Proximal and resolvent averages

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Table of contents

Introduction

The feasibility problem and projection methods

Let $C_1, C_2, \ldots, C_m$ be sets in a Hilbert space X, which we assume to be closed, convex, $\neq \varnothing$. The convex feasibility problem asks to

$$\text{find} \qquad x \in C := C_1 \cap C_2 \cap \cdots \cap C_m.$$

We assume that the sets C_i are "simple" in the sense that the nearest point mappings (projection operators) P_i or

$$P_{C_i} : x \mapsto \operatorname*{argmin}_{c_i \in C_i} \|x - c_i\|$$

are easy to compute.

A projection method combines the projectors in some algorithmic fashion to generate a sequence converging to a solution of the feasibility problem.

Cyclic/alternating projections

The method of cyclic projections generates a sequence $(x_n)_{n\in\mathbb{N}}$ via

$$x_0 \xmapsto{P_1} x_1 \xmapsto{P_2} x_2 \cdots x_{m-1} \xmapsto{P_m} x_m \xmapsto{P_1} x_{m+1} \xmapsto{P_2} x_{m+2} \cdots$$

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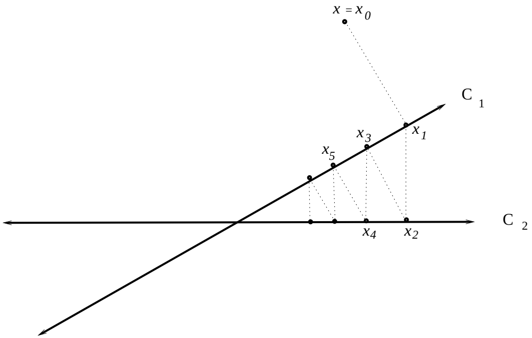
When $m = 2$, this is also called alternating projections:



Method of alternating projections (for $m = 2$ subspaces)

von Neumann's result for subspaces

Theorem. (von Neumann, 1935)
Suppose that C_1 and C_2 are subspaces. The sequence generated by the method of alternating projections converges strongly to the projection of the starting point onto the intersection.

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Remark. (Aronszajn, 1950)
If the angle

$$\arccos \sup_{c_i \in C_i \cap (C_1 \cap C_2)^\perp, \|c_i\| \leq 1} \langle c_1, c_2 \rangle$$

between the subspaces is positive, then the rate of convergence is linear.

Bregman's weak convergence result for convex sets

Theorem. (Bregman, 1965)
Given a starting point $x_0 \in X$, define $(x_n)_{n\in\mathbb{N}}$, the sequence of alternating projections, by

$$x_0 \xrightarrow{P_1} x_1 \xrightarrow{P_2} x_2 \xrightarrow{P_1} x_3 \xrightarrow{P_2} x_4 \xrightarrow{P_1} \cdots .$$

Then

$$x_n \rightharpoonup \bar{c} \in C.$$

Regularity

Remark. (Gubin-Polyak-Raik, 1967)
If $\big(C_1 \cap \mathrm{int}(C_2)\big) \cup \big(C_2 \cap \mathrm{int}(C_1)\big) \neq \varnothing$, then $x_n \to \bar{c} \in C$ strongly (even linearly).

Remark. The results by Aronszajn and by Gubin-Polyak-Raik can be unified: indeed, either assumption implies the Attouch-Brezis constraint qualification

$$\bigcup_{\rho > 0} \rho(C_1 - C_2) \text{ is a closed subspace,}$$

which in turn yields linear convergence (B-Borwein).

Hundal's counterexample

Hundal's counterexample, 2004.
In $X = \ell_2$, there exist two closed convex sets H and K, a vector $f \in X$, and a starting point $y_0 \in K$ so that:

- $\|f\| = 1$;
- H is the hyperplane $\{f\}^\perp$;
- K is a closed convex cone with $\sup\langle f, K\rangle = 0$;
- $H \cap K = \{0\}$.

Then the sequence of alternating projections

converges weakly to 0, but not strongly.

Random projections (for the consistent case)

Rather than projecting cyclically let us "roll a die" instead: let

$$r : \mathbb{N} \to I = \{1, \ldots, m\}$$

be a random map, i.e., $r^{-1}(i)$ is infinite for every $i \in I$, and $x_0 \in X$. Consider the sequence of random projections

$$x_{n+1} = P_{C_{r(n)}} x_n.$$

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Open Problem:
In the convex case, must $(x_n)_{n\in\mathbb{N}}$ converge weakly to a point in C?
(OK if $m = 2$; also OK if $m = 3$ by Dye and Reich, 1992.)

The inconsistent case when $m = 2$

Define the gap vector

$$v := P_{\overline{C_2 - C_1}} 0,$$

and the "generalized solution sets"

$$E_1 := C_1 \cap (C_2 - v) \text{ and } E_2 := (C_1 + v) \cap C_2.$$

(If $C_1 \cap C_2 \neq \varnothing$, then $v = 0$ and $E_1 = E_2 = C_1 \cap C_2$.)
Then $E_1 = \mathrm{Fix}(P_1 \circ P_2)$, $E_2 = \mathrm{Fix}(P_2 \circ P_1)$, and

$$x_{2n+2} - x_{2n+1} \to v, \quad x_{2n+1} - x_{2n} \to -v.$$

Furthermore: Either: $E_1 = E_2 = \varnothing$ and $\|x_n\| \to +\infty$;
Or: $x_{2n+1} \rightharpoonup e_1 \in E_1$ and $x_{2n} \rightharpoonup e_2 \in E_2$,

$$(e_1, e_2) \text{ is a minimizer for } \min_{(y_1, y_2) \in C_1 \times C_2} \|y_1 - y_2\|$$

as well as a cycle: $e_2 = P_2 e_1$ and $e_1 = P_1 e_2$.

