Hilbert Direct Integrals of Monotone Operators*

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Dedicated to the memory of Hédy Attouch

Abstract. Finite Cartesian products of operators play a central role in monotone operator theory and its applications. Extending such products to arbitrary families of operators acting on different Hilbert spaces is an open problem, which we address by introducing the Hilbert direct integral of a family of monotone operators. The properties of this construct are studied and conditions under which the direct integral inherits the properties of the factor operators are provided. The question of determining whether the Hilbert direct integral of a family of subdifferentials of convex functions is itself a subdifferential leads us to introducing the Hilbert direct integral of a family of functions. We establish explicit expressions for evaluating the Legendre conjugate, subdifferential, recession function, Moreau envelope, and proximity operator of such integrals. Next, we propose a duality framework for monotone inclusion problems involving integrals of linearly composed monotone operators and show its pertinence towards the development of numerical solution methods. Applications to inclusion and variational problems are discussed.

Keywords. Integration of set-valued mappings, measurable vector field, monotone operator, optimization, variational analysis.

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§1. Introduction

Let H be a real Hilbert space with scalar product $\langle\cdot\mid\cdot\rangle_{\mathsf{H}}$ and power set 2^{H}. An operator $\mathsf{A}\colon\mathsf{H}\to 2^{\mathsf{H}}$ is monotone if

$$(\forall \mathsf{x}\in\mathsf{H})(\forall \mathsf{y}\in\mathsf{H})(\forall \mathsf{x}^*\in\mathsf{A}\mathsf{x})(\forall \mathsf{y}^*\in\mathsf{A}\mathsf{y})\quad \langle \mathsf{x}-\mathsf{y}\mid \mathsf{x}^*-\mathsf{y}^*\rangle_{\mathsf{H}}\geqslant 0. \tag{1.1}$$

Cartesian products of monotone operators are important constructs that arise in many foundational and practical aspects of the theory [3, 6, 7, 16, 22, 37]. Such products can be defined in a straightforward manner for a finite family $(\mathsf{A}_k)_{1\leqslant k\leqslant p}$ of monotone operators acting, respectively, on real Hilbert spaces $(\mathsf{H}_k)_{1\leqslant k\leqslant p}$. Thus, if one denotes by $\mathcal{H}=\mathsf{H}_1\oplus\cdots\oplus\mathsf{H}_p$ the Hilbert direct sum of $(\mathsf{H}_k)_{1\leqslant k\leqslant p}$ and by $x=(\mathsf{x}_1,\ldots,\mathsf{x}_p)$ a generic vector in $\mathcal{H}$, the product operator is [3]

$$A\colon\mathcal{H}\to 2^{\mathcal{H}}\colon x\mapsto\big\{x^*\in\mathcal{H}\mid \big(\forall k\in\{1,\ldots,p\}\big)\ \mathsf{x}_k^*\in\mathsf{A}_k\mathsf{x}_k\big\}. \tag{1.2}$$

A fundamental instance of an infinite product arises in [6] in the context of evolution equations. There, $(\Omega,\mathcal{F},\mu)$ is a measure space, H is a separable real Hilbert space, $\mathsf{A}\colon\mathsf{H}\to 2^{\mathsf{H}}$ is a monotone operator, $\mathcal{H}=L^2(\Omega,\mathcal{F},\mu;\mathsf{H})$, and a product operator is defined as

$$A\colon\mathcal{H}\to 2^{\mathcal{H}}\colon x\mapsto\big\{x^*\in\mathcal{H}\mid (\forall^{\mu}\omega\in\Omega)\ x^*(\omega)\in\mathsf{A}\big(x(\omega)\big)\big\}, \tag{1.3}$$

where, following [36], the symbol $\forall^{\mu}$ means "for μ-almost every." Another instance of an infinite product appears in [1, Section III.2] in the context of nonautonomous evolution equations, where μ is the Lebesgue measure, $(\mathsf{A}_t)_{t\in[0,T]}$ is a family of monotone operators from H to 2^{H}, $\mathcal{H}=L^2([0,T];\mathsf{H})$, and

$$A\colon\mathcal{H}\to 2^{\mathcal{H}}\colon x\mapsto\big\{x^*\in\mathcal{H}\mid \big(\forall^{\mu}t\in[0,T]\big)\ x^*(t)\in\mathsf{A}_t\big(x(t)\big)\big\}. \tag{1.4}$$

