\begin{array}{l}
\text{for } n = 0,1,\ldots\\
\left\lfloor\begin{array}{l}
\text{for every } i \in I_n\\
\left\lfloor\begin{array}{l}
\text{if } L_i x_n \notin D_i\\
\left\lfloor\; t_{i,n} = x_n - \gamma\dfrac{\phi_i'\big(d_{D_i}(L_i x_n)\big)}{d_{D_i}(L_i x_n)} L_i^*\big(L_i x_n - \mathrm{proj}_{D_i}(L_i x_n)\big)\right.\\
\text{else}\\
\left\lfloor\; t_{i,n} = x_n\right.
\end{array}\right.\\
\text{for every } i \in \{1,\ldots,m\} \smallsetminus I_n\\
\left\lfloor\; t_{i,n} = t_{i,n-1}\right.\\
x_{n+1} = \mathrm{proj}_{C_0}\left(\displaystyle\sum_{i=1}^{m} \omega_i t_{i,n}\right).
\end{array}\right.
\end{array} \tag{4.35}$$

Then the following hold:

(i) *$(x_n)_{n\in\mathbb{N}}$ converges weakly to a solution to Problem 4.14.*

(ii) *Suppose that one of the following holds:*

[e] *Condition [b] is satisfied with the additional assumptions that $\phi_j = \mu_j|\cdot|^2/2$ and D_j is compact.*

[f] *C_0 is boundedly compact.*

Then $(x_n)_{n\in\mathbb{N}}$ converges strongly to a solution to Problem 4.14.

Proof. We first note that (4.32) is an instance of (4.24) with $f_0 = \iota_{C_0}$ and $(\forall i \in \{1,\ldots,m\})$ $f_i = \phi_i \circ d_{D_i} \circ L_i$. Next, we derive from [6, Example 2.7] that, for every $i \in \{1,\ldots,m\}$, f_i is convex and differentiable, and that its gradient

$$\nabla f_i \colon \mathcal{H} \to \mathcal{H} \colon x \mapsto \begin{cases} \dfrac{\phi_i'\big(d_{D_i}(L_i x)\big)}{d_{D_i}(L_i x)} L_i^*\big(L_i x - \operatorname{proj}_{D_i}(L_i x)\big), & \text{if } \; L_i x \notin D_i; \\ 0, & \text{if } \; L_i x \in D_i \end{cases} \tag{4.36}$$

has Lipschitz constant $\mu_i \|L_i\|^2$. Hence (4.35) is an instance of (4.26). Now, in order to apply Proposition 4.12, let us check that (4.25) is satisfied under one of assumptions [a]–[d].

[a]: We have $f_0(x) + \sum_{i=1}^m \omega_i f_i(x) \geqslant \omega_j(\iota_{C_0}(x) + f_j(x)) \to +\infty$ as $\|x\| \to +\infty$.

[b]⇒[a]: In view of [d], we assume that C_0 is unbounded. It follows from Lemma 4.15 that there exists $\rho \in \left]0,+\infty\right[$ such that $(\forall x \in (\ker L_j)^\perp)$ $\|L_j x\| \geqslant \rho\|x\|$. Hence,

$$(\forall x \in C_0) \quad \|L_j x\| \geqslant \rho\|x\|. \tag{4.37}$$

Now let $z \in \mathcal{G}_j$. Then, since D_j is bounded, $\delta = \operatorname{diam}(D_j) + \|\operatorname{proj}_{D_j} z\| < +\infty$ and

$$(\forall y \in \mathcal{G}_j) \quad \|y\| \leqslant \|y - \operatorname{proj}_{D_j} y\| + \|\operatorname{proj}_{D_j} y - \operatorname{proj}_{D_j} z\| + \|\operatorname{proj}_{D_j} z\| \leqslant d_{D_j}(y) + \delta. \tag{4.38}$$

Consequently, $d_{D_j}(y) \to +\infty$ as $\|y\| \to +\infty$ with $y \in \mathcal{G}_j$. Thus, since ϕ_j is coercive by [6, Proposition 16.23], we obtain

$$\phi_j\big(d_{D_j}(y)\big) \to +\infty \quad \text{as} \quad \|y\| \to +\infty \quad \text{with} \quad y \in \mathcal{G}_j. \tag{4.39}$$

We deduce from (4.37) and (4.39) that

$$f_j(x) \to +\infty \quad \text{as} \quad \|x\| \to +\infty \quad \text{with} \quad x \in C_0. \tag{4.40}$$

[c]⇒[b] and [d]⇒[a]: Clear.
We are now ready to use Proposition 4.12 to prove the assertions.

(i): Apply Proposition 4.12(ii).

(ii)[e]: Let x be the weak limit in (i) and set $u_j = \nabla f_j(x)/\mu_j$. Then Proposition 4.12(i) asserts that

$$L_j^*\big(L_j x_n - \operatorname{proj}_{D_j}(L_j x_n)\big) \to u_j. \tag{4.41}$$

We also observe that, since $L_j^* \circ L_j$ is weakly continuous [6, Lemma 2.41], we have $L_j^*(L_j x_n) \rightharpoonup L_j^*(L_j x)$. Therefore, (4.41) yields

$$L_j^*\big(\operatorname{proj}_{D_j}(L_j x_n)\big) \rightharpoonup L_j^*(L_j x) - u_j. \tag{4.42}$$

However, the set $L_j^*(D_j)$ is compact by [6, Lemma 1.20] and it contains $(L_j^*(\operatorname{proj}_{D_j}(L_j x_n)))_{n\in\mathbb{N}}$. This sequence has therefore $L_j^*(L_j x) - u_j$ as its unique strong sequential cluster point. Thus, $L_j^*(\operatorname{proj}_{D_j}(L_j x_n)) \to L_j^*(L_j x) - u_j$ and we deduce from (4.41) that

$$L_j^*(L_j x_n) \to L_j^*(L_j x). \tag{4.43}$$

On the other hand, for every $n \in \mathbb{N}$, since x and x_n lie in $C_0 \subset (\ker L_j)^\perp$, we have $x_n - x \in (\ker L_j)^\perp = (\ker L_j^* \circ L_j)^\perp$. Hence, we deduce from (4.43) and Lemma 4.15 that there exists $\theta \in \left]0,+\infty\right[$ such that

$$\theta\|x_n - x\| \leqslant \|(L_j^* \circ L_j)(x_n - x)\| \to 0. \tag{4.44}$$

We conclude that $x_n \to x$.

(ii)[f]: This follows from Proposition 4.12(iii)(b) since the lower level sets of f_0 are the compact sets $\{\varnothing, C_0\}$. ☐

We conclude by revisiting (1.1) and recovering a classical result on the method of alternating projections.

Example 4.17 [8, Theorem 4(a)] Let C and D be nonempty closed convex subsets of $\mathcal{H}$ such that D is compact. Let $x_0 \in \mathcal{H}$ and set $(\forall n \in \mathbb{N})\ x_{n+1} = \text{proj}_C(\text{proj}_D x_n)$. Then $(x_n)_{n\in\mathbb{N}}$ converges strongly to a point in $x \in C$ such that $x = \text{proj}_C(\text{proj}_D x)$.

Proof. Apply Corollary 4.16(ii)[e] with $m = 1$, $C_0 = C$, $\mathcal{G}_1 = \mathcal{H}$, $L_1 = \mathrm{Id}$, $D_1 = D$, $\gamma = 1$, and $\mu_1 = 1$. ☐

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