# Perspective Functions with Nonlinear Scaling*

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**Abstract**. The classical perspective of a function is a construction which transforms a convex function into one that is jointly convex with respect to an auxiliary scaling variable. Motivated by applications in several areas of applied analysis, we investigate an extension of this construct in which the scaling variable is replaced by a nonlinear term. Our construction is placed in the general context of locally convex spaces and it generates a lower semicontinuous convex function under broad assumptions on the underlying functions. Various convex-analytical properties are established and closed-form expressions are derived. Several applications are presented.

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# 1 Introduction

The objective of this work is to study the following construction, which combines two functions to generate a lower semicontinuous convex function on a product space. Throughout, $\mathcal{X}$ and $\mathcal{Y}$ are real, locally convex, Hausdorff topological vector spaces.

**Definition 1.1** The *preperspective* of a *base* function $\varphi\colon \mathcal{X} \to [-\infty,+\infty]$ with respect to a *scaling* function $s\colon \mathcal{Y} \to [-\infty,+\infty]$ is

$$\begin{aligned} \varphi \ltimes s\colon\ \mathcal{X}\times\mathcal{Y} &\to [-\infty,+\infty] \\ (x,y) &\mapsto \begin{cases} s(y)\varphi\left(\dfrac{x}{s(y)}\right), & \text{if } 0 < s(y) < +\infty; \\ +\infty, & \text{if } -\infty \leqslant s(y) \leqslant 0 \ \text{ or } \ s(y) = +\infty, \end{cases} \end{aligned} \tag{1.1}$$

and the *perspective* of $\varphi$ with respect to $s$ is the largest lower semicontinuous convex function $\varphi \mathbin{\blacktriangleright\!\!<} s$ minorizing $\varphi \ltimes s$.

Definition 1.1 provides a general model for functions found in areas such as mean field games [1], machine learning [4, 43], physics [9, 18], optimal transportation [11, 15, 19, 25, 35], operator theory [16, 26, 42], statistics [23, 44], matrix analysis [24], mathematical programming [29, 36], information theory [34, 51], inverse problems [40], and economics [50]. Although it appears for instance in [9, 34, 51], the preperspective $\varphi \ltimes s$ is of limited use in variational problems due to its lack of lower semicontinuity and convexity.

Let us note that Definition 1.1 covers the classical notion of a linearly scaled perspective. Indeed, let $\Gamma_0(\mathcal{X})$ be the class of proper lower semicontinuous convex functions from $\mathcal{X}$ to $]-\infty,+\infty]$. Take $\varphi \in \Gamma_0(\mathcal{X})$ and let $\operatorname{rec}\varphi$ denote the recession function of $\varphi$. Then the classical perspective of $\varphi$ is

$$\widetilde{\varphi}\colon \mathcal{X}\times\mathbb{R} \to ]-\infty,+\infty] : (x,y) \mapsto \begin{cases} y\varphi\left(\dfrac{x}{y}\right), & \text{if } y > 0; \\ (\operatorname{rec}\varphi)(x), & \text{if } y = 0; \\ +\infty, & \text{if } y < 0. \end{cases} \tag{1.2}$$

Upon letting $\mathcal{Y} = \mathbb{R}$ and $s\colon y \mapsto y$, it follows from [45, Theorem 3E] that $\varphi \mathbin{\blacktriangleright\!\!<} s = \widetilde{\varphi}$. This linear scaling framework is also studied in [21, 22, 28, 46].

The investigation of notions of perspectives with nonlinear scaling functions was initiated in [37, 38, 39] in Euclidean spaces and extensions to infinite-dimensional normed spaces were carried out in [49]. In these papers, $\varphi \in \Gamma_0(\mathcal{X})$ and either $\varphi(0) \leqslant 0$ and $-s \in \Gamma_0(\mathcal{Y})$, or $\varphi \geqslant \operatorname{rec}\varphi$ and $s \in \Gamma_0(\mathcal{Y})$. Such conditions are not fulfilled for perspectives using the elementary base function $\varphi = |\cdot|^2 + \alpha$ ($\alpha \in ]0,+\infty[$) on $\mathcal{X} = \mathbb{R}$, which is used in [2] (see [23, 44] for similar examples). In addition, the construction proposed in [37, 38, 39, 49] provides lower semicontinuous convex functions $f \leqslant \varphi \mathbin{\blacktriangleright\!\!<} s$ and, when $\mathcal{Y} = \mathbb{R}$ and $s\colon y \mapsto y$, it does not capture (1.2) for a general $\varphi \in \Gamma_0(\mathcal{X})$.

The goal of the present work is to build a theory of perspective functions with nonlinear scaling in the context of Definition 1.1 and to derive closed-form expressions for them. We review notation and preliminary results in Section 2. In Section 3, we introduce and study two notions of functional envelopes which will greatly facilitate our analysis and will constitute structuring blocks in subsequent sections. Section 4 is devoted to the derivation of properties of preperspective functions and the computation of their conjugates. Closed-form expressions for perspective functions in the general setting of Definition 1.1 are derived in Section 5, as well as conditions that characterize their properness. Finally, in Section 6, we provide examples and applications of our results and, in Section 7, we make closing statements.

## 2 Notation and preliminary results

### 2.1 Notation

We recall that, throughout, $\mathcal{X}$ and $\mathcal{Y}$ are real, locally convex, Hausdorff topological vector spaces. Let $\mathcal{X}^*$ be the topological dual of $\mathcal{X}$, which is equipped with the weak$*$ topology and is thus also a locally convex Hausdorff topological vector space. In this context, $\mathcal{X}$ and $\mathcal{X}^*$ are placed in compatible duality (see [10]) via the canonical form $\langle \cdot, \cdot \rangle_{\mathcal{X}} \colon \mathcal{X} \times \mathcal{X}^* \to \mathbb{R} \colon (x, x^*) \mapsto x^*(x)$. We denote by $\mathcal{X} \oplus \mathcal{Y}$ the standard product vector space equipped with the product topology and paired with its topological dual $\mathcal{X}^* \times \mathcal{Y}^*$ via

$$\big(\forall (x,y) \in \mathcal{X} \times \mathcal{Y}\big)\big(\forall (x^*,y^*) \in \mathcal{X}^* \times \mathcal{Y}^*\big) \quad \langle (x,y),(x^*,y^*) \rangle_{\mathcal{X}\times\mathcal{Y}} = \langle x, x^* \rangle_{\mathcal{X}} + \langle y, y^* \rangle_{\mathcal{Y}}. \tag{2.1}$$

From now on, we drop the subscripts on the pairing brackets. Let $f \colon \mathcal{X} \to [-\infty,+\infty]$. Then $\operatorname{dom} f = \big\{x \in \mathcal{X} \mid f(x) < +\infty\big\}$ is the domain of $f$, $\overline{\operatorname{dom}}\, f$ the closure of $\operatorname{dom} f$, $\operatorname{lev}_{\leqslant \xi} f = \big\{x \in \mathcal{X} \mid f(x) \leqslant \xi\big\}$ the lower level set of $f$ at height $\xi \in \mathbb{R}$, and $\operatorname{epi} f = \big\{(x,\xi) \in \mathcal{X} \times \mathbb{R} \mid f(x) \leqslant \xi\big\}$ the epigraph of $f$. We say that $f$ is convex if $\operatorname{epi} f$ is convex, lower semicontinuous if $\operatorname{epi} f$ is closed, and proper if $-\infty \notin f(\mathcal{X}) \neq \{+\infty\}$. We denote by $\operatorname{cam} f$ the set of continuous affine minorants of $f$ and put

$$(\forall x \in \mathcal{X}) \quad f^{**}(x) = \sup_{a \in \operatorname{cam} f} a(x). \tag{2.2}$$

In addition, we denote by $\breve{f} \colon \mathcal{X} \to [-\infty,+\infty]$ the largest lower semicontinuous convex function majorized by $f$. The (Legendre) conjugate of $f$ is

$$f^* \colon \mathcal{X}^* \to [-\infty,+\infty] : x^* \mapsto \sup_{x \in \mathcal{X}} \big(\langle x, x^* \rangle - f(x)\big) \tag{2.3}$$

and the conjugate of $g \colon \mathcal{X}^* \to [-\infty,+\infty]$ is

$$g^* \colon \mathcal{X} \to [-\infty,+\infty] : x \mapsto \sup_{x^* \in \mathcal{X}^*} \big(\langle x, x^* \rangle - g(x^*)\big). \tag{2.4}$$

If $f$ is proper and convex, its recession function is

$$\operatorname{rec} f \colon \mathcal{X} \to [-\infty,+\infty] : x \mapsto \sup_{y \in \operatorname{dom} f} \big(f(x+y) - f(y)\big). \tag{2.5}$$

and, if $f \in \Gamma_0(\mathcal{X})$ and $z \in \operatorname{dom} f$, we have

$$(\forall x \in \mathcal{X}) \quad (\operatorname{rec} f)(x) = \lim_{0<\alpha\to+\infty} \frac{f(z+\alpha x) - f(z)}{\alpha} = \sup_{\alpha \in ]0,+\infty[} \frac{f(z+\alpha x) - f(z)}{\alpha}. \tag{2.6}$$

The set of proper lower semicontinuous convex functions from $\mathcal{X}$ to $]-\infty,+\infty]$ is denoted by $\Gamma_0(\mathcal{X})$.

Let $C$ be a subset of $\mathcal{X}$. The indicator function of $C$ is denoted by $\iota_C$, the support function of $C$ by $\sigma_C$, the smallest convex set containing $C$ by $\operatorname{conv} C$, the smallest closed convex set containing $C$ by $\overline{\operatorname{conv}}\, C$, and the recession cone of $C$ by $\operatorname{rec} C$.

### 2.2 Facts from convex analysis

The first three lemmas are standard; see [20, 30, 32, 41, 48].

**Lemma 2.1** *Let $f \in \Gamma_0(\mathcal{X})$, let $x \in \operatorname{dom} f$, and let $y \in \operatorname{dom} f$. Then $f$ is continuous relative to $[x,y]$.*

**Lemma 2.2** *Let $C \subset \mathcal{X}$. Then $\sigma_C = \sigma_{\overline{\mathrm{conv}}\, C}$.*

**Lemma 2.3** *Let $f\colon \mathcal{X} \to [-\infty, +\infty]$. Then the following hold:*

(i) $(f^*)^* = f^{**} \leqslant \breve{f} \leqslant f$.

(ii) $(\breve{f})^* = f^* = f^{***}$.

(iii) $\mathrm{cam}\, f \neq \varnothing \Leftrightarrow -\infty \notin f^{**}(\mathcal{X}) \Leftrightarrow f^{**} \not\equiv -\infty \Leftrightarrow \mathrm{dom}\, f^* \neq \varnothing$.

(iv) $\mathrm{cam}\, f = \varnothing \Rightarrow \breve{f}(\mathcal{X}) \subset \{-\infty, +\infty\}$.

(v) *Suppose that $f \in \Gamma_0(\mathcal{X})$. Then $\mathrm{cam}\, f \neq \varnothing$ and $\breve{f} = f^{**} = f$.*

**Lemma 2.4** *Let $f\colon \mathcal{X} \to \left]-\infty, +\infty\right]$ be such that $\mathrm{cam}\, f \neq \varnothing$. Then the following hold:*

(i) *Suppose that $f \not\equiv +\infty$. Then $f^* \in \Gamma_0(\mathcal{X}^*)$ and $f^{**} \in \Gamma_0(\mathcal{X})$.*

(ii) $\breve{f} = f^{**}$.

(iii) $\overline{\mathrm{conv}}\, \mathrm{dom}\, f = \overline{\mathrm{dom}}\, f^{**}$.

*Proof*. (i)–(ii): See [41].

(iii): We derive from (ii) and [5, Proposition 9.8(iv)] (its proof remains valid in our setting) that $\mathrm{conv}\, \mathrm{dom}\, f \subset \mathrm{dom}\, f^{**} \subset \overline{\mathrm{conv}}\, \mathrm{dom}\, f$. Taking the closure yields the identity. ☐

**Lemma 2.5** *Let $f \in \Gamma_0(\mathcal{X})$. Then the following hold:*

(i) $\mathrm{rec}\, \mathrm{epi}\, f = \mathrm{epi}\, \mathrm{rec}\, f$.

(ii) $\mathrm{rec}\, f = \sigma_{\mathrm{dom}\, f^*} = \mathrm{rec}\, (f^{**})$.

(iii) $f = \mathrm{rec}\, f \Leftrightarrow f^*(\mathrm{dom}\, f^*) = \{0\}$.

*Proof*. (i): See [5, Proposition 9.29] (its proof remains valid in our setting).

(ii): The first identity is from [45, Corollary 3D]. In view of Lemma 2.3(v), it implies the second.

(iii): It follows from Lemma 2.3(v), (ii), and Lemma 2.4(ii) that

$$f = \mathrm{rec}\, f \quad \Leftrightarrow \quad f^* = (\iota_{\mathrm{dom}\, f^*})^{**} = \iota_{\overline{\mathrm{dom}}\, f^*}, \tag{2.7}$$

which implies that $\mathrm{dom}\, f^* = \mathrm{dom}\, \iota_{\overline{\mathrm{dom}}\, f^*} = \overline{\mathrm{dom}}\, f^*$. Thus, since $f^*$ is lower semicontinuous, $f = \mathrm{rec}\, f \Leftrightarrow f^* = \iota_{\overline{\mathrm{dom}}\, f^*} = \iota_{\mathrm{dom}\, f^*} \Leftrightarrow f^*(\mathrm{dom}\, f^*) = \{0\}$. ☐

**Lemma 2.6** *Let $f \in \Gamma_0(\mathcal{X})$. Then the following are equivalent:*

(i) $\mathrm{rec}\, f \leqslant f$.

(ii) $(\forall \lambda \in \left[1, +\infty\right[)(\forall x \in \mathcal{X})\ f(\lambda x) \leqslant \lambda f(x)$.

(iii) $f^*(\mathrm{dom}\, f^*) \subset \left]-\infty, 0\right]$.

*Proof*. (i)⇒(ii): Without loss of generality, let $\lambda \in ]1,+\infty[$ and $x \in \operatorname{dom} f$. Arguing as in the proof of [49, Proposition 8(iii)], we observe that Lemma 2.5(i) yields

$$(\lambda-1)\operatorname{epi} f \subset (\lambda-1)\operatorname{epi}\operatorname{rec} f = (\lambda-1)\operatorname{rec}\operatorname{epi} f = \operatorname{rec}\operatorname{epi} f. \tag{2.8}$$

Hence, $(\lambda x, \lambda f(x)) = (x, f(x)) + (\lambda-1)(x, f(x)) \in \operatorname{epi} f + \operatorname{rec}\operatorname{epi} f = \operatorname{epi} f$. Therefore, $f(\lambda x) \leqslant \lambda f(x)$.

(ii)⇒(iii): Let $x^* \in \operatorname{dom} f^*$. Since

$$\begin{aligned} f^*(x^*) &= \sup_{x\in\mathcal{X}} \big(\langle x, x^*\rangle - f(x)\big) \\ &= \sup_{y\in\mathcal{X}} \sup_{\lambda\in[1,+\infty[} \big(\langle \lambda y, x^*\rangle - f(\lambda y)\big) \\ &\geqslant \sup_{\lambda\in[1,+\infty[} \lambda \sup_{y\in\mathcal{X}} \big(\langle y, x^*\rangle - f(y)\big) \\ &= \sup_{\lambda\in[1,+\infty[} \lambda f^*(x^*), \end{aligned} \tag{2.9}$$

we have $f^*(x^*) > 0 \Rightarrow f^*(x^*) = +\infty$, which contradicts the fact that $x^* \in \operatorname{dom} f^*$.

(iii)⇒(i): In view of Lemma 2.5(ii), $(\forall x^* \in \operatorname{dom} f^*)\ f^*(x^*) \in ]-\infty, 0] \Rightarrow (\forall x^* \in \operatorname{dom} f^*)\ x^* \leqslant f \Rightarrow \sigma_{\operatorname{dom} f^*} \leqslant f \Rightarrow \operatorname{rec} f \leqslant f$. ☐

## 3 The ▾ and ▴ envelopes

We introduce two types of envelope of a function that will be essential in our analysis.

**Definition 3.1** Let $f\colon \mathcal{X} \to [-\infty,+\infty]$. Then

$$f^{\vee}\colon \mathcal{X} \to ]-\infty,+\infty] : x \mapsto \begin{cases} f(x), & \text{if } -\infty < f(x) < 0; \\ +\infty, & \text{otherwise} \end{cases} \tag{3.1}$$

and the ▾ envelope of $f$ is

$$f^{\blacktriangledown} = f^{\vee**}. \tag{3.2}$$

Furthermore,

$$f^{\wedge}\colon \mathcal{X} \to ]-\infty,+\infty] : x \mapsto \begin{cases} f(x), & \text{if } 0 < f(x) < +\infty; \\ +\infty, & \text{otherwise} \end{cases} \tag{3.3}$$

and the ▴ envelope of $f$ is

$$f^{\blacktriangle} = f^{\wedge**}. \tag{3.4}$$

Let us examine some key properties of these envelopes.