Similar examples arise in probability theory [4], circuit theory [15], approximation theory [18], calculus of variations [21], partial differential equations [22], variational analysis [32], convex analysis [35], and evolution systems [37]. In terms of modeling, (1.2) is limited to a finite number of operators, (1.3) requires that all the factor operators be identical to A, and (1.4) imposes that all the factor spaces be identical to H and operates with the standard Lebesgue measure space $[0,T]$. The above examples are not based on a common mathematical setup and the question of defining a unifying theory for arbitrary products of monotone operators acting on different spaces is open. This question is not only of theoretical interest, but it is also motivated by applications in areas such as dynamical systems, stochastic optimization, and inverse problems. It is the objective of the present paper to fill this gap by introducing such a framework, studying the properties of the resulting product operators, and exploring some of their applications.

To support our framework, we bring into play the notion of a direct integral of Hilbert spaces, which is an attempt to extend Hilbert direct sums from finite families to arbitrary ones. This construction originates in papers published around World War II [23, 24, 26, 30]. We follow [20, Section II.§1].

Definition 1.1 ([20, Définition II.§1.1]). Let $(\Omega,\mathcal{F},\mu)$ be a complete σ-finite measure space, let $(\mathsf{H}_\omega)_{\omega\in\Omega}$ be a family of real Hilbert spaces, and let $\prod_{\omega\in\Omega}\mathsf{H}_\omega$ be the usual real vector space of mappings x defined on Ω such that $(\forall\omega\in\Omega)\ x(\omega)\in\mathsf{H}_\omega$. Suppose that $\mathfrak{G}$ is a vector subspace of $\prod_{\omega\in\Omega}\mathsf{H}_\omega$ which satisfies the following:

[A] For every $x\in\mathfrak{G}$, the function $\Omega\to\mathbb{R}\colon\omega\mapsto\|x(\omega)\|_{\mathsf{H}_\omega}$ is $\mathcal{F}$-measurable.

[B] For every $x \in \prod_{\omega\in\Omega} \mathsf{H}_\omega$,

$$\Big[\ (\forall y \in \mathfrak{G})\ \ \Omega \to \mathbb{R}\colon \omega \mapsto \langle x(\omega) \mid y(\omega)\rangle_{\mathsf{H}_\omega} \text{ is } \mathcal{F}\text{-measurable}\ \Big] \quad \Rightarrow \quad x \in \mathfrak{G}. \tag{1.5}$$

[C] There exists a sequence $(e_n)_{n\in\mathbb{N}}$ in $\mathfrak{G}$ such that $(\forall \omega \in \Omega)\ \overline{\operatorname{span}}\{e_n(\omega)\}_{n\in\mathbb{N}} = \mathsf{H}_\omega$.

Then $((\mathsf{H}_\omega)_{\omega\in\Omega}, \mathfrak{G})$ is an *$\mathcal{F}$-measurable vector field of real Hilbert spaces.*

We shall operate within the framework of [20, Section II.§1.5], which revolves around the following assumption.

Assumption 1.2. Let $(\Omega, \mathcal{F}, \mu)$ be a complete σ-finite measure space, let $((\mathsf{H}_\omega)_{\omega\in\Omega}, \mathfrak{G})$ be an $\mathcal{F}$-measurable vector field of real Hilbert spaces, and set

$$\mathfrak{H} = \left\{ x \in \mathfrak{G} \;\middle|\; \int_\Omega \|x(\omega)\|^2_{\mathsf{H}_\omega} \mu(d\omega) < +\infty \right\}. \tag{1.6}$$