The inconsistent case when $m \geq 3$

In striking contrast, Baillon-Combettes-Cominetti (2011) proved:

There exists no function F on X^m such that cycles $(e_1, \ldots, e_m)$ correspond to minimizers for the problem

$$\min_{(y_1, y_2, \ldots, y_m) \in C_1 \times C_2 \times \cdots \times C_m} F(y_1, \ldots, y_m).$$

Underrelaxed projections for the general case

For $\lambda \in \,]0,1]$, consider the composition of underrelaxed projections:

$$Q_\lambda := \big((1-\lambda)\operatorname{Id} + \lambda P_m\big) \circ \cdots \circ \big((1-\lambda)\operatorname{Id} + \lambda P_1\big).$$

Suppose that each $\operatorname{Fix} Q_\lambda \neq \varnothing$, and let

$$\mathcal{L} := \operatorname{Fix}\Big(\sum_{i=1}^{m} \tfrac{1}{m} P_i\Big)$$

be the set of least squares solutions, i.e., the minimizers of the function

$$x \mapsto \sum_{i=1}^{m} d_{C_i}^2(x).$$

De Pierro's Conjecture

Theory of strongly/averaged nonexpansive mappings implies that

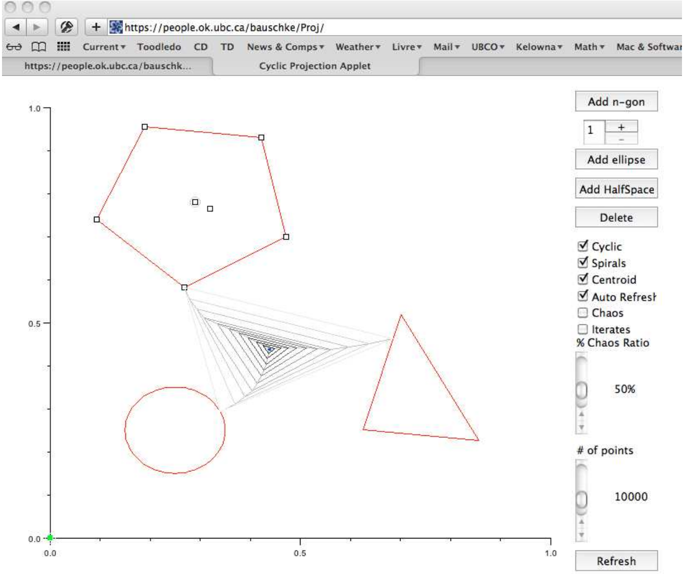
$$x_\lambda := \text{weak} \lim_{n \to +\infty} Q_\lambda^n x$$

exists, for every $x \in X$.

Open Problem: De Pierro's Conjecture
Does the curve $(x_\lambda)_{\lambda \in]0,1]}$ converge to $P_{\mathcal{L}} x$?

Remark. Censor-Eggermont-Gordon (1984) proved this for subspaces in Euclidean space; for further supporting results of this conjecture, see De Pierro (2001) (and also B-Edwards).

Experimental evidence for De Pierro's Conjecture



https://people.ok.ubc.ca/bauschke/Proj/

The arithmetic average

$$\frac{P_1 + P_2 + \cdots + P_m}{m}$$

is a much better behaved object than the composition

$$P_m \circ \cdots \circ P_2 \circ P_1.$$

$\ominus$ The composition is not firmly nonexpansive.
$\oplus$ The average is not a projection; however, it is still a proximal map (Moreau).

In the following, I will advocate the proximal average in the proximal mapping setting and the resolvent average in the general firmly nonexpansive setting (via Minty's correspondence).

The proximal average

Monotone operators

Recall that a set-valued operator $A\colon X \rightrightarrows X$ is *monotone* if

$$\left.\begin{array}{l}(x,u) \in \operatorname{gra} A\\ (y,v) \in \operatorname{gra} A\end{array}\right\} \quad \Rightarrow \quad \langle x-y, u-v\rangle \geq 0,$$

where $\operatorname{gra} A$ is the *graph* of A, and that A is *maximally monotone* if A cannot be properly extended without destroying monotonicity.

Basic examples are the *subdifferential operator* ∂f of $f\colon X \to \left]-\infty,+\infty\right]$, where f is convex, lower semicontinuous, and proper; any *bounded linear operator* $A\colon X \to X$ with a positive symmetric part.

Firmly nonexpansive mappings

Recall that $T\colon X \to X$ is *firmly nonexpansive* if

$$(\forall x \in X)(\forall y \in X) \quad \|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle.$$

Thanks to work by Minty (1962), Reich (1977), and Eckstein and Bertsekas (1992), we have for $T\colon X \to X$ and $A\colon X \rightrightarrows X$:

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- T is firmly nonexpansive
  $\Leftrightarrow$ $T^{-1} - \operatorname{Id}$ is maximally monotone;
- A is maximally monotone
  $\Leftrightarrow$ $\operatorname{Id} + A$ is onto and the *resolvent*

  $$J_A := (\operatorname{Id} + A)^{-1}$$

  is firmly nonexpansive.