**Lemma 3.2** *Let $f\colon \mathcal{X} \mapsto [-\infty,+\infty]$ be such that $E = f^{-1}(]-\infty,0[) \neq \varnothing$. Then the following holds:*

(i) *Suppose that* $\operatorname{cam} f^{\vee} \neq \varnothing$*. Then* $f^{\blacktriangledown} \in \Gamma_0(\mathcal{X})$*,* $\operatorname{dom} f^{\blacktriangledown} \subset \overline{\operatorname{conv}}\, E$*, and* $f^{\blacktriangledown}(\operatorname{dom} f^{\blacktriangledown}) \subset ]-\infty,0]$.

*Now suppose that, in addition, $f \in \Gamma_0(\mathcal{X})$. Then the following are satisfied:*

(ii) $\overline{E} = f^{-1}(]-\infty, 0])$.

(iii) $f^{\blacktriangledown} = f + \iota_{f^{-1}(]-\infty,0])}$.

(iv) $\operatorname{dom} f^{\blacktriangledown} = f^{-1}(]-\infty, 0])$.

(v) $f^{\blacktriangledown -1}(\{0\}) = f^{-1}(\{0\})$.

(vi) $E = (f^{\blacktriangledown})^{-1}(]-\infty, 0[)$ *and* $f^{\blacktriangledown}|_E = f|_E$.

*Proof.* (i): The fact that $f^{\blacktriangledown} \in \Gamma_0(\mathcal{X})$ follows from (3.2) and Lemma 2.4(i). Next, since $f^{\vee}(\mathcal{X}) \subset ]-\infty, 0[ \cup \{+\infty\}$, we deduce from (3.2), Lemma 2.4(iii), and (3.1) that

$$\begin{aligned} \operatorname{dom} f^{\blacktriangledown} &\subset \overline{\operatorname{dom}}\, f^{\blacktriangledown} \\ &= \overline{\operatorname{conv}}\, \operatorname{dom} f^{\vee} \\ &= \overline{\operatorname{conv}}\, (f^{\vee})^{-1}(]-\infty, 0[) \qquad (3.5) \\ &= \overline{\operatorname{conv}}\, E. \qquad (3.6) \end{aligned}$$

On the other hand, Lemma 2.3(i) yields $f^{\blacktriangledown} \leqslant f^{\vee}$. Hence, we derive from (3.5) that

$$\begin{aligned} \operatorname{dom} f^{\blacktriangledown} &\subset \overline{\operatorname{conv}}\, (f^{\blacktriangledown})^{-1}(]-\infty, 0[) \\ &\subset \overline{\operatorname{conv}}\, (f^{\blacktriangledown})^{-1}(]-\infty, 0]) \\ &= (f^{\blacktriangledown})^{-1}(]-\infty, 0]) \qquad (3.7) \end{aligned}$$

since $f^{\blacktriangledown} \in \Gamma_0(\mathcal{X})$.

(ii): Since $f$ is lower semicontinuous, $f^{-1}(]-\infty, 0])$ is closed. Therefore $E \subset f^{-1}(]-\infty, 0]) \Rightarrow \overline{E} \subset f^{-1}(]-\infty, 0])$. Conversely, take $x_0 \in f^{-1}(]-\infty, 0])$ and $x \in E$, and set $(\forall \alpha \in ]0, 1[)\ x_\alpha = \alpha x + (1-\alpha)x_0$. Since $f$ is convex $(\forall \alpha \in ]0, 1[)\ f(x_\alpha) \leqslant \alpha f(x) + (1-\alpha) f(x_0) < 0$, hence $x_\alpha \in E$. Thus $x_0 = \lim_{\alpha \downarrow 0} x_\alpha \in \overline{E}$.

(iii): Let $x^* \in \mathcal{X}^*$ and let us show that $f^{\vee *}(x^*) = \sup(x^* - f)(\overline{E})$. Since $f^{\vee *}(x^*) = \sup(x^* - f)(E)$, we have $f^{\vee *}(x^*) \leqslant \sup(x^* - f)(\overline{E})$. To get the reverse inequality let $x \in \overline{E}$. We need to show that $\langle x, x^* \rangle - f(x) \leqslant f^{\vee *}(x^*)$. It is enough to assume that $x \in \overline{E} \smallsetminus E$, which yields $f(x) = 0$. In addition, since $x^*$ is lower semicontinuous and $f^{\vee}|_E < 0$,

$$\langle x, x^* \rangle - f(x) = \langle x, x^* \rangle \leqslant \sup x^*(\overline{E}) = \sup x^*(E) \leqslant f^{\vee *}(x^*). \qquad (3.8)$$

Thus,

$$f^{\vee *}(x^*) = \sup(x^* - f)(\overline{E}) = \big(f + \iota_{\overline{E}}\big)^*(x^*) = \big(f + \iota_{f^{-1}(]-\infty,0])}\big)^*(x^*). \qquad (3.9)$$

On the other hand, since $E \neq \varnothing$, using (ii), we see that

$$f + \iota_{f^{-1}(]-\infty,0])} = f + \iota_{\overline{E}} \in \Gamma_0(\mathcal{X}). \qquad (3.10)$$

Altogether, (3.10) and Lemma 2.3(v) yield $f^{\blacktriangledown} = f^{\vee **} = (f + \iota_{f^{-1}(]-\infty,0])})^{**} = f + \iota_{f^{-1}(]-\infty,0])}$.

(iv)–(vi): These follow from (iii). ☐

**Lemma 3.3** *Let* $f\colon \mathcal{X} \to [-\infty, +\infty]$ *be such that* $F = f^{-1}(]0, +\infty[) \neq \varnothing$. *Then the following holds:*

(i) $f^{\blacktriangle} \in \Gamma_0(\mathcal{X})$, dom $f^{\blacktriangle} \subset \overline{\operatorname{conv}}\, F$, *and* $f^{\blacktriangle}(\operatorname{dom} f^{\blacktriangle}) \subset [0, +\infty[$.

*Now suppose that, in addition, $f \in \Gamma_0(\mathcal{X})$. Then the following are satisfied:*

(ii) $f^{\blacktriangle} = \max\{f, 0\} + \iota_{\overline{\mathrm{conv}}\, F}$.

(iii) $\mathrm{dom}\, f^{\blacktriangle} = \mathrm{dom}\, f \cap \overline{\mathrm{conv}}\, F \supset F$.

(iv) $(f^{\blacktriangle})^{-1}(\{0\}) = f^{-1}(]{-\infty}, 0]) \cap \overline{\mathrm{conv}}\, F$.

(v) $(f^{\blacktriangle})^{-1}(\{0\}) = \varnothing \Leftrightarrow f^{-1}(]{-\infty}, 0]) = \varnothing$.

(vi) $F = (f^{\blacktriangle})^{-1}(]0, +\infty[)$ *and* $f|_F = f^{\blacktriangle}|_F$.

*Proof.* (i): Set $\theta\colon \mathcal{X} \to \mathbb{R}\colon x \mapsto 0$. Since $f^{\wedge} > \theta \in \mathrm{cam}\, f$ and $F \neq \varnothing$, Lemma 2.4(i) asserts that $f^{\blacktriangle} \in \Gamma_0(\mathcal{X})$. In addition, (3.3), (3.4), and Lemma 2.4(iii) yield

$$\mathrm{dom}\, f^{\blacktriangle} \subset \overline{\mathrm{dom}}\, f^{\blacktriangle} = \overline{\mathrm{conv}}\, \mathrm{dom}\, f^{\wedge} = \overline{\mathrm{conv}}\, F \tag{3.11}$$

and

$$\big(\forall x \in \mathrm{dom}\, f^{\blacktriangle}\big) \quad 0 = \theta(x) = \theta^{**}(x) \leqslant f^{\wedge **}(x) = f^{\blacktriangle}(x) < +\infty. \tag{3.12}$$

(ii): Set $g = \max\{f, 0\}$. Since $F \neq \varnothing$, we have $g \in \Gamma_0(\mathcal{X})$ and $+\infty \not\equiv f^{\wedge} = g + \iota_F \geqslant g + \iota_{\overline{\mathrm{conv}}\, F} \in \Gamma_0(\mathcal{X})$. Hence, appealing to Lemma 2.3(v), we obtain

$$g + \iota_{\overline{\mathrm{conv}}\, F} \leqslant f^{\blacktriangle} \leqslant g + \iota_F. \tag{3.13}$$

Let $x \in \mathcal{X}$. If $x \in F$, then

$$g(x) + \iota_{\overline{\mathrm{conv}}\, F}(x) = f^{\blacktriangle}(x) = g(x) + \iota_F(x) = g(x). \tag{3.14}$$

If $x \notin \overline{\mathrm{conv}}\, F$ or $x \notin \mathrm{dom}\, g$, then $g(x) + \iota_{\overline{\mathrm{conv}}\, F}(x) = f^{\blacktriangle}(x) = g(x) + \iota_F(x) = +\infty$. Now, suppose that $x \in (\mathrm{dom}\, g \cap \overline{\mathrm{conv}}\, F) \smallsetminus F$. Then, since $g(\mathcal{X} \smallsetminus F) \subset \{0, +\infty\}$, we have

$$g(x) = 0. \tag{3.15}$$

It remains to show that $f^{\blacktriangle}(x) = 0$. To this end, fix $\varepsilon \in ]0, +\infty[$. Suppose first that $x \in (\mathrm{conv}\, F) \smallsetminus F$. Since $x \in \mathrm{conv}\, F$, there exist finite families $(x_i)_{i \in I}$ in $F$ and $(\alpha_i)_{i \in I}$ in $]0, 1[$ such that $\sum_{i \in I} \alpha_i = 1$ and $x = \sum_{i \in I} \alpha_i x_i$. Hence, it follows from Lemma 2.1, (3.14), and (3.15) that, for every $i \in I$, there exists $z_i \in ]x, x_i[ \cap F$ such that $f^{\blacktriangle}(z_i) = g(z_i) \in ]0, \varepsilon]$, say $z_i = (1 - \eta_i)x + \eta_i x_i$ for some $\eta_i \in ]0, 1[$. Therefore, for every $i \in I$, $x_i = \eta_i^{-1} z_i + (1 - \eta_i^{-1})x$. In turn, $x = \sum_{i \in I} \alpha_i x_i = \sum_{i \in I} \beta_i z_i$, where, for every $i \in I$, $\beta_i = \alpha_i \eta_i^{-1} / (\sum_{j \in I} \alpha_j \eta_j^{-1}) > 0$. Since $\sum_{i \in I} \beta_i = 1$ and $\{z_i\}_{i \in I} \subset \mathrm{lev}_{\leqslant \varepsilon}\, g$, we have $x \in \mathrm{lev}_{\leqslant \varepsilon}\, g$ and $0 \leqslant f^{\blacktriangle}(x) \leqslant \sum_{i \in I} \beta_i f^{\blacktriangle}(z_i) = \sum_{i \in I} \beta_i g(z_i) \leqslant \varepsilon$. Thus, $f^{\blacktriangle}(x) = 0$. Altogether, in view of (3.15), since $x$ is arbitrarily chosen in $(\mathrm{conv}\, F) \smallsetminus F$, we have

$$\big(\forall u \in (\mathrm{conv}\, F) \smallsetminus F\big) \quad g(u) = 0 \quad \text{and} \quad f^{\blacktriangle}(u) = 0. \tag{3.16}$$

Next, suppose that $x \in (\overline{\mathrm{conv}}\, F) \smallsetminus \mathrm{conv}\, F$. Then there exists a net $(u_a)_{a \in A}$ in $\mathrm{conv}\, F$ such that $u_a \to x$. For every $a \in A$, we consider the following alternatives.

- $u_a \in F$: Since $g(x) = 0$ and $g(u_a) \in ]0, +\infty[$, (3.14) and Lemma 2.1 guarantee the existence of $\widetilde{u}_a \in ]x, u_a[ \cap F$ such that $f^{\blacktriangle}(\widetilde{u}_a) = g(\widetilde{u}_a) \in ]0, \varepsilon]$.

- $u_a \notin F$: Set $\widetilde{u}_a = u_a$. It follows from (3.16) that $g(\widetilde{u}_a) = 0$ and $f^{\blacktriangle}(\widetilde{u}_a) = 0$.

By construction, for every $a \in A$, $\widetilde{u}_a \in \operatorname{conv} F$ and, if $u_a$ is in a convex neighborhood of $x$, so is $\widetilde{u}_a$. Since $\mathcal{X}$ is locally convex, we obtain $\widetilde{u}_a \to x$. By lower semicontinuity of $f^{\blacktriangle}$, we conclude that $0 \leqslant f^{\blacktriangle}(x) \leqslant \underline{\lim}\, f^{\blacktriangle}(\widetilde{u}_a) \leqslant \varepsilon$. This shows that $f^{\blacktriangle}(x) = 0$.

(iii)&(iv): These follow from (ii).

(v): Suppose that $f^{-1}(]-\infty,0]) \neq \varnothing$, let $x \in f^{-1}(]-\infty,0])$, and let $z \in F$. By Lemma 2.1, $[x,z[\, \cap f^{-1}(\{0\}) \cap \overline{F} \neq \varnothing$ and, hence, (iv) yields $(f^{\blacktriangle})^{-1}(\{0\}) \neq \varnothing$ since $\overline{F} \subset \overline{\operatorname{conv}}\, F$. The reverse implication is clear by (iv).

(vi): This follows from (ii). ◻

**Remark 3.4** In the setting of Lemma 3.3, we can have $f \in \Gamma_0(\mathcal{X})$ and $(\overline{\operatorname{conv}}\, F) \cap f^{-1}(]-\infty,0[) \neq \varnothing$. Take, for instance, $\mathcal{X} = \mathbb{R}^2$, and set

$$f\colon \mathcal{X} \to\, ]-\infty,+\infty] : (\xi,\eta) \mapsto \begin{cases} \xi^2/\eta - 1, & \text{if } \eta > 0;\\ -1, & \text{if } \xi = \eta = 0;\\ +\infty, & \text{otherwise.} \end{cases} \tag{3.17}$$

Since $f+1$ is an instance of (1.2), we have $f \in \Gamma_0(\mathcal{X})$. For every $n \in \mathbb{N}$, setting $x_n = (2^{-n}, 2^{-2n-1})$ yields $f(x_n) = 1$. We obtain $F \ni x_n \to (0,0) \in \overline{\operatorname{conv}}\, F$ and $f(0,0) = -1$.

**Lemma 3.5** *Let $f\colon \mathcal{X} \to [-\infty,+\infty]$ be such that $F = f^{-1}(]0,+\infty[) \neq \varnothing$ and assume that* $\operatorname{cam}(-f)^{\vee} \neq \varnothing$. *Then the following hold:*

(i) $-(-f)^{\blacktriangledown} < +\infty$.

(ii) $0 \leqslant f^{\blacktriangle}|_{\overline{\operatorname{conv}}\, F} \leqslant -(-f)^{\blacktriangledown}|_{\overline{\operatorname{conv}}\, F}$.

(iii) $\operatorname{dom} f^{\blacktriangle} = \overline{\operatorname{conv}}\, F$.

*Proof.* (i): Since $\operatorname{cam}(-f)^{\vee} \neq \varnothing$, (3.2) and Lemma 2.3(iii) yield $-\infty \notin (-f)^{\blacktriangledown}(\mathcal{X})$ and therefore $-(-f)^{\blacktriangledown} < +\infty$.

(ii): The first inequality follows from Lemma 3.3(i). We derive from Definition 3.1 and Lemma 2.3(i) that

$$(\forall x \in F) \quad f^{\blacktriangle}(x) \leqslant f^{\wedge}(x) = -\big(-f(x)\big) = -(-f)^{\vee}(x) \leqslant -(-f)^{\blacktriangledown}(x). \tag{3.18}$$

Now set $h = f^{\blacktriangle} + (-f)^{\blacktriangledown}$. Then (3.18) implies that $h|_F \leqslant 0$. Since $F \subset \operatorname{lev}_{\leqslant 0} h$ and $h$ is lower semicontinuous and convex, note that $\overline{\operatorname{conv}}\, F \subset \overline{\operatorname{conv}}\operatorname{lev}_{\leqslant 0} h = \operatorname{lev}_{\leqslant 0} h$.