Let $\mathcal{H}$ be the real vector space of equivalence classes of μ-a.e. equal mappings in $\mathfrak{H}$ equipped with the scalar product

$$\langle \cdot \mid \cdot \rangle_{\mathcal{H}}\colon \mathcal{H} \times \mathcal{H} \to \mathbb{R}\colon (x, y) \mapsto \int_\Omega \langle x(\omega) \mid y(\omega)\rangle_{\mathsf{H}_\omega} \mu(d\omega), \tag{1.7}$$

where we adopt the common practice of designating by x both an equivalence class in $\mathcal{H}$ and a representative of it in $\mathfrak{H}$. Then $\mathcal{H}$ is a Hilbert space [20, Proposition II.§1.5(i)], called the *Hilbert direct integral of* $(\mathsf{H}_\omega)_{\omega\in\Omega}$ *relative to* $\mathfrak{G}$. Following [20, Définition II.§1.3], we write

$$\mathcal{H} = {}^{\mathfrak{G}}\!\!\int_\Omega^{\oplus} \mathsf{H}_\omega \mu(d\omega). \tag{1.8}$$

We are now in a position to propose a definition for an arbitrary product of set-valued operators acting on different Hilbert spaces.

Definition 1.3. Suppose that Assumption 1.2 is in force and, for every $\omega \in \Omega$, let $\mathsf{A}_\omega\colon \mathsf{H}_\omega \to 2^{\mathsf{H}_\omega}$. The *Hilbert direct integral of the operators* $(\mathsf{A}_\omega)_{\omega\in\Omega}$ *relative to* $\mathfrak{G}$ is

$${}^{\mathfrak{G}}\!\!\int_\Omega^{\oplus} \mathsf{A}_\omega \mu(d\omega)\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{ x^* \in \mathcal{H} \mid (\forall^\mu \omega \in \Omega)\ x^*(\omega) \in \mathsf{A}_\omega\big(x(\omega)\big) \big\}. \tag{1.9}$$

In tandem with Definition 1.3, we introduce the following notion of an arbitrary direct sum of functions defined on different Hilbert spaces. In the convex case, the subdifferential operator will serve as a bridge between Definitions 1.3 and 1.4. Indeed, we shall establish in Theorem 4.7 that, under suitable assumptions,

$$\partial\left({}^{\mathfrak{G}}\!\!\int_\Omega^{\oplus} \mathsf{f}_\omega \mu(d\omega) \right) = {}^{\mathfrak{G}}\!\!\int_\Omega^{\oplus} \partial \mathsf{f}_\omega \mu(d\omega). \tag{1.10}$$

Definition 1.4. Suppose that Assumption 1.2 is in force and, for every $\omega \in \Omega$, let $\mathsf{f}_\omega\colon \mathsf{H}_\omega \to [-\infty, +\infty]$. Suppose that, for every $x \in \mathfrak{H}$, the function $\Omega \to [-\infty, +\infty]\colon \omega \mapsto \mathsf{f}_\omega(x(\omega))$ is $\mathcal{F}$-measurable. The *Hilbert direct integral of the functions* $(\mathsf{f}_\omega)_{\omega\in\Omega}$ *relative to* $\mathfrak{G}$ is

$${}^{\mathfrak{G}}\!\!\int_\Omega^{\oplus} \mathsf{f}_\omega \mu(d\omega)\colon \mathcal{H} \to [-\infty, +\infty]\colon x \mapsto \int_\Omega \mathsf{f}_\omega\big(x(\omega)\big)\mu(d\omega), \tag{1.11}$$

where we adopt the customary convention that the integral $\int_\Omega \vartheta d\mu$ of an $\mathcal{F}$-measurable function $\vartheta\colon \Omega \to [-\infty, +\infty]$ is the usual Lebesgue integral, except when the Lebesgue integral $\int_\Omega \max\{\vartheta, 0\} d\mu$ is $+\infty$, in which case $\int_\Omega \vartheta d\mu = +\infty$.

The remainder of the paper is as follows. Section 2 presents our notation and provides preliminary results. The Hilbert direct integral of a family of set-valued operators introduced in Definition 1.3 is studied in Section 3. In particular, we establish conditions under which properties such as monotonicity, maximal monotonicity, cocoercivity, and averagedness are transferable from the factor operators to the Hilbert direct integral. We also establish formulas for the domain, range, inverse, resolvent, and Yosida approximation of this integral. Section 4 focuses on the Hilbert direct integral of functions of Definition 1.4. We provide conditions for evaluating the Legendre conjugate, the subdifferential, the recession function, the Moreau envelope, and the proximity operator of the Hilbert direct integral of a family of functions by applying these operations to each factor and then taking the Hilbert direct integral of the resulting family. In Section 5, the results of Section 3 are used to investigate integral inclusion problems involving a family of linearly composed monotone operators. In this context, we propose a duality theory and discuss some applications.

