Incompressible optimal transport

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IHP nov 2011

OPTIMAL INCOMPRESSIBLE TRANSPORT AND THE EULER EQUATIONS

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INCOMPRESSIBLE FLUID MOTION

One can describe the motion of an incompressible fluid inside a bounded domain D in R^d by a time-dependent family $t \to M_t$ of maps, in the Hilbert space $H = L^2(D, R^d)$, valued in the subset VPM(D) of all Lebesgue measure-preserving maps

$$VPM(D) = \{M \in H, \ \int_D q(M(x))dx = \int_D q(x)dx, \ \forall q \in C(R^d)\}$$

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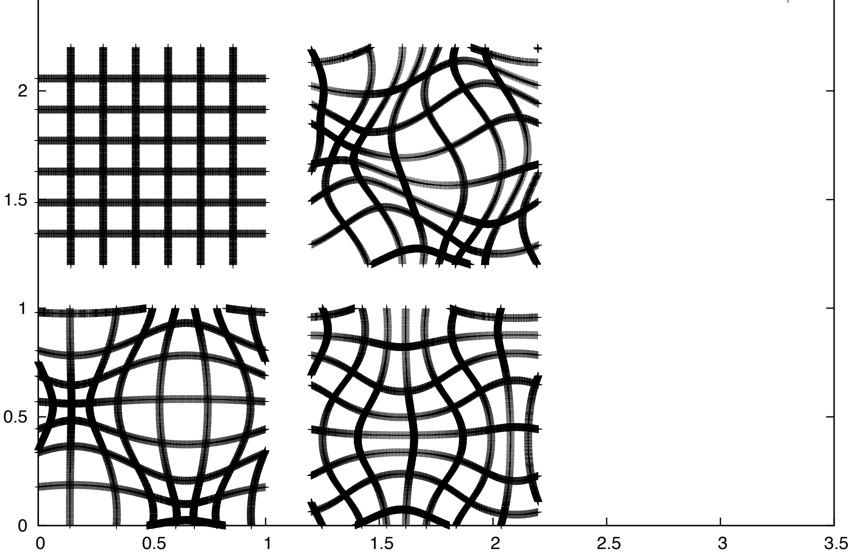
Conventional fluid mechanics further requires these maps to belong to **SDiff(D)**, the subset of all orientation preserving diffeomorphisms in VPM(D). However, as will be shown later, it is useful to give up this additional requirement.



SOUTH-EAST CORNER: A MEASURE-PRESERVING MAP

THE EULER EQUATIONS:

Solutions of the Euler equations, introduced in 1755, correspond to those curves $t \to M_t \in VPM(D)$ for which there exists a time dependent scalar function p_t, called 'pressure field', defined on D, such that

$$\boxed{\frac{d^2 M_t}{dt^2} + (\nabla p_t) \circ M_t = 0}$$

where ∇ is the gradient operator on R^d (with respect to the Euclidean norm $|\cdot|$).

THE PRINCIPLE OF LEAST ACTION: OPTIMAL INCOMPRESSIBLE TRANSPORT

THEOREM **Assume D to be convex. Let $(\mathbf{M_t}, \mathbf{p_t})$ a solution of the Euler equations, with a constant λ such that**

$$\sum_{i,j=1}^{d} \frac{\partial^2 p_t(x)}{\partial x_i \partial x_j} \xi_i \xi_j \leq \lambda |\xi|^2, \quad \forall \xi \in \mathbf{R}^d, \quad \forall x \in D, \quad \forall t$$

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Then, for every $t_0 < t_1$ so that $(t_1 - t_0)^2 \lambda < \pi^2$, M_t is the unique minimizer, among all curves along $VPM(D)$ that coincide with M_t at $t = t_0$, $t = t_1$, of the following **ACTION**

$$\frac{1}{2} \int_{t_0}^{t_1} \int_D \left| \frac{dM_t(x)}{dt} \right|^2 dx dt$$

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In geometric words, such a curve is nothing but a (constant speed) geodesic along VPM(D) , with respect to the metric induced by $H = L^2(D, \mathbf{R}^d)$, cf. Arnold 1966

qui obeïroit à ſon action. Cette idée de l'effort eſt de la derniere importance dans toute la Théorie, tant de l'équilibre que du mouvement, ayant fait voir, que la ſomme de tous les efforts eſt toujours un *maximum* ou *minimum*. Cette belle propriété convient admirablement avec le beau principe de la moindre action; dont nous devons la découverte à notre Illuſtre Préſident, M. de *Maupertuis*.

XXIII. Comme les équations que nous venons de trouver, renferment quatre variables x, y, z, & t, qui ſont abſolument indépendantes entr'elles, vû que la variabilité des trois premières s'étend ſur tous

THE PRINCIPLE OF LEAST ACTION IN EULER'S PAPER

THE DUAL ACTION

Minimizing the action can be written as a saddle point problem, just by using a time-dependent Lagrange multiplier to relax the constraint for M_t to belong to $VPM(D)$

$$\inf_{M} \sup_{p} \int_{t_0}^{t_1} \int_D \{ \frac{1}{2} | \frac{dM_t(x)}{dt} |^2 - p_t(M_t(x)) + p_t(x) \} dxdt$$

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This is trivially bounded from below by

$$\sup_p \inf_M \int_{t_0}^{t_1} \int_D \{\frac{1}{2}|\frac{dM_t(x)}{dt}|^2 - p_t(M_t(x)) + p_t(x)\}dxdt$$

which naturally leads to a dual least action principle

THE DUAL LEAST ACTION PRINCIPLE

THEOREM Under exactly the same conditions (D convex and $(t_1 - t_0)^2\lambda < \pi^2$), the pressure p is the unique maximizer of the **CONCAVE DUAL ACTION**

$$I[p] = \int_D J_p(M_{t_0}(x), M_{t_1}(x))dx + \int_{t_0}^{t_1}\int_D p_t(x)dxdt$$

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As for the previous theorem, the proof is elementary and directly follows from the 1D Poincaré inequality, which explains the role of constant π. Notice that M_t is never assumed to be smooth or one-to-one and the case $d = 1$ is fine.