(iii): This follows from (i), (ii), and Lemma 3.3(i). ◻

**Remark 3.6** Let $f \in \Gamma_0(\mathcal{X})$ be such that $(f^*)^{-1}(]-\infty,0[) \neq \varnothing$. Then $f = \max\{f^{*\blacktriangledown *}, f^{*\blacktriangle *}\}$. Indeed, since Lemma 2.3(v) asserts that $f = f^{**}$, it follows from Lemma 2.4(i), Lemma 3.2(iii), (3.3), Lemma 2.3(ii), and (3.4) that

$$\begin{aligned}(\forall x \in \mathcal{X}) \quad f(x) &= \sup_{x^* \in \mathcal{X}^*} \big(\langle x, x^*\rangle - f^*(x^*)\big)\\ &= \max\left\{ \sup_{x^* \in (f^*)^{-1}(]-\infty,0])} \big(\langle x, x^*\rangle - f^*(x^*)\big), \sup_{x^* \in (f^*)^{-1}(]0,+\infty[)} \big(\langle x, x^*\rangle - f^*(x^*)\big)\right\}\\ &= \max\left\{ \sup_{x^* \in \mathcal{X}^*} \big(\langle x, x^*\rangle - f^{*\blacktriangledown}(x^*)\big), \sup_{x^* \in \mathcal{X}^*} \big(\langle x, x^*\rangle - f^{*\wedge}(x^*)\big)\right\}\\ &= \max\left\{ f^{*\blacktriangledown *}(x), f^{*\blacktriangle *}(x)\right\}. \end{aligned} \tag{3.19}$$

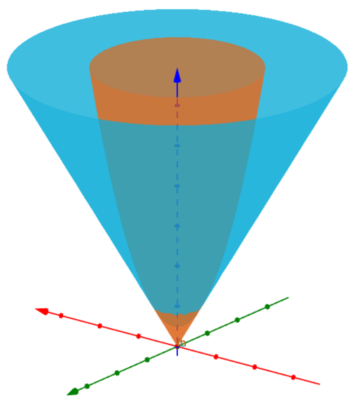


Figure 1: Plots of $f^{*\blacktriangledown *}$ (Huber, blue) and $f^{*\blacktriangle *}$ (Berhu, orange) when $\mathcal{X} = \mathbb{R}^2$ and $f = (\|\cdot\|_2^2 + 1)/2$. We verify that $f$ is the maximum of both functions, as observed in Remark 3.6.

**Example 3.7** Suppose that $(\mathcal{X}, \|\cdot\|)$ is a nonzero real reflexive Banach space with dual norm $\|\cdot\|_*$, let $\alpha \in \left]0, +\infty\right[$, let $p \in \left]1, +\infty\right[$, set $p^* = p/(p-1)$, and set $f = \|\cdot\|^p/p + \alpha^{p^*}/p^*$. Then $f^* = (\|\cdot\|_*^{p^*} - \alpha^{p^*})/p^*$, which yields $(f^*)^{-1}(\left]-\infty, 0\right[) \neq \varnothing$ and $(f^*)^{-1}(\left]0, +\infty\right[) \neq \varnothing$. Therefore, since $\overline{\operatorname{conv}}\,(f^*)^{-1}(\left]0, +\infty\right[) = \mathcal{X}^*$, Lemma 3.2(iii) and Lemma 3.3(ii) imply that

$$f^{*\blacktriangledown}\colon x^* \mapsto \begin{cases} +\infty, & \text{if } \|x^*\|_* > \alpha; \\ \dfrac{\|x^*\|_*^{p^*} - \alpha^{p^*}}{p^*}, & \text{if } \|x^*\|_* \leqslant \alpha \end{cases} \quad \text{and} \quad f^{*\blacktriangle}\colon x^* \mapsto \begin{cases} \dfrac{\|x^*\|_*^{p^*} - \alpha^{p^*}}{p^*}, & \text{if } \|x^*\|_* > \alpha; \\ 0, & \text{if } \|x^*\|_* \leqslant \alpha. \end{cases} \tag{3.20}$$

It is noteworthy that we obtain by conjugation

$$f^{*\blacktriangledown *}\colon x \mapsto \begin{cases} \alpha\|x\|, & \text{if } \|x\| > \alpha^{\frac{1}{p-1}}; \\ \dfrac{\|x\|^p}{p} + \dfrac{\alpha^{p^*}}{p^*}, & \text{if } \|x\| \leqslant \alpha^{\frac{1}{p-1}} \end{cases} \quad \text{and} \quad f^{*\blacktriangle *}\colon x \mapsto \begin{cases} \dfrac{\|x\|^p}{p} + \dfrac{\alpha^{p^*}}{p^*}, & \text{if } \|x\| > \alpha^{\frac{1}{p-1}}; \\ \alpha\|x\|, & \text{if } \|x\| \leqslant \alpha^{\frac{1}{p-1}}. \end{cases} \tag{3.21}$$

We recognize, respectively, the $p$th order Huber and Berhu functions used in [23, 33] (see Figure 1).

# 4 Preperspective functions

Let us first record some direct consequences of Definition 1.1.

**Proposition 4.1** *Let $\varphi\colon \mathcal{X} \to [-\infty, +\infty]$, let $s\colon \mathcal{Y} \to [-\infty, +\infty]$, and set $S = s^{-1}(\left]0, +\infty\right[)$. Then the following hold:*

(i) $\operatorname{dom}(\varphi \ltimes s) = \big\{(x, y) \in \mathcal{X} \times S \;\big|\; x \in s(y)\operatorname{dom}\varphi\big\}$.

(ii) *$\varphi \ltimes s$ is proper if and only if $\varphi$ is proper and $S \neq \varnothing$.*

*Proof.* (i): Clear from Definition 1.1.

(ii): We derive from (1.1) that $-\infty \in (\varphi \ltimes s)(\mathcal{X} \times \mathcal{Y}) \Leftrightarrow -\infty \in \varphi(\mathcal{X})$. Suppose that $\varphi \ltimes s$ is proper and let $(x, y) \in \operatorname{dom}(\varphi \ltimes s)$. In view of (i), $y \in S$ and $x/s(y) \in \operatorname{dom}\varphi$. Now suppose that $\varphi$ is proper and $S \neq \varnothing$, and let $(x, y) \in \operatorname{dom}\varphi \times S$. Then $(s(y)x, y) \in \operatorname{dom}(\varphi \ltimes s)$. ☐

Our first result provides conditions under which the preperspective of a convex function is itself convex.

**Proposition 4.2** *Let $\varphi\colon \mathcal{X} \to [-\infty, +\infty]$ be convex, let $s\colon \mathcal{Y} \to [-\infty, +\infty]$, set $S = s^{-1}(]0, +\infty[)$, and suppose that one of the following holds:*

(i) *$\varphi$ satisfies*

$$(\forall \lambda \in ]1, +\infty[)(\forall x \in \operatorname{dom}\varphi) \quad \varphi(\lambda x) \leqslant \lambda\varphi(x), \tag{4.1}$$

*$s$ is proper and convex, and $S$ is convex.*

(ii) *$\varphi(0) \leqslant 0$ and $-s$ is proper and convex.*

(iii) *$s$ is an affine function.*

*Then $\varphi \ltimes s$ is convex.*

*Proof.* Let $\alpha \in ]0, 1[$, and suppose that $(x_1, y_1) \in \operatorname{dom}(\varphi \ltimes s)$ and $(x_2, y_2) \in \operatorname{dom}(\varphi \ltimes s)$. Set

$$y = \alpha y_1 + (1 - \alpha)y_2. \tag{4.2}$$

Observe that, since $S$ is convex, $y \in S$. Further, set

$$\beta_1 = \frac{\alpha s(y_1)}{s(y)}, \quad \beta_2 = \frac{(1 - \alpha)s(y_2)}{s(y)}, \quad \text{and} \quad \beta = \beta_1 + \beta_2, \tag{4.3}$$

and note that $\beta_1 \in ]0, +\infty[$ and $\beta_2 \in ]0, +\infty[$.

(i): Observe that the convexity of $s$ yields $\beta \in [1, +\infty[$. In view of (4.3), (4.1), and the convexity of $\varphi$, we have

$$\begin{aligned}
(\varphi \ltimes s)\big(\alpha(x_1, y_1) + (1 - \alpha)(x_2, y_2)\big) &= s(y)\varphi\bigg(\frac{\alpha x_1 + (1 - \alpha)x_2}{s(y)}\bigg) \\
&= s(y)\varphi\bigg(\frac{\beta_1 x_1}{s(y_1)} + \frac{\beta_2 x_2}{s(y_2)}\bigg) \\
&= s(y)\varphi\bigg(\beta\bigg(\frac{\beta_1 x_1}{\beta s(y_1)} + \frac{\beta_2 x_2}{\beta s(y_2)}\bigg)\bigg) \\
&\leqslant s(y)\beta\varphi\bigg(\frac{\beta_1}{\beta}\frac{x_1}{s(y_1)} + \frac{\beta_2}{\beta}\frac{x_2}{s(y_2)}\bigg) \\
&\leqslant s(y)\beta_1\varphi\bigg(\frac{x_1}{s(y_1)}\bigg) + s(y)\beta_2\varphi\bigg(\frac{x_2}{s(y_2)}\bigg) \\
&= \alpha s(y_1)\varphi\bigg(\frac{x_1}{s(y_1)}\bigg) + (1 - \alpha)s(y_2)\varphi\bigg(\frac{x_2}{s(y_2)}\bigg) \\
&= \alpha(\varphi \ltimes s)(x_1, y_1) + (1 - \alpha)(\varphi \ltimes s)(x_2, y_2).
\end{aligned} \tag{4.4}$$

(ii)–(iii): By convexity, $s(y) \geqslant \alpha s(y_1) + (1-\alpha)s(y_2) > 0$ and, therefore, (4.3) yields $\beta \in\, ]0,1]$. We have

$$(\varphi \ltimes s)\big(\alpha(x_1,y_1) + (1-\alpha)(x_2,y_2)\big) = s(y)\varphi\bigg(\frac{\alpha x_1 + (1-\alpha)x_2}{s(y)}\bigg) = s(y)\varphi\bigg(\beta_1 \frac{x_1}{s(y_1)} + \beta_2 \frac{x_2}{s(y_2)} + (1-\beta)0\bigg). \tag{4.5}$$

In case (iii) we have $\beta = 1$ and hence, by convexity of $\varphi$,

$$(\varphi \ltimes s)\big(\alpha(x_1,y_1) + (1-\alpha)(x_2,y_2)\big) \leqslant \alpha s(y_1)\varphi\bigg(\frac{x_1}{s(y_1)}\bigg) + (1-\alpha)s(y_2)\varphi\bigg(\frac{x_2}{s(y_2)}\bigg) = \alpha(\varphi \ltimes s)(x_1,y_1) + (1-\alpha)(\varphi \ltimes s)(x_2,y_2). \tag{4.6}$$

We now turn to (ii). If $\beta = 1$, then we obtain (4.6) using (4.5). On the other hand, if $\beta \in\, ]0,1[$, then since $\varphi(0) \leqslant 0$, we have $(1-\beta)s(y)\varphi(0) \leqslant 0$. Hence, it follows from (4.5) and convexity of $\varphi$ that

$$\begin{aligned}(\varphi \ltimes s)\big(\alpha(x_1,y_1) + (1-\alpha)(x_2,y_2)\big) &\leqslant \alpha s(y_1)\varphi\bigg(\frac{x_1}{s(y_1)}\bigg) + (1-\alpha)s(y_2)\varphi\bigg(\frac{x_2}{s(y_2)}\bigg) \\ &\quad + (1-\beta)s(y)\varphi(0) \\ &\leqslant \alpha s(y_1)\varphi\bigg(\frac{x_1}{s(y_1)}\bigg) + (1-\alpha)s(y_2)\varphi\bigg(\frac{x_2}{s(y_2)}\bigg) \\ &\leqslant \alpha(\varphi \ltimes s)(x_1,y_1) + (1-\alpha)(\varphi \ltimes s)(x_2,y_2),\end{aligned} \tag{4.7}$$

which concludes the proof. ☐

Next, we determine the conjugate of the preperspective, using the $\blacktriangledown$ and $\blacktriangle$ envelopes of Definition 3.1. In view of (1.1), if $s^{-1}(]0,+\infty[) = \varnothing$, then $(\varphi \ltimes s)^* \equiv -\infty$ and $\varphi \blacktriangleright\!\!\!\!< s \equiv +\infty$. We therefore rule out this trivial case henceforth.

**Proposition 4.3** *Let $\varphi\colon \mathcal{X} \to [-\infty,+\infty]$, let $s\colon \mathcal{Y} \to [-\infty,+\infty]$, let $x^* \in \mathcal{X}^*$ and $y^* \in \mathcal{Y}^*$, and suppose that $S = s^{-1}(]0,+\infty[) \neq \varnothing$. Then the following hold:*

(i) $(\varphi \ltimes s)^*(x^*,y^*) = \sup_{y\in S}(\langle y,y^*\rangle + s(y)\varphi^*(x^*))$.

(ii) *Suppose that $\varphi^*(x^*) = \pm\infty$. Then $(\varphi \ltimes s)^*(x^*,y^*) = \pm\infty$.*

(iii) *Suppose that $-\infty < \varphi^*(x^*) < 0$. Then $(\varphi \ltimes s)^*(x^*,y^*) = (s^{\blacktriangle *} \ltimes (-\varphi^*))(y^*,x^*)$.*

(iv) *Suppose that $\varphi^*(x^*) = 0$. Then $(\varphi \ltimes s)^*(x^*,y^*) = \sigma_{\overline{\mathrm{conv}}\, S}(y^*)$.*

(v) *Suppose that $0 < \varphi^*(x^*) < +\infty$. Then $(\varphi \ltimes s)^*(x^*,y^*) = ((-s)^{\blacktriangledown *} \ltimes \varphi^*)(y^*,x^*)$.*

*Proof.* (i): It follows from Definition 1.1 and Proposition 4.1(i) that

$$\begin{aligned}(\varphi \ltimes s)^*(x^*,y^*) &= \sup_{\substack{x\in\mathcal{X}\\ y\in\mathcal{Y}}}\big(\langle x,x^*\rangle + \langle y,y^*\rangle - (\varphi \ltimes s)(x,y)\big) \\ &= \sup_{\substack{x\in\mathcal{X}\\ y\in S}}\bigg(\langle x,x^*\rangle + \langle y,y^*\rangle - s(y)\varphi\bigg(\frac{x}{s(y)}\bigg)\bigg) \\ &= \sup_{y\in S}\bigg(\langle y,y^*\rangle + s(y)\bigg(\sup_{x\in\mathcal{X}}\bigg\langle \frac{x}{s(y)},x^*\bigg\rangle - \varphi\bigg(\frac{x}{s(y)}\bigg)\bigg)\bigg) \\ &= \sup_{y\in S}\big(\langle y,y^*\rangle + s(y)\varphi^*(x^*)\big).\end{aligned} \tag{4.8}$$

(ii): This follows from (i).

(iii): It follows from (i), (3.3), (3.4), and Lemma 2.3(ii) that

$$\begin{aligned}(\varphi \ltimes s)^*(x^*,y^*) &= -\varphi^*(x^*)\sup_{y\in S}\left(\left\langle y, \frac{y^*}{-\varphi^*(x^*)}\right\rangle - s(y)\right)\\ &= -\varphi^*(x^*)\sup_{y\in\mathcal{Y}}\left(\left\langle y, \frac{y^*}{-\varphi^*(x^*)}\right\rangle - s^{\wedge}(y)\right)\\ &= -\varphi^*(x^*)s^{\wedge *}\left(\frac{y^*}{-\varphi^*(x^*)}\right)\\ &= \big(s^{\blacktriangle *} \ltimes (-\varphi^*)\big)(y^*,x^*). \end{aligned} \tag{4.9}$$

(iv): We derive from (i) and Lemma 2.2 that $(\varphi \ltimes s)^*(x^*,y^*) = \sigma_S(y^*) = \sigma_{\overline{\text{conv}}\, S}(y^*)$.

(v): It follows from (i), (3.1), (3.2), and Lemma 2.3(ii) that

$$\begin{aligned}(\varphi \ltimes s)^*(x^*,y^*) &= \varphi^*(x^*)\sup_{y\in S}\left(\left\langle y, \frac{y^*}{\varphi^*(x^*)}\right\rangle + s(y)\right)\\ &= \varphi^*(x^*)\sup_{y\in\mathcal{Y}}\left(\left\langle y, \frac{y^*}{\varphi^*(x^*)}\right\rangle - (-s)^{\vee}(y)\right)\\ &= \varphi^*(x^*)(-s)^{\vee *}\left(\frac{y^*}{\varphi^*(x^*)}\right)\\ &= \big((-s)^{\blacktriangledown *} \ltimes \varphi^*\big)(y^*,x^*), \end{aligned} \tag{4.10}$$

as claimed. ☐

As an illustration, we consider the case of affine scaling.