§2. Notation and theoretical tools

2.1. Notation

We follow the notation of [3], to which we refer for a detailed account of the following notions.

Let $\mathcal{H}$ be a real Hilbert space with identity operator $\mathrm{Id}_{\mathcal{H}}$, scalar product $\langle\cdot\mid\cdot\rangle_{\mathcal{H}}$, and associated norm $\|\cdot\|_{\mathcal{H}}$. The weak convergence of a sequence $(x_n)_{n\in\mathbb{N}}$ to x is denoted by $x_n \rightharpoonup x$, and $x_n \to x$ denotes its strong convergence.

Let C be a nonempty closed convex subset of $\mathcal{H}$. Then ι_C is the indicator function of C, d_C is the distance function to C, proj_C is the projection operator onto C, $C^{\ominus}$ is the polar cone of C, and N_C is the normal cone operator of C.

Let $T\colon\mathcal{H}\to\mathcal{H}$ and $\tau\in\left]0,+\infty\right[$. Then T is nonexpansive if it is 1-Lipschitzian, τ-averaged if $\tau\in\left]0,1\right[$ and $\mathrm{Id}_{\mathcal{H}}+\tau^{-1}(T-\mathrm{Id}_{\mathcal{H}})$ is nonexpansive, τ-cocoercive if

$$(\forall x\in\mathcal{H})(\forall y\in\mathcal{H})\quad \langle x-y\mid Tx-Ty\rangle_{\mathcal{H}}\geqslant\tau\|Tx-Ty\|_{\mathcal{H}}^2, \tag{2.1}$$

and firmly nonexpansive if it is 1-cocoercive.

Let $A\colon\mathcal{H}\to 2^{\mathcal{H}}$. The domain of A is $\mathrm{dom}\,A=\{x\in\mathcal{H}\mid Ax\neq\varnothing\}$, the range of A is $\mathrm{ran}\,A=\bigcup_{x\in\mathrm{dom}\,A}Ax$, the set of zeros of A is $\mathrm{zer}\,A=\{x\in\mathcal{H}\mid 0\in Ax\}$, and the graph of A is $\mathrm{gra}\,A=\{(x,x^*)\in\mathcal{H}\times\mathcal{H}\mid x^*\in Ax\}$. The inverse of A is the operator $A^{-1}\colon\mathcal{H}\to 2^{\mathcal{H}}$ with graph $\mathrm{gra}\,A^{-1}=\{(x^*,x)\in\mathcal{H}\times\mathcal{H}\mid x^*\in Ax\}$. The resolvent of A is $J_A=(\mathrm{Id}_{\mathcal{H}}+A)^{-1}$, and the Yosida approximation of A of index $\gamma\in\left]0,+\infty\right[$ is ${}^{\gamma}\!A=(\mathrm{Id}_{\mathcal{H}}-J_{\gamma A})/\gamma=(\gamma\mathrm{Id}_{\mathcal{H}}+A^{-1})^{-1}$. Suppose that A is monotone (see (1.1)). Then A is maximally monotone if there exists no monotone operator $B\colon\mathcal{H}\to 2^{\mathcal{H}}$ such that $\mathrm{gra}\,A\subset\mathrm{gra}\,B\neq\mathrm{gra}\,A$. In this case, $\mathrm{dom}\,J_A=\mathcal{H}$, J_A is firmly nonexpansive, and for every $x\in\mathrm{dom}\,A$, Ax is nonempty, closed, and convex, and we set ${}^{0}\!Ax=\mathrm{proj}_{Ax}\,0$.