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  is firmly nonexpansive.
- Critical vs fixed points: $0 \in Ax \Leftrightarrow x = J_A x$, i.e., $x \in \operatorname{Fix} J_A$.

Proximal mappings

We also have the equivalence

$$\begin{aligned} & T \text{ is firmly nonexpansive} \\ \Leftrightarrow\ & 2T - \mathrm{Id} \text{ is nonexpansive (Lipschitz-1).} \end{aligned} \qquad (*)$$

Moreau's *proximal map* (or *proximity operator*, early 1960s) is

$$J_{\partial f} x = \operatorname{Prox}_f x;$$

in fact, $\operatorname{Prox}_f x$ is the unique minimizer of $y \mapsto f(y) + \frac{1}{2}\|x - y\|^2$.

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Suppose each $(T_i)_{i \in I}$ is firmly nonexpansive and $(\lambda_i)_{i \in I}$ are convex coefficients (weights): each $\lambda_i > 0$ and $\sum_{i \in I} \lambda_i = 1$. Set

$$T := \sum_{i \in I} \lambda_i T_i.$$

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$$T := \sum_{i \in I} \lambda_i T_i.$$

Then T is firmly nonexpansive since $2T - \mathrm{Id}$ is nonexpansive:

$$2T - \mathrm{Id} = 2\sum_{i \in I} \lambda_i T_i - \mathrm{Id} = \sum_{i \in I} \lambda_i (2T_i - \mathrm{Id}).$$

Proximal mappings form a convex set

Moreau showed that if each T_i is even a proximal map, then so is the average T.

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Put differently, given functions $(f_i)_{i \in I}$ in Γ, there exists $f \in \Gamma$ such that

$$\mathrm{Prox}_f = \sum_{i \in I} \lambda_i \, \mathrm{Prox}_{f_i} .$$

The function f is unique up to an additive constant; among all these functions, the proximal average that we shall formally define does have beautiful and useful properties.

Handy notation

- $\mathfrak{q}\colon x \mapsto \frac{1}{2}\langle x, x\rangle$ quadratic energy function
- $\Gamma =$ functions from X to $]-\infty, +\infty]$ that are convex, lower semicontinuous, and proper
- $\mathbf{f} = (f_1, \ldots, f_m) \in \Gamma^m$
- $\mathbf{f}^* = (f_1^*, \ldots, f_m^*)$
- $\mathbf{f}^{**} = (f_1^{**}, \ldots, f_m^{**}) = \mathbf{f}$
- $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}_+^m$
- $\lambda_1 + \cdots + \lambda_m = 1$
- $\mu > 0$

Definition of the proximal average

Definition. (B-Goebel-Lucet-Wang) The $\boldsymbol{\lambda}$*-weighted proximal average of* $\mathbf{f}$ *with parameter* μ is defined by

$$p_\mu(\mathbf{f}, \boldsymbol{\lambda}) = \lambda_1 \bullet (f_1 + \mu \bullet \mathfrak{q}) \,\Box \cdots \Box\, \lambda_m \bullet (f_m + \mu \bullet \mathfrak{q}) - \mu \bullet \mathfrak{q},$$

where *epi-addition* and *epi-multiplication* are

$$\big(f \,\Box\, g\big)(x) = \inf_{y+z=x} \big(f(y) + g(z)\big);$$

and $\alpha \bullet f = \alpha f(\cdot/\alpha)$, if $\alpha > 0$; $\alpha \bullet f = \iota_{\{0\}}$, if $\alpha = 0$.

Reformulations

If $I = \big\{ i \in \{1, \ldots, m\} \;\big|\; \lambda_i > 0 \big\}$, then

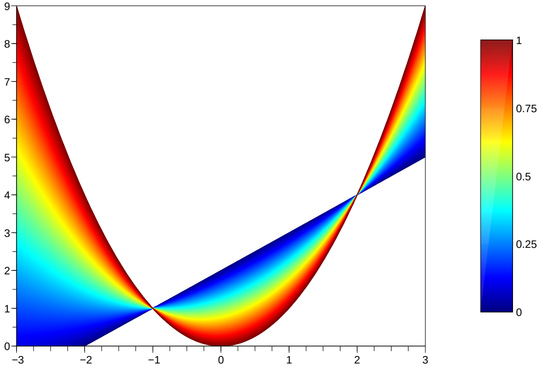
$$p_\mu(\mathbf{f}, \boldsymbol{\lambda})(x) = \frac{1}{\mu}\Big(-\tfrac{1}{2}\|x\|^2 + \inf_{\sum_{i\in I} x_i = x} \sum_{i\in I} \lambda_i\big(\mu f_i(x_i/\lambda_i) + \tfrac{1}{2}\|x_i/\lambda_i\|^2\big)\Big).$$

Furthermore,

$$\begin{aligned} p_\mu(\mathbf{f}, \boldsymbol{\lambda}) &= \big(\lambda_1(f_1^* \,\square\, \mu\,\mathfrak{q}) + \cdots + \lambda_m(f_m^* \,\square\, \mu\,\mathfrak{q})\big)^* - \mu^{-1}\,\mathfrak{q} \\ &= \big(\lambda_1(f_1 + \mu^{-1}\,\mathfrak{q})^* + \cdots + \lambda_m(f_m + \mu^{-1}\,\mathfrak{q})^*\big)^* - \mu^{-1}\,\mathfrak{q}. \end{aligned}$$

Remark. This was first studied explicitly (for $m = 2$ and $\mu = 1$) by B-Matoušková-Reich to obtain a Güler-like counterexample for the proximal point algorithm — based on Hundal's counterexample!