**Example 4.4** Let $\varphi \in \Gamma_0(\mathcal{X})$, let $w^* \in \mathcal{Y}^* \smallsetminus \{0\}$, let $\overline{y} \in \mathcal{Y}$, set $s = w^* - \langle \overline{y}, w^*\rangle$, set $S = \big\{y \in \mathcal{Y} \;\big|\; \langle y - \overline{y}, w^*\rangle > 0\big\}$, and set $K = \big\{y \in \mathcal{Y} \;\big|\; \langle y, w^*\rangle \geqslant 0\big\}$. Let $x^* \in \mathcal{X}^*$ and $y^* \in \mathcal{Y}^*$. If $\varphi^*(x^*) = +\infty$, Proposition 4.3(ii) yields $(\varphi \ltimes s)^*(x^*,y^*) = +\infty$. Otherwise, $\varphi^*(x^*) \in \mathbb{R}$ and, since $S \neq \varnothing$, it follows from Proposition 4.3(i) that

$$\begin{aligned}(\varphi \ltimes s)^*(x^*,y^*) &= \sup_{y\in S}\big(\langle y,y^*\rangle + \varphi^*(x^*)\langle y-\overline{y}, w^*\rangle\big)\\ &= \langle \overline{y}, y^*\rangle + \sup_{y\in S}\langle y - \overline{y}, y^* + \varphi^*(x^*)w^*\rangle\\ &= \langle \overline{y}, y^*\rangle + \sup_{y\in K}\langle y, y^* + \varphi^*(x^*)w^*\rangle\\ &= \begin{cases}\langle \overline{y}, y^*\rangle, & \text{if } \big(\exists\, \beta \in \left]-\infty, -\varphi^*(x^*)\right]\big)\;\; y^* = \beta w^*;\\ +\infty, & \text{otherwise.}\end{cases}\end{aligned} \tag{4.11}$$

In particular, suppose that $\mathcal{Y} = \mathbb{R}$, $w^* = 1$, and $\overline{y} = 0$, i.e., $s\colon y \mapsto y$. Then $\varphi \ltimes s$ is the standard preperspective of (1.1) and (4.11) yields

$$(\varphi \ltimes s)^* = \iota_C, \quad \text{where} \quad C = \big\{(x^*,y^*) \in \mathcal{X}^* \times \mathbb{R} \;\big|\; \varphi^*(x^*) + y^* \leqslant 0\big\}, \tag{4.12}$$

which recovers the expression given in [45].

Next, we derive a variant of Proposition 4.3 that will be more readily applicable.

**Theorem 4.5** *Let $\varphi\colon \mathcal{X} \to ]-\infty,+\infty]$ be proper, let $s\colon \mathcal{Y} \to [-\infty,+\infty]$ be such that $S = s^{-1}(]0,+\infty[) \neq \varnothing$, let $x^* \in \mathcal{X}^*$, and let $y^* \in \mathcal{Y}^*$. Then the following hold:*

(i) *Suppose that $\varphi^*(\mathcal{X}^*) \subset ]-\infty,0] \cup \{+\infty\}$ and $(\varphi^*)^{-1}(]-\infty,0[) \neq \varnothing$. Then*

$$(\varphi \ltimes s)^*(x^*,y^*) = \begin{cases} -\varphi^*(x^*)s^{\blacktriangle *}\left(\dfrac{y^*}{-\varphi^*(x^*)}\right), & \text{if } -\infty < \varphi^*(x^*) < 0; \\ \sigma_{\overline{\text{conv}}\, S}(y^*), & \text{if } \varphi^*(x^*) = 0; \\ +\infty, & \text{if } \varphi^*(x^*) = +\infty. \end{cases} \tag{4.13}$$

(ii) *Suppose that $\varphi^*(\mathcal{X}^*) \subset \{0,+\infty\}$. Then*

$$(\varphi \ltimes s)^*(x^*,y^*) = \iota_{(\varphi^*)^{-1}(\{0\})}(x^*) + \sigma_{\overline{\text{conv}}\, S}(y^*). \tag{4.14}$$

(iii) *Suppose that $\varphi^*(\mathcal{X}^*) \subset [0,+\infty]$ and $(\varphi^*)^{-1}(]0,+\infty[) \neq \varnothing$. Then*

$$(\varphi \ltimes s)^*(x^*,y^*) = \begin{cases} \varphi^*(x^*)(-s)^{\blacktriangledown *}\left(\dfrac{y^*}{\varphi^*(x^*)}\right), & \text{if } 0 < \varphi^*(x^*) < +\infty; \\ \sigma_{\overline{\text{conv}}\, S}(y^*), & \text{if } \varphi^*(x^*) = 0; \\ +\infty, & \text{if } \varphi^*(x^*) = +\infty. \end{cases} \tag{4.15}$$

(iv) *Suppose that $(\varphi^*)^{-1}(]-\infty,0[) \neq \varnothing$ and $(\varphi^*)^{-1}(]0,+\infty[) \neq \varnothing$. Then the following hold:*

a) $$(\varphi \ltimes s)^*(x^*,y^*) = \begin{cases} -\varphi^*(x^*)s^{\blacktriangle *}\left(\dfrac{y^*}{-\varphi^*(x^*)}\right), & \text{if } -\infty < \varphi^*(x^*) < 0; \\ \sigma_{\overline{\text{conv}}\, S}(y^*), & \text{if } \varphi^*(x^*) = 0; \\ \varphi^*(x^*)(-s)^{\blacktriangledown *}\left(\dfrac{y^*}{\varphi^*(x^*)}\right), & \text{if } 0 < \varphi^*(x^*) < +\infty; \\ +\infty, & \text{if } \varphi^*(x^*) = +\infty. \end{cases}$$

b) $(\varphi \ltimes s)^*(x^*,y^*) = \min\big\{\big(\varphi^{*\blacktriangledown *} \ltimes s\big)^*(x^*,y^*), \big(\varphi^{*\blacktriangle *} \ltimes s\big)^*(x^*,y^*)\big\}$.

*Proof*. Claims (i)–(iv)a) follow from Proposition 4.3(ii)–(v) and Definition 1.1. It remains to show (iv)b). Since $\varphi$ is proper, $-\infty \notin \varphi^*(\mathcal{X}^*)$. Moreover, $\operatorname{dom}\varphi^* \neq \varnothing$ and hence $\varphi^* \in \Gamma_0(\mathcal{X}^*)$. Therefore, applying items (i), (vi), and (v) in Lemma 3.2 to $\varphi^*$ and invoking Lemma 2.3(v) and (i) yield

$$\big(\varphi^{*\blacktriangledown *} \ltimes s\big)^*(x^*,y^*) = \begin{cases} -\varphi^*(x^*)s^{\blacktriangle *}\left(\dfrac{y^*}{-\varphi^*(x^*)}\right), & \text{if } -\infty < \varphi^*(x^*) < 0; \\ \sigma_{\overline{\text{conv}}\, S}(y^*), & \text{if } \varphi^*(x^*) = 0; \\ +\infty, & \text{if } 0 < \varphi^*(x^*) \leqslant +\infty. \end{cases} \tag{4.16}$$

Likewise, using Lemma 3.3 and (iii), we arrive at

$$(\varphi^{*\blacktriangle *} \ltimes s)^*(x^*,y^*) = \begin{cases} \varphi^*(x^*)(-s)^{\blacktriangledown *}\left(\dfrac{y^*}{\varphi^*(x^*)}\right), & \text{if } 0 < \varphi^*(x^*) < +\infty; \\ \sigma_{\overline{\text{conv}}\, S}(y^*), & \text{if } -\infty < \varphi^*(x^*) \leqslant 0 \\ & \quad \text{and } x^* \in \overline{\text{conv}}\,(\varphi^*)^{-1}(]0,+\infty[); \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.17}$$

If $0 \leqslant \varphi^*(x^*) \leqslant +\infty$, we deduce the identity from (iv)a), (4.16), and (4.17). Now assume that $-\infty < \varphi^*(x^*) < 0$. Lemma 3.3(i) asserts that $s^\blacktriangle(\operatorname{dom} s^\blacktriangle) \subset [0,+\infty[$ and $\operatorname{dom} s^\blacktriangle \subset \overline{\operatorname{conv}}\, S$. Hence,

$$\begin{aligned} -\varphi^*(x^*)s^{\blacktriangle*}\left(\frac{y^*}{-\varphi^*(x^*)}\right) &= -\varphi^*(x^*) \sup_{y\in\operatorname{dom} s^\blacktriangle}\left(\left\langle y, \frac{y^*}{-\varphi^*(x^*)}\right\rangle - s^\blacktriangle(y)\right) \\ &= \sup_{y\in\overline{\operatorname{conv}}\, S}\left(\langle y,y^*\rangle + \varphi^*(x^*)s^\blacktriangle(y)\right) \\ &\leqslant \sup_{y\in\overline{\operatorname{conv}}\, S}\langle y,y^*\rangle \\ &= \sigma_{\overline{\operatorname{conv}}\, S}(y^*), \end{aligned} \tag{4.18}$$

which yields

$$\min\{(\varphi^{*\blacktriangledown*} \ltimes s)^*(x^*,y^*), (\varphi^{*\blacktriangle*} \ltimes s)^*(x^*,y^*)\} = -\varphi^*(x^*)s^{\blacktriangle*}\left(\frac{y^*}{-\varphi^*(x^*)}\right). \tag{4.19}$$

Thus, the conclusion follows from (iv)a). ☐

We conclude this section by establishing conditions under which the preperspective admits a continuous affine minorant. Note that, in view of Lemma 2.3(iii) and Theorem 4.5(ii), $\operatorname{cam}\varphi = \varnothing \Rightarrow \operatorname{cam}(\varphi \ltimes s) = \varnothing$.

**Corollary 4.6** *Let* $\varphi\colon \mathcal{X} \to\, ]-\infty,+\infty]$ *be proper and such that* $\operatorname{cam}\varphi \neq \varnothing$ *and let* $s\colon \mathcal{Y} \to [-\infty,+\infty]$ *be such that* $S = s^{-1}(]0,+\infty[) \neq \varnothing$. *Then*

$$\operatorname{cam}(\varphi \ltimes s) \neq \varnothing \quad\Leftrightarrow\quad \left[\,(\varphi^*)^{-1}(]-\infty,0]) \neq \varnothing \quad \textit{or} \quad \operatorname{cam}(-s)^\vee \neq \varnothing\,\right]. \tag{4.20}$$

*Proof*. Lemma 2.3(iii) asserts that $\operatorname{cam}(\varphi \ltimes s) = \varnothing$ if and only if $(\varphi \ltimes s)^* \equiv +\infty$. In view of Theorem 4.5(iii),

$$\left[\varphi^*(\mathcal{X}^*) \subset\, ]0,+\infty] \quad\text{and}\quad (-s)^{\blacktriangledown*} \equiv +\infty\right] \quad\Rightarrow\quad (\varphi \ltimes s)^* \equiv +\infty. \tag{4.21}$$

An inspection of items (i)–(iv)a) in Theorem 4.5 shows that the converse implication also holds. Altogether, (4.20) follows from (4.21) and Lemma 2.3(ii)–(iii). ☐

**Example 4.7** Let $\varphi\colon \mathcal{X} \to\, ]-\infty,+\infty]$ be proper and convex, and let $s\colon \mathcal{Y} \to [-\infty,+\infty]$ be such that $S = s^{-1}(]0,+\infty[) \neq \varnothing$. Suppose that one of the following holds:

(i) $\Gamma_0(\mathcal{X}) \ni \varphi \geqslant \operatorname{rec}\varphi$.

(ii) $\varphi$ is lower semicontinuous at $0$ and $\varphi(0) \in\, ]0,+\infty[$.

(iii) $\operatorname{cam}\varphi \neq \varnothing$ and $-s \in \Gamma_0(\mathcal{Y})$.

Then $\operatorname{cam}(\varphi \ltimes s) \neq \varnothing$.

*Proof*. (i): This follows from Lemma 2.3(v), Corollary 4.6, and Lemma 2.6.

(ii): As in [5, Proposition 13.44], we have $\inf\varphi^*(\mathcal{X}^*) = -\varphi^{**}(0) = -\varphi(0) \in\, ]-\infty,0[$, which yields $(\varphi^*)^{-1}(]-\infty,0]) \neq \varnothing$. Hence the conclusion follows from Lemma 2.3(iii) and Corollary 4.6.

(iii): According to Lemma 2.3(v), $\varnothing \neq \operatorname{cam}(-s) \subset \operatorname{cam}(-s)^\vee$. Therefore, the conclusion follows from Corollary 4.6. ☐

# 5 Perspective functions

We investigate the properties of the perspective function introduced in Definition 1.1. We preface our analysis with the case of affine scaling.

**Example 5.1** Let $\varphi \in \Gamma_0(\mathcal{X})$, suppose that $w^* \in \mathcal{Y}^* \smallsetminus \{0\}$, let $\overline{y} \in \mathcal{Y}$, and set $s = w^* - \langle \overline{y}, w^* \rangle$. Let $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Then

$$(\varphi \lftimes s)(x,y) = \begin{cases} \langle y-\overline{y}, w^*\rangle \varphi\left(\dfrac{x}{\langle y-\overline{y}, w^*\rangle}\right), & \text{if } \langle y-\overline{y}, w^*\rangle > 0;\\ (\operatorname{rec}\varphi)(x), & \text{if } \langle y-\overline{y}, w^*\rangle = 0;\\ +\infty, & \text{otherwise.} \end{cases} \tag{5.1}$$

In particular, if $\mathcal{Y} = \mathbb{R}$, $w^* = 1$, and $\overline{y} = 0$, we recover the fact that $\varphi \lftimes s = \widetilde{\varphi}$ mentioned in Section 1 (see (1.2)).

*Proof*. Since $-s \in \Gamma_0(\mathcal{Y})$, it follows from Lemma 2.3(v) and Example 4.7(iii) that $\operatorname{cam}(\varphi \ltimes s) \neq \varnothing$. Therefore, we deduce from Definition 1.1, Lemma 2.4(ii), and Example 4.4 that

$$\begin{aligned}(\varphi \lftimes s)(x,y) &= \breve{(\varphi \ltimes s)}(x,y)\\ &= (\varphi \ltimes s)^{**}(x,y)\\ &= \sup_{\substack{x^*\in\mathcal{X}^*\\ y^*\in\mathcal{Y}^*}} \big(\langle x, x^*\rangle + \langle y, y^*\rangle - (\varphi \ltimes s)^*(x^*, y^*)\big)\\ &= \sup_{\substack{x^*\in\operatorname{dom}\varphi^*\\ \beta\in]-\infty,-\varphi^*(x^*)]}} \big(\langle x, x^*\rangle + \beta\langle y-\overline{y}, w^*\rangle\big)\\ &= \begin{cases} \sup\limits_{x^*\in\operatorname{dom}\varphi^*} \big(\langle x, x^*\rangle - \varphi^*(x^*)\langle y-\overline{y}, w^*\rangle\big), & \text{if } \langle y-\overline{y}, w^*\rangle > 0;\\ \sigma_{\operatorname{dom}\varphi^*}(x), & \text{if } \langle y-\overline{y}, w^*\rangle = 0;\\ +\infty, & \text{if } \langle y-\overline{y}, w^*\rangle < 0, \end{cases}\end{aligned} \tag{5.2}$$

which, by virtue of Lemma 2.3(v) and Lemma 2.5(ii), yields (5.1). ☐

We are now ready to present our main result, which provides explicit expressions of the perspective function in the general case of nonlinear scaling. We state our theorem in a setting that avoids the degenerate case when $(\varphi \lftimes s)(\mathcal{X} \times \mathcal{Y}) \subset \{-\infty, +\infty\}$.

**Theorem 5.2** *Let $\varphi\colon \mathcal{X} \to\, ]-\infty,+\infty]$ be proper and such that $\operatorname{cam}\varphi \neq \varnothing$, let $s\colon \mathcal{Y} \to [-\infty,+\infty]$ be such that $S = s^{-1}(]0,+\infty[) \neq \varnothing$, and suppose that*

$$(\varphi^*)^{-1}(]-\infty,0]) \neq \varnothing \quad \textit{or} \quad \operatorname{cam}(-s)^{\vee} \neq \varnothing. \tag{5.3}$$

*Then*

(i) $\varphi \lftimes s \in \Gamma_0(\mathcal{X} \oplus \mathcal{Y})$.