We denote by $\Gamma_0(\mathcal{H})$ the class of functions $f\colon\mathcal{H}\to\left]-\infty,+\infty\right]$ which are lower semicontinuous, convex, and such that $\mathrm{dom}\,f=\{x\in\mathcal{H}\mid f(x)<+\infty\}\neq\varnothing$. Let $f\in\Gamma_0(\mathcal{H})$. The conjugate of f is $\Gamma_0(\mathcal{H})\ni f^*\colon x^*\mapsto\sup_{x\in\mathcal{H}}(\langle x\mid x^*\rangle_{\mathcal{H}}-f(x))$ and the subdifferential of f is the maximally monotone operator

$$\partial f\colon\mathcal{H}\to 2^{\mathcal{H}}\colon x\mapsto\{x^*\in\mathcal{H}\mid(\forall y\in\mathcal{H})\ \langle y-x\mid x^*\rangle_{\mathcal{H}}+f(x)\leqslant f(y)\}. \tag{2.2}$$

The proximity operator $\mathrm{prox}_f=J_{\partial f}$ of f maps every $x\in\mathcal{H}$ to the unique minimizer of the function $\mathcal{H}\to\left]-\infty,+\infty\right]\colon y\mapsto f(y)+\|x-y\|_{\mathcal{H}}^2/2$, the Moreau envelope of f of index $\gamma\in\left]0,+\infty\right[$ is ${}^{\gamma}\!f\colon\mathcal{H}\to\mathbb{R}\colon x\mapsto\min_{y\in\mathcal{H}}(f(y)+\|x-y\|_{\mathcal{H}}^2/(2\gamma))$, and $\mathrm{rec}\,f$ is the recession function of f.

2.2. Integrals of set-valued mappings

Let $(\Omega, \mathcal{F}, \mu)$ be a complete σ-finite measure space and let H be a separable real Hilbert space. For every $p \in [1, +\infty[$, set

$$\mathscr{L}^p(\Omega, \mathcal{F}, \mu; \mathsf{H}) = \left\{ x \colon \Omega \to \mathsf{H} \;\middle|\; x \text{ is } (\mathcal{F}, \mathcal{B}_{\mathsf{H}})\text{-measurable and } \int_\Omega \|x(\omega)\|_{\mathsf{H}}^p \mu(d\omega) < +\infty \right\}, \tag{2.3}$$

where $\mathcal{B}_{\mathsf{H}}$ stands for the Borel σ-algebra of H. The Lebesgue (also called Bochner [25]) integral of a mapping $x \in \mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathsf{H})$ is denoted by $\int_\Omega x(\omega)\mu(d\omega)$. We denote by $L^p(\Omega, \mathcal{F}, \mu; \mathsf{H})$ the space of equivalence classes of μ-a.e. equal mappings in $\mathscr{L}^p(\Omega, \mathcal{F}, \mu; \mathsf{H})$; see [36, Section V.§7] for background. The Aumann integral of a set-valued mapping $X \colon \Omega \to 2^{\mathsf{H}}$ is

$$\int_\Omega X(\omega)\mu(d\omega) = \left\{ \int_\Omega x(\omega)\mu(d\omega) \;\middle|\; x \in \mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathsf{H}) \text{ and } (\forall^\mu \omega \in \Omega)\ x(\omega) \in X(\omega) \right\}. \tag{2.4}$$

2.3. Hilbert direct integrals of Hilbert spaces

Going back to Definition 1.1 and Assumption 1.2, the following examples of Hilbert direct integrals will be used repeatedly.

Example 2.1. Here are instances of measurable vector fields and Hilbert direct integrals based on [20, Exemples on pp. 142–143 and 148].

(i) Let $p \in \mathbb{N} \smallsetminus \{0\}$ and let $(\alpha_k)_{1 \leqslant k \leqslant p} \in\,]0, +\infty[^p$. Set

$$\Omega = \{1, \dots, p\}, \quad \mathcal{F} = 2^{\{1,\dots,p\}}, \quad \text{and} \quad \big(\forall k \in \{1, \dots, p\}\big)\ \ \mu\big(\{k\}\big) = \alpha_k. \tag{2.5}$$