Visualizing the arithmetic average from $2x + 2$ to x^2



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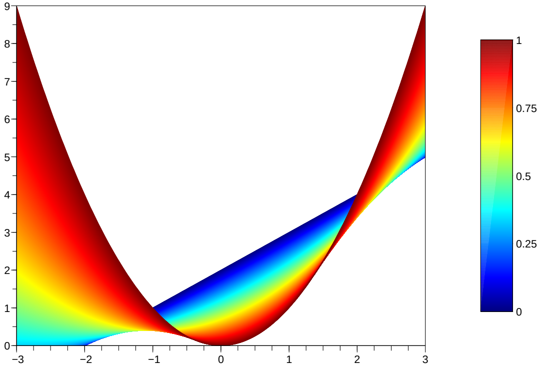
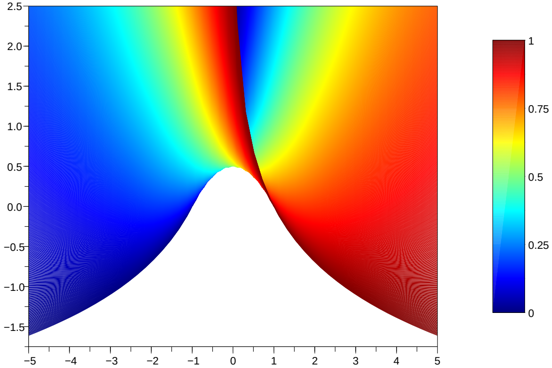


From $-\ln(-x)$ *to* $-\ln(x)$



Basic results — a selection

Theorem. $\operatorname{dom} p_\mu(\mathbf{f}, \boldsymbol{\lambda}) = \lambda_1 \operatorname{dom} f_1 + \cdots + \lambda_m \operatorname{dom} f_m$, and the epi-sum for $p_\mu(\mathbf{f}, \boldsymbol{\lambda})$ is always exact (i.e., the infimum is attained).

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Corollary. If some f_i has full domain and $\lambda_i > 0$, then $p_\mu(\mathbf{f}, \boldsymbol{\lambda})$ has full domain as well.

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Corollary. $p_\mu(\mathbf{f}, \boldsymbol{\lambda})$ is convex, lower semicontinuous, and proper.

Proof. Applying the last theorem twice, we deduce that

$$\begin{aligned}\big(p_\mu(\mathbf{f}, \boldsymbol{\lambda})\big)^{**} &= \big(p_{\mu^{-1}}(\mathbf{f}^*, \boldsymbol{\lambda})\big)^* = p_{(\mu^{-1})^{-1}}(\mathbf{f}^{**}, \boldsymbol{\lambda}) \\ &= p_\mu(\mathbf{f}, \boldsymbol{\lambda}).\end{aligned}$$

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Example. $p_1\big(\mathbf{f},\mathbf{f}^*,1/(2m)\big) = \mathfrak{q}$.

Moreau envelope

Recall that the *Moreau envelope* of f with parameter μ is

$$e_\mu f = f \,\square\, \mu \bullet \mathfrak{q} = (f^* + \mu\,\mathfrak{q})^*.$$

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Corollary.

$$\operatorname{argmin} p_\mu(\mathbf{f}, \boldsymbol{\lambda}) = \operatorname{argmin}\big(\lambda_1 e_\mu f_1 + \cdots + \lambda_m e_\mu f_m\big).$$

Example. (*least squares solutions* revisited)
If each $f_i = \iota_{C_i}$, where C_i is closed, convex, nonempty, then

$$\operatorname{argmin} p_\mu(\mathbf{f}, \boldsymbol{\lambda}) = \operatorname{argmin}(\lambda_1 d_{C_1}^2 + \cdots + \lambda_m d_{C_m}^2).$$

Proximal mapping

Recall that the *proximal mapping* of f with parameter μ is

$$P_\mu f := \mathrm{Prox}_{\mu f} = \big(\,\mathrm{Id} + \mu\partial f\big)^{-1};$$

it satisfies

$$(P_\mu f) \circ (\mu\,\mathrm{Id}) = \nabla(e_{\mu^{-1}}(f^*)).$$

Finally, we are able to motivate the term "proximal average":

Theorem.

$$P_\mu\big(p_\mu(\mathbf{f}, \boldsymbol{\lambda})\big) = \lambda_1 P_\mu f_1 + \cdots + \lambda_m P_\mu f_m.$$

Proof.