SHNIRELMAN'S DENSITY RESULT

It is customary to consider the subset SDiff(D) of VPM(D) made of Lebesgue-measure preserving maps that are, in addition, orientation preserving diffeomorphisms. For d $\geq$ 2, VPM(D) is precisely the L^2 closure of SDiff(D). This is a relatively easy result.

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The identification of the closure of SDiff(D) for the a priori finer geodesic distance induced by L^2 is a much more difficult issue.
For simple (say contractile) domains D, this closure is still VPM(D) for d $\geq$ 3 (but definitely not for d $=$ 2) as shown by Shnirelman in his landmark paper (Math USSR Sb 1985).

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These results have striking consequences: in particular maps of form

$$\boxed{M(x) = (h(x_1), x_2, x_3)}$$

where h is any Lebesgue-measure preserving map of the unit interval, are in the closure of $\mathrm{SDiff}([0,1]^3)$.

EXISTENCE OF OPTIMAL INCOMPRESSIBLE TRANSPORT?

MINIMIZING GEODESICS AND OPTIMAL INCOMPRESSIBLE TRANSPORT Shnirelman has proven (Math USSR Sb 1986) that existence of an optimal transport (i.e. a minimizing geodesics) along SDiff(D) may fail when d $\geq$ 3. Remarkably enough, as we will see, the case d $\geq$ 3 turns out to be "easy", with a crucial use of the convex structure of the dual problem. The case d $=$ 2 is clearly linked to symplectic geometry and seems extremely difficult: a fascinating strategy has been developed by Shnirelman, by adding braid constraints to the minimization problem, which certainly deserves further investigations.

APPROXIMATELY OPTIMAL INCOMPRESSIBLE TRANSPORT

DEFINITION Let us assume D to be convex, fix $t_0 = 0,\ t_1 = 1$ and consider two maps $M_0, M_1 \in VPM(D)$. We say that $(M_t^\epsilon) \in SDiff(D)$ is an ϵ-optimal incompressible transport if

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where $\frac{1}{2}d(M_0, M_1)^2$ denotes the maximal dual action. The existence of such approximations is in no way trivial and is a consequence of the key density results due to Shnirelman (GAFA 1994).

OPTIMAL INCOMPRESSIBLE TRANSPORT: EXISTENCE OF A UNIQUE PRESSURE GRADIENT

MAIN THEOREM Let us assume D to be convex, with $d \geq 3$, fix $t_0 = 0,\ t_1 = 1$ and consider two maps $M_0, M_1 \in VPM(D)$. Then, there is a UNIQUE pressure-gradient ∇p_t such that for all (M_t^ϵ) ϵ-optimal incompressible transport, we have in the sense of distributions

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This result essentially goes back to YB CPAM 1999, with important improvements in Ambrosio-Figalli ARMA 2008. It is a combination of solving the dual least action problem and using Shnirelman's density result for "generalized flows", GAFA 1994.

QUALITATIVE RESULTS

1) UNIQUENESS OF THE PRESSURE GRADIENT **This is a remarkable feature of the theory. There is no equivalent result for finite dimensional configuration spaces such as SO(3), on which geodesic curves (for appropriate metrics) correspond to the motion of solid bodies in classical mechanics. We believe this strange phenomenon to be the consequence of the "hidden convexity" of the problem in dimension 3 and more.**

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2) LIMITED REGULARITY OF THE PRESSURE GRADIENT The pressure gradient was proven first (YB CPAM 1999) to be a locally bounded measure. Later, Ambrosio and Figalli have shown a better L^2 integrability with respect to the time variable (with measure values in space). Recently, I found an explicit example (that goes back to Duchon and Robert) of solutions with a pressure field semi-concave in the space variable and not more. The optimal regularity, and its dependence on the data, are clearly challenging analytic issues.

QUALITATIVE RESULTS

3) COMPARISON WITH "CLASSICAL" OPTIMAL TRANSPORT

Classical optimal transport ignores compressibility effects and the resulting optimality equations just describe the motion of a pressure-less potential flow. In the incompressible version, the pressure field plays the role of the "Kantorovich" potential in the classical theory.The existence and uniqueness of the pressure field is a natural counterpart of the classical theory. However, there is a definite lack of both existence and uniqueness for the transport part. Finally, classical optimal transport has turned out to be a powerful tool for functional inequalities and geometric analysis (cf. Villani's books). Nothing similar is known about the potential applications of incompressible optimal transport.

DENSITY OF PERMUTATIONS IN VPM(D)

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The set of all such permutations, denoted P(D) turns out to be L^2 dense in VPM$([0,1]^3)$ for all dimensions.
Combined with the previous result, this justifies the use of 1D calculations by (simple) combinatorial optimization methods to understand the geometry of 3D volume preserving maps. They actually give useful insights.


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SOME REFERENCES

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L. Euler, opera omnia, seria secunda 12, p. 274 (in french, english translation available)

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