Let $(\mathsf{H}_k)_{1 \leqslant k \leqslant p}$ be separable real Hilbert spaces and let $\mathfrak{G} = \mathsf{H}_1 \times \cdots \times \mathsf{H}_p$ be the usual Cartesian product vector space. Then $((\mathsf{H}_k)_{1 \leqslant k \leqslant p}, \mathfrak{G})$ is an $\mathcal{F}$-measurable vector field of real Hilbert spaces and ${}^{\mathfrak{G}}\!\int_\Omega^\oplus \mathsf{H}_\omega \mu(d\omega)$ is the weighted Hilbert direct sum of $(\mathsf{H}_k)_{1 \leqslant k \leqslant p}$, that is, the Hilbert space obtained by equipping $\mathfrak{G}$ with the scalar product

$$\big((\mathsf{x}_k)_{1 \leqslant k \leqslant p}, (\mathsf{y}_k)_{1 \leqslant k \leqslant p}\big) \mapsto \sum_{k=1}^p \alpha_k \langle \mathsf{x}_k \mid \mathsf{y}_k \rangle_{\mathsf{H}_k}. \tag{2.6}$$

(ii) In the setting of (i), suppose that $(\forall k \in \{1, \dots, p\})\ \alpha_k = 1$. Then

$${}^{\mathfrak{G}}\!\int_\Omega^\oplus \mathsf{H}_\omega \mu(d\omega) = \mathsf{H}_1 \oplus \cdots \oplus \mathsf{H}_p \tag{2.7}$$

is the standard Hilbert direct sum of $(\mathsf{H}_k)_{1 \leqslant k \leqslant p}$.

(iii) Let $(\alpha_k)_{k \in \mathbb{N}}$ be a sequence in $]0, +\infty[$ and set

$$\Omega = \mathbb{N}, \quad \mathcal{F} = 2^{\mathbb{N}}, \quad \text{and} \quad (\forall k \in \mathbb{N})\ \ \mu\big(\{k\}\big) = \alpha_k. \tag{2.8}$$

Let $(\mathsf{H}_k)_{k \in \mathbb{N}}$ be separable real Hilbert spaces and set $\mathfrak{G} = \prod_{k \in \mathbb{N}} \mathsf{H}_k$. Then $((\mathsf{H}_k)_{k \in \mathbb{N}}, \mathfrak{G})$ is an $\mathcal{F}$-measurable vector field of real Hilbert spaces and ${}^{\mathfrak{G}}\!\int_\Omega^\oplus \mathsf{H}_\omega \mu(d\omega)$ is the Hilbert space obtained by equipping the vector space

$$\mathfrak{H} = \left\{ (\mathsf{x}_k)_{k \in \mathbb{N}} \in \mathfrak{G} \;\middle|\; \sum_{k \in \mathbb{N}} \alpha_k \|\mathsf{x}_k\|_{\mathsf{H}_k}^2 < +\infty \right\} \tag{2.9}$$

with the scalar product

$$\big((\mathsf{x}_k)_{k\in\mathbb{N}},(\mathsf{y}_k)_{k\in\mathbb{N}}\big)\mapsto\sum_{k\in\mathbb{N}}\alpha_k\langle\mathsf{x}_k\mid\mathsf{y}_k\rangle_{\mathsf{H}_k}. \tag{2.10}$$

(iv) Let $(\Omega,\mathcal{F},\mu)$ be a complete σ-finite measure space, let H be a separable real Hilbert space, and set

$$\big[\,(\forall\omega\in\Omega)\ \ \mathsf{H}_\omega=\mathsf{H}\,\big]\quad\text{and}\quad\mathfrak{G}=\big\{x\colon\Omega\to\mathsf{H}\mid x\text{ is }(\mathcal{F},\mathcal{B}_{\mathsf{H}})\text{-measurable}\big\}. \tag{2.11}$$

Then $((\mathsf{H}_\omega)_{\omega\in\Omega},\mathfrak{G})$ is an $\mathcal{F}$-measurable vector field of real Hilbert spaces and

$$\prescript{\mathfrak{G}}{}{\int_\Omega^\oplus}\mathsf{H}_\omega\mu(d\omega)=L^2\big(\Omega,\mathcal{F},\mu;\mathsf{H}\big). \tag{2.12}$$

The following results are given as remarks in [20, Section II.§1.3]. We provide proofs for completeness.