We have

$$\begin{aligned} e_{\mu^{-1}}\big((p_\mu(\mathbf{f},\boldsymbol{\lambda}))^*\big) &= e_{\mu^{-1}}\big(p_{\mu^{-1}}(\mathbf{f}^*,\boldsymbol{\lambda})\big) \\ &= \lambda_1 e_{\mu^{-1}}(f_1^*) + \cdots + \lambda_m e_{\mu^{-1}}(f_m^*). \end{aligned}$$

Taking gradients yields

$$\nabla\big(e_{\mu^{-1}}\big((p_\mu(\mathbf{f},\boldsymbol{\lambda}))^*\big)\big) = \lambda_1 \nabla(e_{\mu^{-1}}(f_1^*)) + \cdots + \lambda_m \nabla\big(e_{\mu^{-1}}(f_m^*)\big);$$

in turn, this is equivalent to

$$\big(P_\mu\big(p_\mu(\mathbf{f},\boldsymbol{\lambda})\big)\big)\circ(\mu\,\mathrm{Id}) = \lambda_1(P_\mu f_1)\circ(\mu\,\mathrm{Id}) + \cdots + \lambda_m(P_\mu f_m)\circ(\mu\,\mathrm{Id}),$$

i.e., to

$$P_\mu\big(p_\mu(\mathbf{f},\boldsymbol{\lambda})\big) = \lambda_1(P_\mu f_1) + \cdots + \lambda_m(P_\mu f_m). \qquad \blacksquare$$

Cones

Example. Let $K_1, \ldots, K_m$ be closed subspaces that are pairwise orthogonal and such that $K_1 \oplus \cdots \oplus K_m = X$, and suppose that each $f_i = \iota_{K_i}$ and $\lambda_i > 0$. Then

$$p_\mu(\mathbf{f}, \boldsymbol{\lambda}) = \mu^{-1} \sum_{i=1}^{m} (\lambda_i^{-1} - 1)(\mathfrak{q} \circ P_{K_i}).$$

Example. Let K be a nonempty closed convex cone in X and let $\lambda \in]0,1[$. Then

$$p_1\big((\iota_K, \iota_{K^\ominus}), (1-\lambda, \lambda)\big)(x) = \frac{\lambda^2 \|P_K x\|^2 + (1-\lambda)^2 \|P_{K^\ominus} x\|^2}{2(1-\lambda)\lambda},$$

where $K^\ominus = \big\{u \in X \;\big|\; \sup\langle u, K\rangle \le 0\big\}$ is the *polar cone* of K.

Legendre functions

Let $g \in \Gamma$. The following generalizes classical notions in $\mathbb{R}^N$:

- g is *essentially smooth* if ∂g is at most single-valued and $\operatorname{int} \operatorname{dom} g$ is nonempty;
- g is *essentially strictly convex* if g^* is essentially smooth;
- g is *Legendre* if g is both essentially smooth and essentially strictly convex.

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Corollary. (Inheritance) Suppose each $\lambda_i > 0$.

- If some f_i is essentially smooth, then so is $p_\mu(\mathbf{f}, \boldsymbol{\lambda})$.
- If some f_j is essentially strictly convex, then so is $p_\mu(\mathbf{f}, \boldsymbol{\lambda})$.
- If some f_i is essentially smooth and some f_j is essentially strictly convex (where not necessarily $i = j$), then $p_\mu(\mathbf{f}, \boldsymbol{\lambda})$ is Legendre.

Varying the parameter μ

Theorem. (pointwise limits) Let $x \in X$. Then the function

$$\mathbb{R}_{++} \to \,]-\infty, +\infty] : \mu \mapsto p_\mu(\mathbf{f}, \boldsymbol{\lambda})(x) \quad \text{is decreasing.}$$

In fact,

$$\lim_{\mu \to 0^+} p_\mu(\mathbf{f}, \boldsymbol{\lambda})(x) = \big(\lambda_1 f_1 + \cdots + \lambda_m f_m\big)(x)$$

and

$$\lim_{\mu \to +\infty} p_\mu(\mathbf{f}, \boldsymbol{\lambda})(x) = \big(\lambda_1 \bullet f_1 \,\Box \cdots \Box\, \lambda_m \bullet f_m\big)(x).$$

Antiderivatives

Recall that $f \in \Gamma$ is an antiderivative of A if $\operatorname{gra} A \subseteq \operatorname{gra} \partial f$.

Fact. (Rockafellar, 1970) Let A be cyclically monotone, i.e., $\sum_{i=1}^{n} \langle a_{i+1} - a_i, a_i^* \rangle \leq 0$ for $n \geq 2$, $(a_i, a_i^*) \in \operatorname{gra} A$ and $a_{n+1} = a_1$. Then the following hold:

- The *Rockafellar functions* $R_{A,(a,a^*)}(x)$ defined by

$$\sup_{2 \leq n,\, (a_i, a_i^*) \in \operatorname{gra} A} \left(\sum_{i=1}^{n-2} \langle a_{i+1} - a_i, a_i^* \rangle + \langle x - a_{n-1}, a_{n-1}^* \rangle \right)$$

  (with $(a, a^*) = (a_1, a_1^*) \in \operatorname{gra} A$ fixed) are antiderivatives of A.
- Maximally cyclically monotone operators are precisely subdifferential operators of functions in Γ.
- If A is maximally cyclically monotone, then antiderivatives of A differ only by constants.

Rockafellar's question

In 2005, R.T. Rockafellar asked the following.