*Furthermore, let $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Then the following are satisfied:*

(ii) *Suppose that $\varphi^*(\mathcal{X}^*) \subset\, ]-\infty,0] \cup \{+\infty\}$ and $(\varphi^*)^{-1}(]-\infty,0[) \neq \varnothing$. Then*

$$(\varphi \lftimes s)(x,y) = \begin{cases} s^{\blacktriangle}(y)\breve{\varphi}\left(\dfrac{x}{s^{\blacktriangle}(y)}\right), & \textit{if } 0 < s^{\blacktriangle}(y) < +\infty;\\ (\operatorname{rec}\breve{\varphi})(x), & \textit{if } s^{\blacktriangle}(y) = 0;\\ +\infty, & \textit{if } s^{\blacktriangle}(y) = +\infty. \end{cases} \tag{5.4}$$

(iii) *Suppose that* $\varphi^*(\mathcal{X}^*) \subset \{0, +\infty\}$. *Then*

$$\big(\varphi \ltimes s\big)(x, y) = (\operatorname{rec} \breve{\varphi})(x) + \iota_{\overline{\operatorname{conv}} S}(y). \tag{5.5}$$

(iv) *Suppose that* $\varphi^*(\mathcal{X}^*) \subset [0, +\infty]$. *Then the following are satisfied:*

a) *Suppose that* $(\varphi^*)^{-1}(\{0\}) \neq \varnothing$ *and* cam $(-s)^\vee = \varnothing$. *Then*

$$\big(\varphi \ltimes s\big)(x, y) = \sigma_{(\varphi^*)^{-1}(\{0\})}(x) + \iota_{\overline{\operatorname{conv}} S}(y). \tag{5.6}$$

b) *Suppose that* $(\varphi^*)^{-1}(]0, +\infty[) \neq \varnothing$ *and* cam $(-s)^\vee \neq \varnothing$. *Then*

$$\big(\varphi \ltimes s\big)(x, y) = \begin{cases} -(-s)^{\blacktriangledown}(y)\breve{\varphi}\left(\dfrac{x}{-(-s)^{\blacktriangledown}(y)}\right), & \text{if } -\infty < (-s)^{\blacktriangledown}(y) < 0; \\ (\operatorname{rec} \breve{\varphi})(x), & \text{if } (-s)^{\blacktriangledown}(y) = 0; \\ +\infty, & \text{if } (-s)^{\blacktriangledown}(y) = +\infty. \end{cases} \tag{5.7}$$

(v) *Suppose that* $(\varphi^*)^{-1}(]-\infty, 0[) \neq \varnothing$ *and that* $(\varphi^*)^{-1}(]0, +\infty[) \neq \varnothing$. *Then the following are satisfied:*

a) $\big(\varphi \ltimes s\big)(x, y) = \max\big\{\big(\varphi^{*\blacktriangledown *} \ltimes s\big)(x, y), \big(\varphi^{*\blacktriangle *} \ltimes s\big)(x, y)\big\}$.

b) *Suppose that* cam $(-s)^\vee = \varnothing$. *Then* $\varphi^{*\blacktriangledown *} \ltimes s \geqslant \varphi^{*\blacktriangle *} \ltimes s$ *and*

$$\big(\varphi \ltimes s\big)(x, y) = \begin{cases} s^{\blacktriangle}(y)\varphi^{*\blacktriangledown *}\left(\dfrac{x}{s^{\blacktriangle}(y)}\right), & \text{if } 0 < s^{\blacktriangle}(y) < +\infty; \\ (\operatorname{rec} \varphi^{*\blacktriangledown *})(x), & \text{if } s^{\blacktriangle}(y) = 0; \\ +\infty, & \text{if } s^{\blacktriangle}(y) = +\infty. \end{cases} \tag{5.8}$$

c) *Suppose that* cam $(-s)^\vee \neq \varnothing$. *Then*

$$\big(\varphi \ltimes s\big)(x, y) =$$
$$\begin{cases} \max\left\{s^{\blacktriangle}(y)\varphi^{*\blacktriangledown *}\left(\dfrac{x}{s^{\blacktriangle}(y)}\right), -(-s)^{\blacktriangledown}(y)\varphi^{*\blacktriangle *}\left(\dfrac{x}{-(-s)^{\blacktriangledown}(y)}\right)\right\}, & \text{if } 0 < s^{\blacktriangle}(y) < +\infty; \\ \max\left\{(\operatorname{rec} \varphi^{*\blacktriangledown *})(x), -(-s)^{\blacktriangledown}(y)\varphi^{*\blacktriangle *}\left(\dfrac{x}{-(-s)^{\blacktriangledown}(y)}\right)\right\}, & \text{if } (-s)^{\blacktriangledown}(y) < 0 = s^{\blacktriangle}(y); \\ (\operatorname{rec} \breve{\varphi})(x), & \text{if } (-s)^{\blacktriangledown}(y) = 0 = s^{\blacktriangle}(y); \\ +\infty, & \text{if } s^{\blacktriangle}(y) = +\infty, \end{cases} \tag{5.9}$$

*where all the possible cases are exhausted.*

*Proof*. Since cam $\varphi \neq \varnothing$ and $\varphi \not\equiv +\infty$, by virtue of Lemma 2.4(i)–(ii), we have

$$\varphi^* \in \Gamma_0(\mathcal{X}^*) \quad \text{and} \quad \varphi^{**} = \breve{\varphi} \in \Gamma_0(\mathcal{X}). \tag{5.10}$$

In turn, it follows from (5.3), Corollary 4.6, Definition 1.1, and Lemma 2.4(ii) that

$$\operatorname{cam}(\varphi \ltimes s) \neq \varnothing \quad \text{and} \quad \varphi \ltimes s = (\varphi \ltimes s)\breve{} = (\varphi \ltimes s)^{**}. \tag{5.11}$$

We also derive from Lemma 2.3(ii), (5.10), and Lemma 2.5(ii) that

$$\sigma_{\mathrm{dom}\,\varphi^*} = \sigma_{\mathrm{dom}\,(\breve{\varphi})^*} = \mathrm{rec}\,\breve{\varphi} \tag{5.12}$$

and from Proposition 4.1(ii) that

$$\mathrm{dom}\,(\varphi \ltimes s) \neq \varnothing. \tag{5.13}$$

(i): This follows from (5.11), (5.13), and Lemma 2.4(i).

(ii): Theorem 4.5(i) implies that $\mathrm{dom}\,(\varphi \ltimes s)^* \subset (\varphi^*)^{-1}(]-\infty, 0]) \times \mathcal{Y}^*$. Consequently,

$$(\varphi \ltimes s)^{**}(x,y) = \max\left\{ \sup_{\substack{x^* \in (\varphi^*)^{-1}(]-\infty,0[)\\ y^* \in \mathcal{Y}^*}} \big(\langle x, x^*\rangle + \langle y, y^*\rangle - (\varphi \ltimes s)^*(x^*, y^*)\big),\right.$$
$$\left.\sup_{\substack{x^* \in (\varphi^*)^{-1}(\{0\})\\ y^* \in \mathcal{Y}^*}} \big(\langle x, x^*\rangle + \langle y, y^*\rangle - (\varphi \ltimes s)^*(x^*, y^*)\big)\right\}. \tag{5.14}$$

Moreover, by Theorem 4.5(i), (3.4), and Lemma 2.3(ii),

$$\begin{aligned}
&\sup_{\substack{x^* \in (\varphi^*)^{-1}(]-\infty,0[)\\ y^* \in \mathcal{Y}^*}} \big(\langle x, x^*\rangle + \langle y, y^*\rangle - (\varphi \ltimes s)^*(x^*, y^*)\big)\\
&\quad= \sup_{\substack{x^* \in (\varphi^*)^{-1}(]-\infty,0[)\\ y^* \in \mathcal{Y}^*}} \left(\langle x, x^*\rangle + \langle y, y^*\rangle + \varphi^*(x^*) s^{\blacktriangle *}\left(\frac{y^*}{-\varphi^*(x^*)}\right)\right)\\
&\quad= \sup_{x^* \in (\varphi^*)^{-1}(]-\infty,0[)} \left(\langle x, x^*\rangle - \varphi^*(x^*) \sup_{y^* \in \mathcal{Y}^*}\left(\left\langle y, \frac{y^*}{-\varphi^*(x^*)}\right\rangle - s^{\blacktriangle *}\left(\frac{y^*}{-\varphi^*(x^*)}\right)\right)\right)\\
&\quad= \sup_{x^* \in (\varphi^*)^{-1}(]-\infty,0[)} \big(\langle x, x^*\rangle - \varphi^*(x^*) s^{\blacktriangle}(y)\big)
\end{aligned} \tag{5.15}$$

and

$$\begin{aligned}
\sup_{\substack{x^* \in (\varphi^*)^{-1}(\{0\})\\ y^* \in \mathcal{Y}^*}} \big(\langle x, x^*\rangle + \langle y, y^*\rangle - (\varphi \ltimes s)^*(x^*, y^*)\big) &= \sup_{\substack{x^* \in (\varphi^*)^{-1}(\{0\})\\ y^* \in \mathcal{Y}^*}} \big(\langle x, x^*\rangle + \langle y, y^*\rangle - \sigma_{\overline{\mathrm{conv}}\,S}(y^*)\big)\\
&= \sup_{x^* \in (\varphi^*)^{-1}(\{0\})} \big(\langle x, x^*\rangle + \iota_{\overline{\mathrm{conv}}\,S}(y)\big).
\end{aligned} \tag{5.16}$$

Hence, in view of (5.14) and (5.15),

$$(\varphi \ltimes s)^{**}(x,y) = \max\left\{ \sup_{x^* \in (\varphi^*)^{-1}(]-\infty,0[)} \big(\langle x, x^*\rangle - \varphi^*(x^*) s^{\blacktriangle}(y)\big), \sup_{x^* \in (\varphi^*)^{-1}(\{0\})} \big(\langle x, x^*\rangle + \iota_{\overline{\mathrm{conv}}\,S}(y)\big)\right\}. \tag{5.17}$$

In addition, Lemma 3.3(i) yields $s^{\blacktriangle}(y) \in [0, +\infty]$. If $s^{\blacktriangle}(y) = +\infty$, since $(\varphi^*)^{-1}(]-\infty, 0[) \neq \varnothing$, then it follows from (5.17) that $(\varphi \ltimes s)^{**}(x, y) = +\infty$. Now assume that $s^{\blacktriangle}(y) \in [0, +\infty[$. Then Lemma 3.3(i) yields

$$y \in \overline{\mathrm{conv}}\,S. \tag{5.18}$$

Thus, if $s^\blacktriangle(y) \in \,]0,+\infty[$, we deduce from (5.17), (5.18), and (5.10) that

$$
\begin{aligned}
(\varphi \ltimes s)^{**}(x,y) &= \sup_{x^* \in (\varphi^*)^{-1}(]-\infty,0])} \big(\langle x, x^*\rangle - \varphi^*(x^*)s^\blacktriangle(y)\big) \\
&= \sup_{x^* \in \operatorname{dom} \varphi^*} \big(\langle x, x^*\rangle - \varphi^*(x^*)s^\blacktriangle(y)\big) \\
&= s^\blacktriangle(y)\breve{\varphi}\left(\frac{x}{s^\blacktriangle(y)}\right).
\end{aligned}
\tag{5.19}
$$

Now, if $s^\blacktriangle(y) = 0$, we infer from (5.17) and (5.12) that

$$
(\varphi \ltimes s)^{**}(x,y) = \max\big\{\sigma_{(\varphi^*)^{-1}(]-\infty,0[)}(x), \sigma_{(\varphi^*)^{-1}(\{0\})}(x)\big\} = \sigma_{\operatorname{dom}\varphi^*}(x) = \big(\operatorname{rec}\breve{\varphi}\big)(x). \tag{5.20}
$$

Hence, (5.4) holds.

(iii): Theorem 4.5(ii) implies that $\varnothing \neq \operatorname{dom}(\varphi \ltimes s)^* \subset (\varphi^*)^{-1}(\{0\}) \times \mathcal{Y}^*$. Hence, we have

$$
\begin{aligned}
(\varphi \ltimes s)^{**}(x,y) &= \sup_{\substack{x^* \in (\varphi^*)^{-1}(\{0\}) \\ y^* \in \mathcal{Y}^*}} \big(\langle x, x^*\rangle + \langle y, y^*\rangle - (\varphi \ltimes s)^*(x^*,y^*)\big) \\
&= \sup_{\substack{x^* \in (\varphi^*)^{-1}(\{0\}) \\ y^* \in \mathcal{Y}^*}} \big(\langle x, x^*\rangle + \langle y, y^*\rangle - \sigma_{\overline{\operatorname{conv}}\, S}(y^*)\big) \\
&= \sup_{x^* \in \operatorname{dom}\varphi^*} \big(\langle x, x^*\rangle + \iota_{\overline{\operatorname{conv}}\, S}(y)\big) \\
&= \sigma_{\operatorname{dom}\varphi^*}(x) + \iota_{\overline{\operatorname{conv}}\, S}(y),
\end{aligned}
\tag{5.21}
$$

and we obtain (5.5) from (5.12).

(iv): Theorem 4.5(iii) implies that $\operatorname{dom}(\varphi \ltimes s)^* \subset (\varphi^*)^{-1}([0,+\infty[) \times \mathcal{Y}^*$, which yields

$$
\begin{aligned}
(\varphi \ltimes s)^{**}(x,y) = \max\Bigg\{ &\sup_{\substack{x^* \in (\varphi^*)^{-1}(\{0\}) \\ y^* \in \mathcal{Y}^*}} \big(\langle x, x^*\rangle + \langle y, y^*\rangle - (\varphi \ltimes s)^*(x^*,y^*)\big), \\
&\sup_{\substack{x^* \in (\varphi^*)^{-1}(]0,+\infty[) \\ y^* \in \mathcal{Y}^*}} \big(\langle x, x^*\rangle + \langle y, y^*\rangle - (\varphi \ltimes s)^*(x^*,y^*)\big) \Bigg\}.
\end{aligned}
\tag{5.22}
$$

Moreover, by Theorem 4.5(iii), as in (5.16),

$$
\sup_{\substack{x^* \in (\varphi^*)^{-1}(\{0\}) \\ y^* \in \mathcal{Y}^*}} \big(\langle x, x^*\rangle + \langle y, y^*\rangle - (\varphi \ltimes s)^*(x^*,y^*)\big) = \sup_{x^* \in (\varphi^*)^{-1}(\{0\})} \big(\langle x, x^*\rangle + \iota_{\overline{\operatorname{conv}}\, S}(y)\big) \tag{5.23}
$$

and, using (3.2) and Lemma 2.3(ii),

$$
\begin{aligned}
&\sup_{\substack{x^* \in (\varphi^*)^{-1}(]0,+\infty[) \\ y^* \in \mathcal{Y}^*}} \big(\langle x, x^*\rangle + \langle y, y^*\rangle - (\varphi \ltimes s)^*(x^*,y^*)\big) \\
&\quad = \sup_{\substack{x^* \in (\varphi^*)^{-1}(]0,+\infty[) \\ y^* \in \mathcal{Y}^*}} \left(\langle x, x^*\rangle + \langle y, y^*\rangle - \varphi^*(x^*)(-s)^{\blacktriangledown *}\left(\frac{y^*}{\varphi^*(x^*)}\right)\right) \\
&\quad = \sup_{x^* \in (\varphi^*)^{-1}(]0,+\infty[)} \left(\langle x, x^*\rangle + \varphi^*(x^*) \sup_{y^* \in \mathcal{Y}^*} \left(\left\langle y, \frac{y^*}{\varphi^*(x^*)}\right\rangle - (-s)^{\blacktriangledown *}\left(\frac{y^*}{\varphi^*(x^*)}\right)\right)\right) \\
&\quad = \sup_{x^* \in (\varphi^*)^{-1}(]0,+\infty[)} \big(\langle x, x^*\rangle + \varphi^*(x^*)(-s)^{\blacktriangledown}(y)\big).
\end{aligned}
\tag{5.24}
$$

Combining (5.22), (5.23), and (5.24), we get

$$(\varphi \ltimes s)^{**}(x,y) = \\ \max\left\{ \sup_{x^*\in(\varphi^*)^{-1}(\{0\})} \big(\langle x,x^*\rangle + \iota_{\overline{\operatorname{conv}}\,S}(y)\big), \sup_{x^*\in(\varphi^*)^{-1}(]0,+\infty[)} \big(\langle x,x^*\rangle + \varphi^*(x^*)(-s)^{\blacktriangledown}(y)\big)\right\}. \tag{5.25}$$

(iv)a): Lemma 2.3(iii) asserts that $(-s)^{\blacktriangledown} \equiv -\infty$. Therefore, since $(\varphi^*)^{-1}(\{0\}) \neq \varnothing$, we deduce from (5.25) that

$$(\varphi \ltimes s)^{**}(x,y) = \sup_{x^*\in(\varphi^*)^{-1}(\{0\})} \big(\langle x,x^*\rangle + \iota_{\overline{\operatorname{conv}}\,S}(y)\big) = \sigma_{(\varphi^*)^{-1}(\{0\})}(x) + \iota_{\overline{\operatorname{conv}}\,S}(y), \tag{5.26}$$

as announced in (5.6).