> *Given a cyclically monotone operator with finite graph, find a method that produces an antiderivative of A that preserves the "natural symmetry" induced by convex duality.*

Neither Rockafellar's antiderivatives $R_{A,(a,a^*)}$ nor their pointwise maximum

$$m_A(x) := \sup_{(a,a^*)\in \operatorname{gra} A} R_{A,(a,a^*)}(x)$$

have this property.

An answer

Let $\mathcal{A}$ be the set of cyclically monotone operators with finite graph.

Theorem. (B-Lucet-Wang) The method

$$\mathfrak{m}\colon \mathcal{A} \to \Gamma\colon A \mapsto p_1\big(m_A, m^*_{A^{-1}}, \tfrac{1}{2}, \tfrac{1}{2}\big)$$

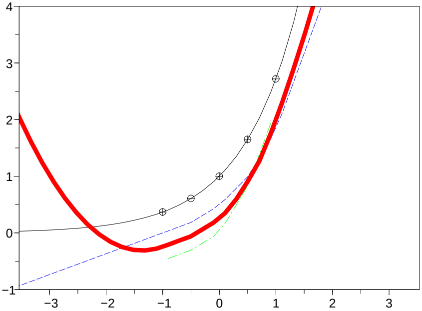
produces *primal-dual symmetric antiderivatives* in the sense that

$$(\mathfrak{m}_A)^* = \mathfrak{m}_{A^{-1}}.$$

In other words, the following is a commutative diagram.

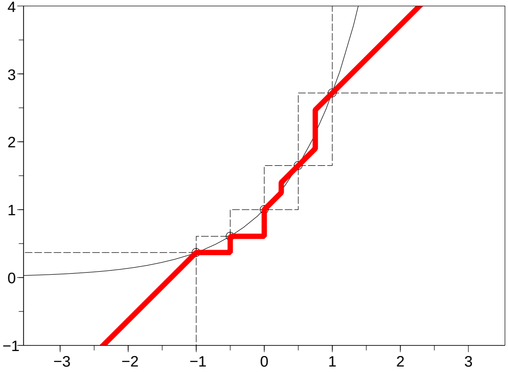
$$\begin{array}{ccc} A & \xrightarrow{\;*\;} & A^{-1} \\ {\scriptstyle\mathfrak{m}}\big\downarrow & & {\scriptstyle\mathfrak{m}}\big\downarrow \\ \mathfrak{m}_A & \xrightarrow{\;*\;} & \mathfrak{m}_{A^{-1}} \end{array}$$

An example - 5 points sampled from exp



thick black — exp; five circled points — the sample;
dashed blue — m_A; dashed-dotted green — $m^*_{A^{-1}}$;
thick red — $\mathfrak{m}_A$.

∂(primal-dual symmetric extension)



Note the "slope one" property of $\partial \mathfrak{m}_A$ outside the rectangle $\operatorname{conv} \operatorname{dom} A \times \operatorname{conv} \operatorname{ran} A$.

The resolvent average

Near equality and near convexity

We now assume for a while that

$$\boxed{X \text{ is finite-dimensional}} \quad \text{and that } I = \{1, 2, \ldots, m\},$$

because then the "relative interior" calculus works particularly well.

Definition. Let A and B be subsets of X. We say that A and B are *nearly equal* if

$$A \approx B \quad :\Leftrightarrow \quad \overline{A} = \overline{B} \quad \text{and} \quad \operatorname{ri} A = \operatorname{ri} B.$$

Proposition. Let $A \subseteq X$. Then A is *nearly convex* (in the sense of Rockafellar and Wets), i.e., there exists a convex subset C of X such that $C \subseteq A \subseteq \overline{C}$ if and only if

$$A \approx \operatorname{conv} A.$$

Calculus

Proposition. Assume that $A, A_1, \ldots, A_m$ are nearly convex subsets of X and that $B, B_1, \ldots, B_m$ are just subsets of X, all $\neq \varnothing$. Then:

- $A \approx \operatorname{conv} A \approx \overline{A} \approx \operatorname{ri} A$.
- If $A \approx B$, then B is nearly convex.
- If $(\forall i \in I)\ A_i \approx B_i$, then $\sum_{i \in I} A_i$ is nearly convex and

$$\sum_{i \in I} A_i \approx \sum_{i \in I} B_i.$$

- If B is compact and $A_1 + B \approx A_2 + B$, then $A_1 \approx A_2$.

Relevance for maximally monotone operators

Fact. Let $A\colon X \rightrightarrows X$ be maximally monotone. Then

$$\operatorname{dom} A \quad \text{and} \quad \operatorname{ran} A \quad \text{are nearly convex.}$$

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Recall that $A\colon X \rightrightarrows X$ is *rectangular* (a.k.a. 3^* or star monotone; Brezis-Haraux 1976) if the *Fitzpatrick function* satisfies

$$\big(\forall x \in \operatorname{dom} A\big)\big(\forall x^* \in \operatorname{ran} A\big)$$
$$F_A(x,x^*) := \sup_{(a,a^*)\in \operatorname{gra} A} \big(\langle x, a^*\rangle + \langle a, x^*\rangle - \langle a, a^*\rangle\big) < +\infty.$$

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Examples.

- The skew rotator by $\pi/2$ in the plane is *not rectangular*.
- ∂f is rectangular.
- $J_A = (\operatorname{Id} + A)^{-1}$ is rectangular.

On the range

Theorem. Let $(A_i)_{i\in I}$ be a family of maximally monotone rectangular operators such that $\bigcap_{i\in I} \operatorname{ri}\operatorname{dom} A_i \neq \varnothing$, let $(\lambda_i)_{i\in I}$ be a family in $\mathbb{R}_{++}$, and let $j \in I$. Then $A := \sum_{i\in I} \lambda_i A_i$ is maximally monotone, rectangular,

$$\operatorname{ran} A = \operatorname{ran} \sum_{i\in I} \lambda_i A_i \approx \sum_{i\in I} \lambda_i \operatorname{ran} A_i \quad \text{is nearly convex,}$$

and the following hold:

- If $\sum_{i\in I} \lambda_i \operatorname{ran} A_i = X$, then A is surjective.
- If A_j is surjective, then A is surjective.
- If $0 \in \bigcap_{i\in I} \overline{\operatorname{ran}}\, A_i$, then $0 \in \overline{\operatorname{ran}}\, A$.
- If $0 \in (\operatorname{int}\operatorname{ran} A_j) \cap \bigcap_{i\in I\smallsetminus\{j\}} \overline{\operatorname{ran}}\, A_i$, then $0 \in \operatorname{int}\operatorname{ran} A$.

Application to firmly nonexpansive mappings

Corollary. Let $(T_i)_{i\in I}$ be a family of firmly nonexpansive mappings on X, let $(\lambda_i)_{i\in I}$ be a family in $\mathbb{R}_{++}$ such that $\sum_{i\in I}\lambda_i = 1$, and let $j \in I$. Set

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- T is firmly nonexpansive and $\operatorname{ran} T \approx \sum_{i\in I}\lambda_i \operatorname{ran} T_i$ is nearly convex.
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Proof. As a resolvent, each T_i is rectangular. Now apply the last result. ■

Back to projections

Example. Let $(C_i)_{i\in I}$ be a family of nonempty closed convex subsets of X with associated projection operators P_i, and let $(\lambda_i)_{i\in I}$ be a family of strictly positive real numbers such that $\sum_{i\in I}\lambda_i = 1$. Then

$$\operatorname{ran}\sum_{i\in I}\lambda_i P_i \approx \sum_{i\in I}\lambda_i C_i.$$

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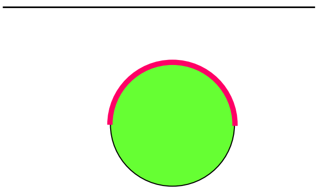
Example. Suppose that $X = \mathbb{R}^2$, $m = 2$, $C_1 = \mathbb{R}\times\{2\}$, and $C_2 =$ unit ball centered at 0 of radius 1. The composition $P_2 \circ P_1$ is nonexpansive but ran $P_2 \circ P_1$ is not even nearly convex:



Asymptotic regularity

Theorem. Let $(T_i)_{i \in I}$ be a family of firmly nonexpansive mappings on X, and let $(\lambda_i)_{i \in I}$ be a family of strictly positive real numbers such that $\sum_{i \in I} \lambda_i = 1$. Suppose that each T_i is asymptotically regular, i.e., $0 \in \overline{\operatorname{ran}(\operatorname{Id} - T_i)}$, i.e., T_i has—or "almost" has—a fixed point.

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Example. Suppose that $X = \mathbb{R}^2$, $m = 2$, $C_1 = \mathbb{R}\times\{0\}$ and $C_2 = \operatorname{epi}\exp$, with corresponding projectors P_1 and P_2. Then each $\operatorname{Fix} P_i = C_i \neq \varnothing$, yet $\operatorname{Fix}\left(\frac{1}{2}P_1 + \frac{1}{2}P_2\right) = \varnothing$.

The resolvent average

Theorem. Let $(A_i)_{i\in I}$ be a family of maximally monotone—not necessarily rectangular—operators, let $(\lambda_i)_{i\in I}$ be in $\mathbb{R}_{++}$ such that $\sum_{i\in I}\lambda_i = 1$, let $j \in I$, and define the resolvent average by

$$A := \Big(\sum_{i\in I}\lambda_i\big(\operatorname{Id} + A_i\big)^{-1}\Big)^{-1} - \operatorname{Id}.$$

Then the following hold.

The resolvent average

(i) A is maximally monotone and

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(iii) If $\operatorname{dom} A_j = X$, then $\operatorname{dom} A = X$.

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item (iv) is abstract supercoercivity;
item (vi) is abstract coercivity;
"if some A_j is good, then so is A" (B-Moffat-Wang, forthcoming).