(iv)b): Lemma 3.2(i) yields $(-s)^{\blacktriangledown}(y) \in\, ]-\infty,0]\cup\{+\infty\}$. If $(-s)^{\blacktriangledown}(y) = +\infty$, since $(\varphi^*)^{-1}(]0,+\infty[) \neq \varnothing$, it follows from (5.25) that $(\varphi \ltimes s)^{**}(x,y) = +\infty$. Now assume that $-\infty < (-s)^{\blacktriangledown}(y) \leqslant 0$. Then Lemma 3.2(i) yields

$$y \in \overline{\operatorname{conv}}\,S. \tag{5.27}$$

Thus, if $(-s)^{\blacktriangledown}(y) = 0$, we infer from (5.25) and (5.12) that

$$(\varphi \ltimes s)^{**}(x,y) = \max\left\{\sigma_{(\varphi^*)^{-1}(\{0\})}(x), \sigma_{(\varphi^*)^{-1}(]0,+\infty[)}(x)\right\} = \sigma_{\operatorname{dom}\varphi^*}(x) = \big(\operatorname{rec}\breve{\varphi}\big)(x). \tag{5.28}$$

Next, assume that $(-s)^{\blacktriangledown}(y) < 0$. Then we deduce from (5.27), (5.25), and (5.10) that

$$\begin{aligned}(\varphi \ltimes s)^{**}(x,y) &= \sup_{x^*\in(\varphi^*)^{-1}([0,+\infty[)} \big(\langle x,x^*\rangle + \varphi^*(x^*)(-s)^{\blacktriangledown}(y)\big)\\ &= \sup_{x^*\in\operatorname{dom}\varphi^*} \big(\langle x,x^*\rangle + \varphi^*(x^*)(-s)^{\blacktriangledown}(y)\big)\\ &= -(-s)^{\blacktriangledown}(y)\breve{\varphi}\left(\frac{x}{-(-s)^{\blacktriangledown}(y)}\right).\end{aligned} \tag{5.29}$$

This verifies that (5.7) holds.

(v): We deduce from Lemma 3.2(i) and Lemma 3.3(ii) that $\varphi^{*\blacktriangledown} \in \Gamma_0(\mathcal{X}^*)$ and $\varphi^{*\blacktriangle} \in \Gamma_0(\mathcal{X}^*)$. In turn, Lemma 2.3(v) yields

$$(\varphi^{*\blacktriangledown *})\breve{} = \varphi^{*\blacktriangledown *} \in \Gamma_0(\mathcal{X}) \quad \text{and} \quad (\varphi^{*\blacktriangle *})\breve{} = \varphi^{*\blacktriangle *} \in \Gamma_0(\mathcal{X}). \tag{5.30}$$

Note also that (5.10), Lemma 3.2(vi), and Lemma 3.3(v) imply that

$$(\varphi^{*\blacktriangledown})^{-1}(]-\infty,0[) = (\varphi^*)^{-1}(]-\infty,0[) \neq \varnothing \quad \text{and} \quad (\varphi^{*\blacktriangle})^{-1}(\{0\}) \neq \varnothing. \tag{5.31}$$

We derive from Corollary 4.6, Lemma 2.3(ii), and (5.31) that $\operatorname{cam}(\varphi^{*\blacktriangledown *} \ltimes s) \neq \varnothing$ and $\operatorname{cam}(\varphi^{*\blacktriangle *} \ltimes s) \neq \varnothing$. Therefore we deduce from Lemma 2.4(ii) that

$$\varphi^{*\blacktriangledown *} \blacktriangleright\!\!\!\!\ltimes\, s = \big(\varphi^{*\blacktriangledown *} \ltimes s\big)^{**} \quad \text{and} \quad \varphi^{*\blacktriangle *} \blacktriangleright\!\!\!\!\ltimes\, s = \big(\varphi^{*\blacktriangle *} \ltimes s\big)^{**}. \tag{5.32}$$

(v)a): It follows from Theorem 4.5(iv)b) that

$$\big(\varphi \ltimes s\big)^{**}(x,y) = \max\left\{\big(\varphi^{*\blacktriangledown *} \ltimes s\big)^{**}(x,y), \big(\varphi^{*\blacktriangle *} \ltimes s\big)^{**}(x,y)\right\}. \tag{5.33}$$

Thus, the claim follows from (5.11) and (5.32).

(v)b): According to (5.10) and Lemma 3.2(iv), $\operatorname{dom}\varphi^{*\blacktriangledown} = (\varphi^*)^{-1}(]-\infty,0])$. Hence, using Theorem 4.5(i), Lemma 2.3(ii), and (5.31), we arrive at $\operatorname{dom}(\varphi^{*\blacktriangledown *} \ltimes s)^* \subset \operatorname{dom}\varphi^{*\blacktriangledown} \times \mathcal{Y}^* = (\varphi^*)^{-1}(]-\infty,0]) \times \mathcal{Y}^*$. Therefore, it follows from (5.32) that

$$\big(\varphi^{*\blacktriangledown *} \ltimes s\big)(x,y) = \max\left\{ \sup_{\substack{x^*\in(\varphi^*)^{-1}(]-\infty,0[)\\ y^*\in\mathcal{Y}^*}} \big(\langle x,x^*\rangle + \langle y,y^*\rangle - (\varphi^{*\blacktriangledown *} \ltimes s)^*(x^*,y^*)\big), \right.$$
$$\left. \sup_{\substack{x^*\in(\varphi^*)^{-1}(\{0\})\\ y^*\in\mathcal{Y}^*}} \big(\langle x,x^*\rangle + \langle y,y^*\rangle - (\varphi^{*\blacktriangledown *} \ltimes s)^*(x^*,y^*)\big) \right\}. \tag{5.34}$$

On the one hand, Theorem 4.5(i) applied to $\varphi^{*\blacktriangledown *}$ and $s$, Lemma 3.2(vi) applied to $\varphi^*$, Lemma 2.3(ii), and Lemma 3.3(i) applied to $s$ yield

$$\begin{aligned}
&\sup_{\substack{x^*\in(\varphi^*)^{-1}(]-\infty,0[)\\ y^*\in\mathcal{Y}^*}} \big(\langle x,x^*\rangle + \langle y,y^*\rangle - (\varphi^{*\blacktriangledown *} \ltimes s)^*(x^*,y^*)\big)\\
&\qquad= \sup_{\substack{x^*\in(\varphi^*)^{-1}(]-\infty,0[)\\ y^*\in\mathcal{Y}^*}} \left(\langle x,x^*\rangle + \langle y,y^*\rangle + \varphi^{*\blacktriangledown}(x^*) s^{\blacktriangle *}\left(\frac{y^*}{-\varphi^{*\blacktriangledown}(x^*)}\right)\right)\\
&\qquad= \sup_{x^*\in(\varphi^*)^{-1}(]-\infty,0[)} \left(\langle x,x^*\rangle - \varphi^*(x^*) \sup_{y^*\in\mathcal{Y}^*}\left(\left\langle y, \frac{y^*}{-\varphi^*(x^*)}\right\rangle - s^{\blacktriangle *}\left(\frac{y^*}{\varphi^*(x^*)}\right)\right)\right)\\
&\qquad= \sup_{x^*\in(\varphi^*)^{-1}(]-\infty,0[)} \big(\langle x,x^*\rangle - \varphi^*(x^*) s^{\blacktriangle}(y)\big)\\
&\qquad\geqslant \sup_{x^*\in(\varphi^*)^{-1}(]-\infty,0[)} \big(\langle x,x^*\rangle + \iota_{\overline{\operatorname{conv}}\, S}(y)\big).
\end{aligned} \tag{5.35}$$

On the other hand, with the help of Lemma 3.2(v), Theorem 4.5(i) applied to $\varphi^{*\blacktriangledown *}$ implies that

$$\begin{aligned}
&\sup_{\substack{x^*\in(\varphi^*)^{-1}(\{0\})\\ y^*\in\mathcal{Y}^*}} \big(\langle x,x^*\rangle + \langle y,y^*\rangle - (\varphi^{*\blacktriangledown *} \ltimes s)^*(x^*,y^*)\big)\\
&\qquad= \sup_{\substack{x^*\in(\varphi^*)^{-1}(\{0\})\\ y^*\in\mathcal{Y}^*}} \big(\langle x,x^*\rangle + \langle y,y^*\rangle - \sigma_{\overline{\operatorname{conv}}\, S}(y^*)\big)\\
&\qquad= \sup_{x^*\in(\varphi^*)^{-1}(\{0\})} \big(\langle x,x^*\rangle + \iota_{\overline{\operatorname{conv}}\, S}(y)\big).
\end{aligned} \tag{5.36}$$

Combining (5.34), (5.35), (5.36), Lemma 3.3(iv), (5.31), and (iv)a) we obtain

$$\begin{aligned}
\big(\varphi^{*\blacktriangledown *} \ltimes s\big)(x,y) &\geqslant \sigma_{(\varphi^*)^{-1}(]-\infty,0])}(x) + \iota_{\overline{\operatorname{conv}}\, S}(y)\\
&\geqslant \sigma_{(\varphi^{*\blacktriangle})^{-1}(\{0\})}(x) + \iota_{\overline{\operatorname{conv}}\, S}(y)\\
&= \big(\varphi^{*\blacktriangle *} \ltimes s\big)(x,y).
\end{aligned} \tag{5.37}$$

Altogether, the result follows from (5.31), (v)a), and (ii) applied to $\varphi^{*\blacktriangledown *}$ and $s$.

(v)c): Using Lemma 3.3(i) and Lemma 3.5, we partition $\mathcal{Y}$ as

$$\mathcal{Y} = (s^{\blacktriangle})^{-1}(]0,+\infty[) \bigcup \Big((s^{\blacktriangle})^{-1}(\{0\}) \cap ((-s)^{\blacktriangledown})^{-1}(]-\infty,0[)\Big)$$
$$\bigcup \Big((s^{\blacktriangle})^{-1}(\{0\}) \cap ((-s)^{\blacktriangledown})^{-1}(\{0\})\Big) \bigcup (s^{\blacktriangle})^{-1}(\{+\infty\}), \tag{5.38}$$

which corresponds to the cases in (5.9). Therefore, it follows from (5.30), Lemma 2.5(ii), Lemma 3.2(iv), Lemma 3.3(iii), Lemma 2.3(ii), and (5.12) that

$$\begin{aligned}\max\{(\operatorname{rec}\varphi^{*\blacktriangledown *})(x), (\operatorname{rec}\varphi^{*\blacktriangle *})(x)\} &= \max\{\sigma_{\operatorname{dom}\varphi^{*\blacktriangledown}}(x), \sigma_{\operatorname{dom}\varphi^{*\blacktriangle}}(x)\}\\ &= \sigma_{\operatorname{dom}\varphi^*}(x)\\ &= (\operatorname{rec}\breve{\varphi})(x).\end{aligned} \tag{5.39}$$

Altogether, (5.9) follows from (v)a), by applying (ii) to $\varphi^{*\blacktriangledown *} \ltimes s$ and (iv)b) to $\varphi^{*\blacktriangle *} \ltimes s$, and invoking (5.38), (5.30), and (5.39). ☐

Next, we focus on the case when $\varphi \in \Gamma_0(\mathcal{X})$ and $\pm s \in \Gamma_0(\mathcal{Y})$. We express the results in terms of recession functions via Lemma 2.6, which does not involve the sign of $\varphi^*$.

**Corollary 5.3** *Let $\varphi \in \Gamma_0(\mathcal{X})$ and let $s\colon \mathcal{Y} \to [-\infty, +\infty]$ be such that $S = s^{-1}(]0, +\infty[) \neq \varnothing$. Let $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Then the following hold:*

(i) *Suppose that $\varphi \geqslant \operatorname{rec}\varphi \neq \varphi$ and $s \in \Gamma_0(\mathcal{Y})$. Then*

$$(\varphi \ltimes s)(x, y) = \begin{cases} s(y)\varphi\left(\dfrac{x}{s(y)}\right), & \text{if } 0 < s(y) < +\infty;\\ (\operatorname{rec}\varphi)(x), & \text{if } y \in \overline{\operatorname{conv}}\, S \text{ and } s(y) \leqslant 0;\\ +\infty, & \text{otherwise.}\end{cases} \tag{5.40}$$

(ii) *Suppose that $\varphi = \operatorname{rec}\varphi$. Then $(\varphi \ltimes s)(x, y) = \varphi(x) + \iota_{\overline{\operatorname{conv}}\, S}(y)$.*

(iii) *Suppose that $\varphi \neq \operatorname{rec}\varphi$, $\varphi(0) \leqslant 0$, and $-s \in \Gamma_0(\mathcal{Y})$. Then*

$$(\varphi \ltimes s)(x, y) = \begin{cases} s(y)\varphi\left(\dfrac{x}{s(y)}\right), & \text{if } 0 < s(y) < +\infty;\\ (\operatorname{rec}\varphi)(x), & \text{if } s(y) = 0;\\ +\infty, & \text{otherwise.}\end{cases} \tag{5.41}$$

*Furthermore, in each case, $\varphi \ltimes s \in \Gamma_0(\mathcal{X} \oplus \mathcal{Y})$.*

*Proof*. We first observe that Lemma 2.3(v) yields $\breve{\varphi} = \varphi$. Furthermore, by Lemma 2.5(iii), we have

$$\varphi = \operatorname{rec}\varphi \quad \Leftrightarrow \quad \varphi^*(\mathcal{X}^*) \subset \{0, +\infty\}. \tag{5.42}$$

(i): Lemma 2.6 and (5.42) yield

$$\operatorname{dom}\varphi^* = (\varphi^*)^{-1}(]-\infty, 0]) \quad \text{and} \quad (\varphi^*)^{-1}(]-\infty, 0[) \neq \varnothing. \tag{5.43}$$

Hence, (5.40) follows from Theorem 5.2(ii) and Lemma 3.3 applied to $s$.

(ii): This assertion follows from (5.42) and Theorem 5.2(iii).

(iii): We have $(\forall x^* \in \mathcal{X}^*)\ \varphi^*(x^*) \geqslant \langle 0, x^*\rangle - \varphi(0) \geqslant 0$. Thus, (5.42) yields

$$\operatorname{dom}\varphi^* = (\varphi^*)^{-1}([0, +\infty[) \quad \text{and} \quad (\varphi^*)^{-1}(]0, +\infty[) \neq \varnothing. \tag{5.44}$$

Thus, since $\operatorname{cam}(-s)^{\vee} \neq \varnothing$ by Lemma 2.3(v), (5.41) follows from Theorem 5.2(iv)b) and Lemma 3.2 applied to $-s$.

Finally, since (5.3) holds in each case, we deduce from Theorem 5.2(i) that $\varphi \ltimes s \in \Gamma_0(\mathcal{X} \oplus \mathcal{Y})$. ☐

**Remark 5.4** As mentioned in the Introduction, in the context of Corollary 5.3, alternative notions of perspective functions with nonlinear scaling were proposed in [37, 49] under additional restrictions on the scaling function. Specically, these papers deal with operations $\Delta_1$ and $\Delta_2$ between functions $\varphi \in \Gamma_0(\mathcal{X})$ and $\psi \in \Gamma_0(\mathcal{Y})$ in the following scenarios.

(i) Suppose that $\varphi \geqslant \operatorname{rec}\varphi \neq \varphi$ and $\psi(\operatorname{dom}\psi) \subset [0,+\infty[$. In view of (1.2),

$$\varphi \,\Delta_2\, \psi\colon \mathcal{X}\oplus\mathcal{Y} \to \left]-\infty,+\infty\right] : (x,y) \mapsto \begin{cases} \widetilde{\varphi}(x,\psi(y)) & \text{if } y \in \operatorname{dom}\psi;\\ +\infty, & \text{if } y \notin \operatorname{dom}\psi. \end{cases} \tag{5.45}$$

Now suppose that $\psi^{-1}(]0,+\infty[) \neq \varnothing$. It follows from Corollary 5.3(i) that

$$\varphi \,\Delta_2\, \psi \leqslant \varphi \ltimes \psi\colon (x,y) \mapsto (\varphi \,\Delta_2\, \psi)(x,y) + \iota_{\overline{\operatorname{conv}}\,\psi^{-1}(]0,+\infty[)}(y). \tag{5.46}$$

Let us note that, since equality fails above, the $\varphi \,\Delta_2\, \psi$ is not the largest minorant of $\varphi \ltimes \psi$ in $\Gamma_0(\mathcal{X}\oplus\mathcal{Y})$. For instance, suppose that

$$\mathcal{Y} = \mathbb{R} \quad\text{and}\quad \psi\colon y \mapsto \max\{0,y\}. \tag{5.47}$$

Then $\overline{\operatorname{conv}}\,\psi^{-1}(]0,+\infty[) = [0,+\infty[$ and therefore, if $y \in \left]-\infty,0\right[$ and $0 \in \operatorname{dom}\varphi$, we have $\psi(y) = 0$ and $0 = (\varphi \,\Delta_2\, \psi)(0,y) < (\varphi \ltimes \psi)(0,y) = +\infty$.