Positive semidefinite matrices

As an illustration of both the proximal and the resolvent average, consider the following set up:

- $\mathbf{A} = (A_1, \dots, A_m) \in (\mathbb{S}^N_+)^m$
- $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m_+$
- $\lambda_1 + \cdots + \lambda_m = 1$
- $\mu > 0$

Now set

$$\mathcal{R}_\mu(\mathbf{A}, \boldsymbol{\lambda}) := \\ \left(\lambda_1(A_1 + \mu^{-1}\operatorname{Id})^{-1} + \cdots + \lambda_m(A_m + \mu^{-1}\operatorname{Id})^{-1}\right)^{-1} - \mu^{-1}\operatorname{Id},$$

so that

$$J_{\mu\mathcal{R}_\mu(\mathbf{A},\boldsymbol{\lambda})} = \lambda_1 J_{\mu A_1} + \cdots + \lambda_m J_{\mu A_m}.$$

The bridge to the proximal average

For $B \in \mathbb{S}^N$, set

$$q_B \colon x \mapsto \tfrac{1}{2}\langle x, Bx \rangle.$$

If each $f_i = q_{A_i}$, then

$$p_\mu(\mathbf{f}, \boldsymbol{\lambda}) = q_{\mathcal{R}_\mu(\mathbf{A}, \boldsymbol{\lambda})}$$

and hence

$$\nabla p_\mu(\mathbf{f}, \boldsymbol{\lambda}) = \mathcal{R}_\mu(\mathbf{A}, \boldsymbol{\lambda}).$$

Thus the results on the proximal average are applicable!

Averages: harmonic vs resolvent vs arithmetic

Recall that the harmonic and arithmetic averages are defined by

$$\mathcal{H}(\mathbf{A}, \boldsymbol{\lambda}) = \left(\lambda_1 A_1^{-1} + \cdots + \lambda_m A_m^{-1}\right)^{-1},$$

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$$\mathcal{A}(\mathbf{A}, \boldsymbol{\lambda}) = \lambda_1 A_1 + \cdots + \lambda_m A_m,$$

respectively.

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$$\mathcal{H}(\mathbf{A},\boldsymbol{\lambda}) = \left(\lambda_1 A_1^{-1} + \cdots + \lambda_m A_m^{-1}\right)^{-1},$$

and

$$\mathcal{A}(\mathbf{A},\boldsymbol{\lambda}) = \lambda_1 A_1 + \cdots + \lambda_m A_m,$$

respectively.

Theorem. We have

$$\mathcal{H}(\mathbf{A},\boldsymbol{\lambda}) \preceq \mathcal{R}_\mu(\mathbf{A},\boldsymbol{\lambda}) \preceq \mathcal{A}(\mathbf{A},\boldsymbol{\lambda}),$$

$$\lim_{\mu\to 0^+} \mathcal{R}_\mu(\mathbf{A},\boldsymbol{\lambda}) = \mathcal{A}(\mathbf{A},\boldsymbol{\lambda}), \quad \lim_{\mu\to +\infty} \mathcal{R}_\mu(\mathbf{A},\boldsymbol{\lambda}) = \mathcal{H}(\mathbf{A},\boldsymbol{\lambda}),$$

and

$$\left(\mathcal{R}_\mu(\mathbf{A},\boldsymbol{\lambda})\right)^{-1} = \mathcal{R}_{\mu^{-1}}(\mathbf{A}^{-1},\boldsymbol{\lambda}).$$

Back to the general setting

We now let X be possibly infinite-dimensional again. Let

$$A$$

be a—not necessarily maximally—monotone operator on X.
Our aim is to find an **explicit** maximally monotone extension of A.
Recall the corresponding Fitzpatrick function is

$$F_A\colon (x,x^*) \mapsto \sup_{(a,a^*)\in \operatorname{gra} A} \big(\langle x,a^*\rangle + \langle a,x^*\rangle - \langle a,a^*\rangle\big).$$

Given $F \in \Gamma(X \times X)$, it will be convenient to define

$$F^\intercal(x^*,x) = F(x,x^*)$$

and also to define $G(F)\colon X \rightrightarrows X$ via

$$x^* \in G(F)x \quad \Leftrightarrow \quad (x^*,x) \in \partial F(x,x^*).$$

Explicit maximally monotone extension

Theorem. Let $A\colon X \rightrightarrows X$ be monotone and set

$$E_A := p_1\big(F_A, F_A^{*\intercal}, \tfrac{1}{2}, \tfrac{1}{2}\big).$$

Then $E_A^* = E_A^\intercal$, $G(E_A)$ is a *maximally monotone extension* of A that is primal-dual symmetric in the sense that

$$\big(G(E_A)\big)^{-1} = G(E_{A^{-1}}).$$

Explicit maximally monotone extension

Theorem. Let $A\colon X \rightrightarrows X$ be monotone and set

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Remark.
This provided an answer to a problem of Fitzpatrick from 1988 (and it also works in reflexive spaces).
Note that this construction does not require Zorn's Lemma!
Similarly, via Minty, we also obtain Zorn's-Lemma-free extensions of (firmly) nonexpansive mappings in the spirit of Kirszbraun-Valentine!

Current/future work and open problems

Current/future work

- More basic theory for the resolvent average ✓
- Asymptotic regularity of compositions of resolvents ✓
- Extend resolvent average to nonreflexive Banach spaces and Bregman-distance like settings ?
- **Numerical convex analysis** (Lucet et al., on-going) ...
- **Numerical monotone operator theory** ?

Open problems

- Strong convergence of random projections for subspaces ?
- Weak convergence of random projections ?
- De Pierro's conjecture ?

Bibliographical starting points

For further information...

- Please email me at heinz.bauschke@ubc.ca if you wish to obtain detailed pointers to specific results.
- The interplay of maximally monotone operators and firmly nonexpansive mappings is a central theme in

Some classical references. . .

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Merci beaucoup!