(ii) Suppose that $\varphi \neq \operatorname{rec}\varphi$, $\varphi(0) \leqslant 0$, and $\psi(\operatorname{dom}\psi) \subset \left]-\infty,0\right]$. Then, using (1.2),

$$\varphi \,\Delta_1\, \psi\colon \mathcal{X}\oplus\mathcal{Y} \to \left]-\infty,+\infty\right] : (x,y) \mapsto \begin{cases} \widetilde{\varphi}(x,-\psi(y)), & \text{if } y \in \operatorname{dom}\psi;\\ +\infty, & \text{if } y \notin \operatorname{dom}\psi. \end{cases} \tag{5.48}$$

Now suppose that $\psi^{-1}(]-\infty,0[) \neq \varnothing$. Then it follows from Corollary 5.3(iii) that $\varphi \,\Delta_1\, \psi = \varphi \ltimes (-\psi)$. In turn, Definition 1.1 asserts that, in this particular scenario, $\varphi \,\Delta_1\, \psi$ is the largest minorant of $\varphi \ltimes (-\psi)$ in $\Gamma_0(\mathcal{X}\oplus\mathcal{Y})$.

The construction proposed in Definition 1.1 covers a much broader range of functions $(\varphi, s)$ that those employed above. Concrete instances will be presented in Section 6.

**Remark 5.5** Let $\varphi \in \Gamma_0(\mathcal{X})$ and let $s\colon \mathcal{Y} \to [-\infty,+\infty]$ be such that $s^{-1}(]0,+\infty[) \neq \varnothing$. The above remark reveals some particular instances in which $\varphi \ltimes s$ can be expressed in terms of the classical perspective of (1.2) applied to certain transformations of $\varphi$ and $s$. Let us clarify these identities and, in particular, address the natural question that arises as to the validity of the identity

$$(\varphi \ltimes s)(x,y) = \begin{cases} \widetilde{\varphi}(x,s(y)) & \text{if } s(y) \in \mathbb{R};\\ +\infty, & \text{otherwise} \end{cases} \tag{5.49}$$

beyond the classical case already discussed in Section 1 in which $\mathcal{Y} = \mathbb{R}$ and $s\colon y \mapsto y$. It turns out that (5.49) is true only in very specific instances, some of which are provided below. Let $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Then it follows from Theorem 5.2 that the following hold:

(i) Suppose that $\varphi^*(\mathcal{X}^*) \subset \left]-\infty,0\right] \cup \{+\infty\}$. Then

$$(\varphi \ltimes s)(x,y) = \begin{cases} \widetilde{\varphi}(x,s^{\blacktriangle}(y)), & \text{if } y \in \operatorname{dom} s^{\blacktriangle};\\ +\infty, & \text{if } y \notin \operatorname{dom} s^{\blacktriangle}. \end{cases} \tag{5.50}$$

If we assume additionally that $s = s^{\blacktriangle}$, then it follows from Lemma 2.6 that (5.49) holds. This corresponds to the setting of Remark 5.4(i).

(ii) Suppose that $\varphi^*(\mathcal{X}^*) \subset [0,+\infty]$, $(\varphi^*)^{-1}(]0,+\infty[) \neq \varnothing$, and $\operatorname{cam}(-s)^{\vee} \neq \varnothing$. Then

$$(\varphi \ltimes s)(x,y) = \begin{cases} \widetilde{\varphi}\big(x, -(-s)^{\blacktriangledown}(y)\big), & \text{if } y \in \operatorname{dom}(-s)^{\blacktriangledown};\\ +\infty, & \text{if } y \notin \operatorname{dom}(-s)^{\blacktriangledown}. \end{cases} \tag{5.51}$$

If we assume additionally that $s = -(-s)^{\blacktriangledown}$, then (5.49) holds. This corresponds to the setting of Remark 5.4(ii).

(iii) Suppose that $w^* \in \mathcal{Y}^* \smallsetminus \{0\}$, $\overline{y} \in \mathcal{Y}$, and $s = w^* - \langle \overline{y}, w^* \rangle$. Then Example 5.1 implies that (5.49) holds.

(iv) Suppose that $(\varphi^*)^{-1}(]-\infty,0[) \neq \varnothing$ and that $(\varphi^*)^{-1}(]0,+\infty[) \neq \varnothing$. Then

$$(\varphi \ltimes s)(x,y) = \begin{cases} \max\big\{\widetilde{\varphi^{*\blacktriangledown *}}\big(x, s^{\blacktriangle}(y)\big), \widetilde{\varphi^{*\blacktriangle *}}\big(x, -(-s)^{\blacktriangledown}(y)\big)\big\}, & \text{if } y \in \operatorname{dom} s^{\blacktriangle};\\ +\infty, & \text{if } y \notin \operatorname{dom} s^{\blacktriangle}. \end{cases} \tag{5.52}$$

If $s = s^{\blacktriangle} = -(-s)^{\blacktriangledown}$, it follows from Remark 3.6 that (5.49) holds.

## 6 Examples and applications

We illustrate various cases that arise in Theorem 5.2.

**Example 6.1** Suppose that $\mathcal{X}$ is a nonzero real reflexive Banach space, let $\alpha \in ]0,+\infty[$, let $p \in ]1,+\infty[$, set $p^* = p/(p-1)$, and set

$$\varphi_1 \colon \mathcal{X} \to \mathbb{R} \colon x \mapsto \begin{cases} \alpha\|x\|, & \text{if } \|x\| > \alpha^{\frac{1}{p-1}};\\ \dfrac{\|x\|^p}{p} + \dfrac{\alpha^{p^*}}{p^*}, & \text{if } \|x\| \leqslant \alpha^{\frac{1}{p-1}}. \end{cases} \tag{6.1}$$

Suppose that $\mathcal{Y} = \mathbb{R}$, let $\beta \in [0,1[$, and set

$$s \colon \mathbb{R} \to ]-\infty,+\infty] \colon y \mapsto \begin{cases} y - \dfrac{\beta^2+1}{2}, & \text{if } y > 1;\\ \dfrac{|y|^2-\beta^2}{2}, & \text{if } -1 \leqslant y \leqslant 1;\\ +\infty, & \text{if } y < -1. \end{cases} \tag{6.2}$$

It follows from Example 3.7 that $\varphi_1^*(\mathcal{X}^*) \subset ]-\infty,0] \cup \{+\infty\}$ and $(\varphi_1^*)^{-1}(]-\infty,0[) \neq \varnothing$. Furthermore, Lemma 3.3(ii) yields

$$s^{\blacktriangle} \colon y \mapsto \begin{cases} y - \dfrac{\beta^2+1}{2}, & \text{if } y > 1;\\ \dfrac{|y|^2-\beta^2}{2}, & \text{if } -1 < y \leqslant -\beta \text{ or } \beta < y \leqslant 1;\\ 0, & \text{if } -\beta \leqslant y \leqslant \beta;\\ +\infty, & \text{if } y < -1. \end{cases} \tag{6.3}$$

We thus derive $\varphi_1 \ltimes s$ from Theorem 5.2(ii); see Figure 2.

**Example 6.2** Suppose that $\mathcal{X}$ is a nonzero real reflexive Banach space, let $\alpha \in ]0,+\infty[$, let $p \in ]1,+\infty[$, set $p^* = p/(p-1)$, and set

$$\varphi_2 \colon \mathcal{X} \to \mathbb{R} \colon x \mapsto \begin{cases} \dfrac{\|x\|^p}{p} + \dfrac{\alpha^{p^*}}{p^*}, & \text{if } \|x\| > \alpha^{\frac{1}{p-1}}; \\ \alpha\|x\|, & \text{if } \|x\| \leqslant \alpha^{\frac{1}{p-1}}. \end{cases} \tag{6.4}$$

Let $\mathcal{Y}$ and $s$ be as in Example 6.1. In view of Example 3.7, we have $\varphi_2^*(\mathcal{X}^*) \subset [0,+\infty[$ and $(\varphi_2^*)^{-1}(]0,+\infty[) \neq \varnothing$. Additionally, $\operatorname{cam}(-s)^{\vee} \neq \varnothing$ and (3.2) yields

$$-(-s)^{\blacktriangledown} \colon y \mapsto \begin{cases} y + \dfrac{3-\beta^2}{2}, & \text{if } y \geqslant -1; \\ -\infty, & \text{if } y < -1. \end{cases} \tag{6.5}$$

We thus derive $\varphi_2 \ltimes s$ from Theorem 5.2(iv)b); see Figure 2.

**Example 6.3** Suppose that $\mathcal{X}$ is a nonzero real reflexive Banach space, let $\alpha \in ]0,+\infty[$, let $p \in ]1,+\infty[$, set $p^* = p/(p-1)$, and set

$$\varphi_3 \colon \mathcal{X} \to \mathbb{R} \colon x \mapsto \|x\|^p/p + \alpha^{p^*}/p^*. \tag{6.6}$$

Let $\mathcal{Y}$ and $s$ be as in Example 6.1. Then, as seen in Example 3.7, $(\varphi_3^*)^{-1}(]-\infty,0[) \neq \varnothing$, $(\varphi_3^*)^{-1}(]0,+\infty[) \neq \varnothing$, and it follows from (3.21), (6.1), and (6.4) that $\varphi_3{}^{*\blacktriangledown *} = \varphi_1$ and $\varphi_3{}^{*\blacktriangle *} = \varphi_2$. Hence, we derive $\varphi_3 \ltimes s$ from Theorem 5.2(v)a); see Figure 2.

**Example 6.4** Let $\mathcal{X}$ and $\varphi_3$ be as in Example 6.3, let $\varphi_1$ be as in Example 6.1, and let $\varphi_2$ be as in Example 6.2. Recall that $(\varphi_3^*)^{-1}(]-\infty,0[) \neq \varnothing$, $(\varphi_3^*)^{-1}(]0,+\infty[) \neq \varnothing$, $\varphi_3{}^{*\blacktriangledown *} = \varphi_1$, and $\varphi_3{}^{*\blacktriangle *} = \varphi_2$. Suppose that $\mathcal{Y} = \mathbb{R}$, let $1 \neq q \in ]0,+\infty[$, and set

$$s \colon \mathbb{R} \to ]-\infty,+\infty] \colon y \mapsto \begin{cases} y^q, & \text{if } y \geqslant 0; \\ +\infty, & \text{if } y < 0. \end{cases} \tag{6.7}$$

Since $\operatorname{cam}(-s)^{\vee} = \varnothing$ for $q > 1$, it follows from (3.2), (3.4), Lemma 3.2(iii), and Lemma 3.3(ii) that

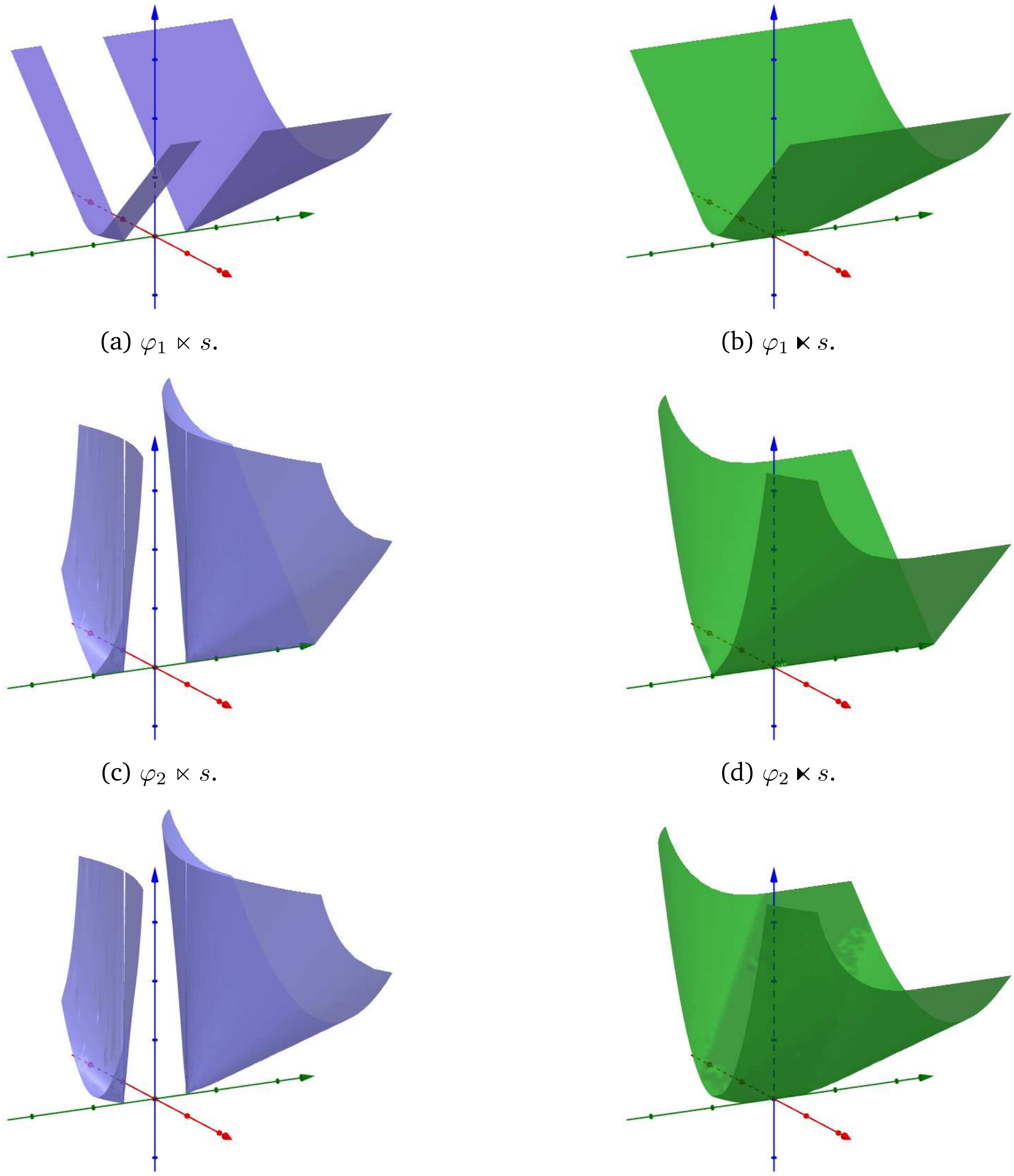
$$s^{\blacktriangle} \colon y \mapsto \begin{cases} 0, & \text{if } y \geqslant 0 \text{ and } q < 1; \\ y^q, & \text{if } y \geqslant 0 \text{ and } q > 1; \\ +\infty, & \text{if } y < 0 \end{cases} \quad \text{and} \quad -(-s)^{\blacktriangledown} \colon y \mapsto \begin{cases} y^q, & \text{if } y \geqslant 0 \text{ and } q < 1; \\ -\infty, & \text{if } y < 0 \text{ and } q < 1; \\ +\infty, & \text{if } q > 1. \end{cases} \tag{6.8}$$

Hence, we derive $\varphi_3 \ltimes s$ from Theorem 5.2(v)c) for $q < 1$, and from Theorem 5.2(v)b) for $q > 1$ (see Figure 3).

We now turn our attention to specific applications by considering integral functions of the form

$$(\mathsf{x},\mathsf{y}) \mapsto \int_\Omega \big(\varphi_\omega \ltimes s_\omega\big)\big(\mathsf{x}(\omega),\mathsf{y}(\omega)\big)\mu(d\omega), \tag{6.9}$$

where the integrand is a perspective function with nonlinear scaling in the sense of Definition 1.1.

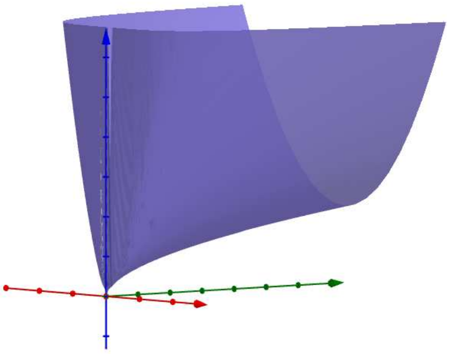


(a) $\varphi_1 \ltimes s$. (b) $\varphi_1 \boldsymbol{\ltimes} s$.

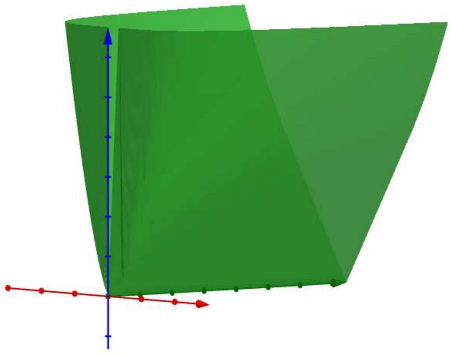
(c) $\varphi_2 \ltimes s$. (d) $\varphi_2 \boldsymbol{\ltimes} s$.

(e) $\varphi_3 \ltimes s$. (f) $\varphi_3 \boldsymbol{\ltimes} s$.

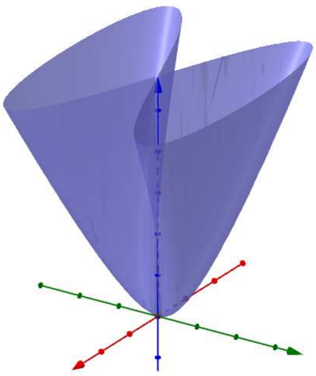
Figure 2: Plots of $\varphi_i \ltimes s$ (left) and $\varphi_i \boldsymbol{\ltimes} s$ (right) for $i \in \{1, 2, 3\}$ in Examples 6.1-6.3 with $p = 2$, $\alpha = 1$, and $\beta = 1/2$. The $x$-axis is in red and the $y$-axis in green.



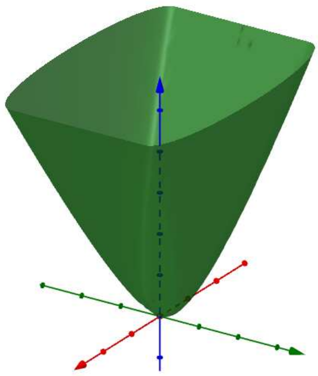
(a) $\varphi_3 \ltimes s$ with $p = 1/q = 2$.



(b) $\varphi_3 \pmb{\ltimes} s$ with $p = 1/q = 2$.



(c) $\varphi_3 \ltimes s$ with $p = q = 2$.



(d) $\varphi_3 \pmb{\ltimes} s$ with $p = q = 2$.

Figure 3: Plots of $\varphi_3 \ltimes s$ (left) and $\varphi_3 \pmb{\ltimes} s$ (right) in Example 6.4. The $x$-axis is in red and the $y$-axis in green.

**Example 6.5** Let $p \in ]1,+\infty[$ and $q \in ]0,1]$. Suppose that $\mathcal{X} = \mathbb{R}^N$ is normed with $\|\cdot\|$, $\mathcal{Y} = \mathbb{R}$, $\varphi\colon \mathcal{X} \to ]-\infty,+\infty] : x \mapsto \|x\|^p/p$, and

$$s\colon \mathcal{Y} \to [-\infty,+\infty] : y \mapsto \begin{cases} y^q, & \text{if } y \geqslant 0; \\ -\infty, & \text{if } y < 0. \end{cases} \tag{6.10}$$

Let $T \in ]0,+\infty[$, set $\mathcal{M} = (L^1([0,T] \times \mathbb{R}^d))^N$, set $\mathcal{R} = L^1([0,T] \times \mathbb{R}^d)$, and consider the integral function

$$\Phi\colon \mathcal{M} \oplus \mathcal{R} \to ]-\infty,+\infty] : (m,\varrho) \mapsto \int_0^T \int_{\mathbb{R}^d} (\varphi \ltimes s)\big(m(t,\xi),\varrho(t,\xi)\big)\,dt d\xi. \tag{6.11}$$

In optimal mass transportation theory, $m$ and $\varrho$ represent the momentum and the density of particles, respectively, and $m/\varrho$ represents their velocity [6, 47]. In the case when $p = 2$ and $q = 1$, $\varphi \ltimes s$ is a classical perspective (see (1.2)) and the function (6.11) is related to the dynamical formulation of the 2-Wasserstein distance [6, 47]. Based on this formulation, convex optimization methods are proposed in [8, 17] to approximate the iterates of the so-called JKO scheme [31] for gradient flows in the space of probability measures. When $q \neq 1$, (6.11) appears in optimal transportation based on $p$-Wasserstein distances with nonlinear mobilities [15, 19, 25] and in the optimal control of McKean–Vlasov systems with congestion [1]. Space-dependent potentials $(\varphi_\xi)_{\xi\in\Xi}$, where $\Xi \subset \mathbb{R}^d$, are also found [7, 13, 14], where they lead to functions of the form

$$\Psi\colon \mathcal{M} \oplus \mathcal{R} \to ]-\infty,+\infty] : (m,\varrho) \mapsto \int_0^T \int_{\Xi} (\varphi_\xi \ltimes s)\big(m(t,\xi),\varrho(t,\xi)\big)\,dt d\xi. \tag{6.12}$$

Theorem 5.2 provides conditions under which $(\varphi_\xi \ltimes s)_{\xi\in\Xi}$ is a family of functions in $\Gamma_0(\mathcal{X} \oplus \mathcal{Y})$. Note that in [7, 13, 14], $q = 1$ and we are therefore dealing with classical perspectives (see Example 5.1). Our nonlinear setting allows us to employ (6.12) with $q < 1$ and more structured space-dependent potentials. For instance, in the context of optimal transport theory, consider

$$(\forall \xi \in \Xi) \quad \varphi_\xi\colon \mathbb{R}^N \to ]-\infty,+\infty] : x \mapsto \|x\|^p/p + \iota_{C(\xi)}(\|x\|) + h(\xi), \tag{6.13}$$

where $C(\xi) \subset [0,+\infty[$ is an interval representing a constraint on the speed of particles located at $\xi$ and $h$ is a spatial penalization term. For every $\xi \in \Xi$ such that $\inf C(\xi) > 0$, we have $(\varphi_\xi^*)^{-1}(]0,+\infty[) \neq \varnothing$ and $(\varphi_\xi^*)^{-1}(]-\infty,0[) \neq \varnothing$, and Theorem 5.2 is needed to compute $\varphi_\xi \ltimes s$. An illustration is provided in Figure 4. Another type of scaling function in (6.11) is proposed in [11], namely the concave function

$$s\colon \mathcal{Y} \to [-\infty,+\infty] : y \mapsto \begin{cases} \dfrac{y(1-y)}{\alpha(1-y)+\beta y}, & \text{if } y \in [0,1]; \\ -\infty, & \text{otherwise}, \end{cases} \tag{6.14}$$

where $(\alpha,\beta) \in ]0,+\infty[^2$.

**Example 6.6** Let $U$ be a finite set, suppose that $\mathcal{X} = \mathbb{R}$ and $\mathcal{Y} = \mathbb{R}^2$, let $s\colon \mathcal{Y} \to [-\infty,+\infty]$, and, for every $(u_1,u_2) \in U^2$, let $\varphi_{u_1,u_2} \in \Gamma_0(\mathcal{X})$. Furthermore, set $\mathcal{M} = L^2([0,1];\mathbb{R}^{U\times U})$, $\mathcal{R} = L^1([0,1];\mathbb{R}^U)$, and

$$\begin{aligned} &\Phi\colon \mathcal{M} \oplus \mathcal{R} \to ]-\infty,+\infty] \\ &\quad (m,\varrho) \mapsto \int_0^1 \sum_{u_1\in U} \sum_{u_2\in U} (\varphi_{u_1,u_2} \ltimes s)\big(m(t,u_1,u_2),\varrho(t,u_1),\varrho(t,u_2)\big)\,dt. \end{aligned} \tag{6.15}$$

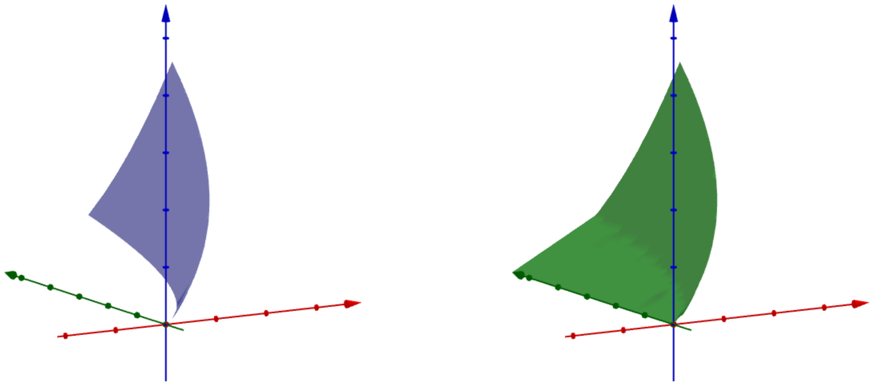


Figure 4: Plot of $\varphi \ltimes s$ (left) and $\varphi \pmb{\ltimes} s$ (right) in Example 6.5 for $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $\varphi = |\cdot|^2/2 + \iota_{[1,2]}$, and $s\colon y \mapsto \sqrt{y}$ if $y \geqslant 0$. The $x$-axis is in red and the $y$-axis in green.

Theorem 5.2 provides conditions under which, for every $(u_1, u_2) \in U^2$, $\varphi_{u_1,u_2} \pmb{\ltimes} s \in \Gamma_0(\mathcal{X} \oplus \mathcal{Y})$. In the particular case when, for every $(u_1, u_2) \in U^2$, $\varphi_{u_1,u_2} = K(u_1, u_2)\pi(u_1)\varphi$, where $\varphi\colon \mathcal{X} \to \left]-\infty, +\infty\right] : x \mapsto |x|^2/2$, $K\colon U \times U \to \mathbb{R}$ is an irreducible and reversible Markov kernel on $U$, and $\pi\colon U \to \mathbb{R}$ is the associated stationary distribution, (6.15) reduces to

$$\Phi\colon (m, \varrho) \mapsto \int_0^1 \sum_{u_1 \in U} \sum_{u_2 \in U} (\varphi \pmb{\ltimes} s)\big(m(t, u_1, u_2), \varrho(t, u_1), \varrho(t, u_2)\big) K(u_1, u_2)\pi(u_1)dt, \tag{6.16}$$

which appears in [35]. Under some additional conditions on $s$, satisfied for instance by the logarithmic mean

$$s\colon (y_1, y_2) \mapsto \begin{cases} 0, & \text{if } (y_1, y_2) \in (\{0\} \times [0, +\infty[) \cup (]0, +\infty[ \times \{0\}); \\ y_1, & \text{if } y_1 = y_2 \in \left]0, +\infty\right[; \\ \dfrac{y_2 - y_1}{\log(y_2) - \log(y_1)}, & \text{if } (y_1, y_2) \in \left]0, +\infty\right[ \times \left]0, +\infty\right[ \text{ and } y_1 \neq y_2; \\ -\infty, & \text{otherwise,} \end{cases} \tag{6.17}$$

and by the geometric mean

$$s\colon (y_1, y_2) \mapsto \begin{cases} \sqrt{y_1 y_2}, & \text{if } (y_1, y_2) \in \left[0, +\infty\right[ \times \left[0, +\infty\right[; \\ -\infty, & \text{otherwise,} \end{cases} \tag{6.18}$$

the function $\Phi$ is used in [35] to construct a distance on the set of probability densities on $U$ with respect to $\pi$.

**Example 6.7** One of the oldest instances involving standard perspective functions is the Fisher information of a differentiable probability density $\mathsf{y}\colon \mathbb{R}^N \to \left]0, +\infty\right[$ [27], that is,

$$\Psi(\mathsf{y}) = \int_{\mathbb{R}^N} \frac{\|\nabla \mathsf{y}(\omega)\|_2^2}{\mathsf{y}(\omega)} d\omega, \tag{6.19}$$

where $\|\cdot\|_2$ is the standard Euclidean norm on $\mathbb{R}^N$. Going back to Definition 1.1, given a nonempty open set $\Omega\subset\mathbb{R}^N$, (6.19) can be formalized as an instance of the function

$$\Psi\colon W^{1,r}(\Omega)\to\left]-\infty,+\infty\right]\colon \mathsf{y}\mapsto\int_\Omega(\varphi\ltimes s)\big(\nabla\mathsf{y}(\omega),\mathsf{y}(\omega)\big)\,d\omega, \tag{6.20}$$

where $r\in\left[1,+\infty\right[$, $\mathcal{X}=\mathbb{R}^N$, $\mathcal{Y}=\mathbb{R}$, $\varphi=\|\cdot\|_2^2$, and $s\colon y\mapsto y$. More generally, assume that $\Gamma_0(\mathcal{X})\ni\varphi\geqslant 0$ and that $\Gamma_0(\mathcal{Y})\ni -s\leqslant 0$ satisfies $s^{-1}(\left]0,+\infty\right[)\neq\varnothing$. Then $\operatorname{cam}(-s)^{\vee}\neq\varnothing$ and Theorem 5.2(i) asserts that $\varphi\ltimes s\in\Gamma_0(\mathcal{X}\oplus\mathcal{Y})$. In turn, the linearity and the continuity of $\mathsf{y}\mapsto(\nabla\mathsf{y},\mathsf{y})$ imply that $\Psi\in\Gamma_0(W^{1,r}(\Omega))$. For instance, let $\|\cdot\|$ be a norm on $\mathbb{R}^N$, let $p\in\left]1,+\infty\right[$, take $\gamma\in\left]1/p,1\right]$, set $q=(\gamma p-1)/(p-1)\in\left]0,1\right]$, and define

$$\varphi=\|\cdot\|^p\quad\text{and}\quad s\colon\mathcal{Y}\to\left[-\infty,+\infty\right[\colon y\mapsto\begin{cases}y^q, & \text{if } y\geqslant 0;\\ -\infty, & \text{if } y<0.\end{cases} \tag{6.21}$$

To make (6.20) explicit in this scenario, let us introduce

$$\ln_\gamma\colon\mathbb{R}\to\left[-\infty,+\infty\right[\colon y\mapsto\begin{cases}\dfrac{y^{1-\gamma}-1}{1-\gamma}, & \text{if } \gamma\neq 1 \text{ and } y\in\left]0,+\infty\right[;\\ \ln y, & \text{if } \gamma=1 \text{ and } y\in\left]0,+\infty\right[;\\ -\infty, & \text{if } y\in\left]-\infty,0\right]\end{cases} \tag{6.22}$$

and note that $(\forall y\in\left]0,+\infty\right[)\ (\ln_\gamma)'(y)=1/y^\gamma$. Let $\mathsf{y}\in W^{1,r}(\Omega)$, set $\Omega_0=\big\{\omega\in\Omega\mid\mathsf{y}(\omega)=0\big\}$, and set $\Omega_+=\big\{\omega\in\Omega\mid\mathsf{y}(\omega)>0\big\}$. Then, by Corollary 5.3(iii) and [3, Proposition 5.8.2], if $\mathsf{y}\geqslant 0$ a.e.,

$$\begin{aligned}\int_\Omega(\varphi\ltimes s)\big(\nabla\mathsf{y}(\omega),\mathsf{y}(\omega)\big)\,d\omega&=\int_{\Omega_0}(\operatorname{rec}\varphi)(\nabla\mathsf{y}(\omega))d\omega+\int_{\Omega_+}s(\mathsf{y}(\omega))\varphi\left(\frac{\nabla\mathsf{y}(\omega)}{s(\mathsf{y}(\omega))}\right)d\omega\\&=\int_{\Omega_0}\iota_{\{0\}}(\nabla\mathsf{y}(\omega))d\omega+\int_{\Omega_+}\mathsf{y}(\omega)^q\left\|\frac{\nabla\mathsf{y}(\omega)}{\mathsf{y}(\omega)^q}\right\|^p d\omega\\&=\int_{\Omega_+}\mathsf{y}(\omega)\left\|\frac{\nabla\mathsf{y}(\omega)}{\mathsf{y}(\omega)^\gamma}\right\|^p d\omega\\&=\int_{\Omega_+}\mathsf{y}(\omega)\|\nabla\ln_\gamma\mathsf{y}(\omega)\|^p d\omega.\end{aligned} \tag{6.23}$$

Altogether, it follows from Corollary 5.3(iii) that

$$\int_\Omega(\varphi\ltimes s)\big(\nabla\mathsf{y}(\omega),\mathsf{y}(\omega)\big)\,d\omega=\begin{cases}\displaystyle\int_{\Omega_+}\mathsf{y}(\omega)\|\nabla\ln_\gamma\mathsf{y}(\omega)\|^p d\omega, & \text{if } \mathsf{y}\geqslant 0 \text{ a.e.};\\ +\infty, & \text{otherwise.}\end{cases} \tag{6.24}$$

This type of integral shows up in information theory and in thermostatistics [9, 34]. In view of Corollary 5.3(iii), our construction (6.24) is guaranteed to be in $\Gamma_0(W^{1,r}(\Omega))$, which opens a path to solve variational problems such as those in [9] rigorously. In the case when $\|\cdot\|=\|\cdot\|_2$, $p=2$, and $\gamma=q=1$, this recovers a result of [21] on the Fisher information (6.19).

# 7 Concluding remarks

We have proposed several contributions to the theory of perspective functions with nonlinear scaling. First, we introduce the notion of a preperspective function and define the perspective as its largest lower semicontinuous minorant. This construction captures the standard case of linear scaling and guarantees properness, lower semicontinuity, and convexity regardless of the sign of the conjugate of the base function and of the nature of the scaling function. Our construction necessitate the introduction of new envelopes, called the ▾ and ▴ envelopes, which we have thoroughly investigated. We then compute the Legendre conjugate of the proposed nonlinear scaled perspectives. These conjugation formulas are central in duality methods but they also proved to be essential to the computation of proximity operators of perspective functions in the follow-up paper [12]. Our next contribution is to provide explicit formulas for the computation of perspective functions in a broad range of scenarios. Finally, these notions are illustrated by examples as well as through applications touching on areas such as mean-field games, optimal transportation, and information theory.

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