Lemma 2.2. *Let $(\Omega,\mathcal{F},\mu)$ be a complete σ-finite measure space and let $((\mathsf{H}_\omega)_{\omega\in\Omega},\mathfrak{G})$ be an $\mathcal{F}$-measurable vector field of Hilbert spaces. Then the following hold:*

(i) *Let x and y be in $\mathfrak{G}$. Then the function $\Omega\to\mathbb{R}\colon\omega\mapsto\langle x(\omega)\mid y(\omega)\rangle_{\mathsf{H}_\omega}$ is $\mathcal{F}$-measurable.*

(ii) *Let $x\in\prod_{\omega\in\Omega}\mathsf{H}_\omega$ and $y\in\mathfrak{G}$ be such that $x=y$ μ-a.e. Then $x\in\mathfrak{G}$.*

(iii) *Let $\xi\colon\Omega\to\mathbb{R}$ be $\mathcal{F}$-measurable and let $x\in\mathfrak{G}$. Then the mapping $\xi x\colon\omega\mapsto\xi(\omega)x(\omega)$ lies in $\mathfrak{G}$.*

(iv) *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{G}$ and let $x\in\prod_{\omega\in\Omega}\mathsf{H}_\omega$. Suppose that $(\forall^\mu\omega\in\Omega)$ $x_n(\omega)\rightharpoonup x(\omega)$. Then $x\in\mathfrak{G}$.*

(v) *There exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathfrak{G}$ such that*

$$\begin{cases}(\forall n\in\mathbb{N})\ \ \displaystyle\int_\Omega\|u_n(\omega)\|^2_{\mathsf{H}_\omega}\mu(d\omega)<+\infty\\[2mm](\forall\omega\in\Omega)\ \ \overline{\big\{u_n(\omega)\big\}_{n\in\mathbb{N}}}=\mathsf{H}_\omega.\end{cases} \tag{2.13}$$

Proof. (i): Since $\mathfrak{G}$ is a vector subspace of $\prod_{\omega\in\Omega}\mathsf{H}_\omega$, $x+y\in\mathfrak{G}$ and $x-y\in\mathfrak{G}$. Hence, by property [A] in Definition 1.1, the functions $\Omega\to\mathbb{R}\colon\omega\mapsto\|x(\omega)+y(\omega)\|_{\mathsf{H}_\omega}$ and $\Omega\to\mathbb{R}\colon\omega\mapsto\|x(\omega)-y(\omega)\|_{\mathsf{H}_\omega}$ are $\mathcal{F}$-measurable. Therefore, the assertion follows from the polarization identity $(\forall\omega\in\Omega)$ $4\langle x(\omega)\mid y(\omega)\rangle_{\mathsf{H}_\omega}=\|x(\omega)+y(\omega)\|^2_{\mathsf{H}_\omega}-\|x(\omega)-y(\omega)\|^2_{\mathsf{H}_\omega}$.

(ii): Take $z\in\mathfrak{G}$. Then $(\forall^\mu\omega\in\Omega)$ $\langle x(\omega)\mid z(\omega)\rangle_{\mathsf{H}_\omega}=\langle y(\omega)\mid z(\omega)\rangle_{\mathsf{H}_\omega}$. At the same time, since y and z lie in $\mathfrak{G}$, we deduce from (i) that the function $\Omega\to\mathbb{R}\colon\omega\mapsto\langle y(\omega)\mid z(\omega)\rangle_{\mathsf{H}_\omega}$ is $\mathcal{F}$-measurable. Hence, the completeness of $(\Omega,\mathcal{F},\mu)$ implies that the function $\Omega\to\mathbb{R}\colon\omega\mapsto\langle x(\omega)\mid z(\omega)\rangle_{\mathsf{H}_\omega}$ is also $\mathcal{F}$-measurable. Consequently, property [B] in Definition 1.1 forces $x\in\mathfrak{G}$.

(iii): We have $\xi x\in\prod_{\omega\in\Omega}\mathsf{H}_\omega$. On the other hand, for every $y\in\mathfrak{G}$, it results from (i) that the function $\omega\mapsto\langle\xi(\omega)x(\omega)\mid y(\omega)\rangle_{\mathsf{H}_\omega}=\xi(\omega)\langle x(\omega)\mid y(\omega)\rangle_{\mathsf{H}_\omega}$ is $\mathcal{F}$-measurable. Hence, we conclude via property [B] in Definition 1.1 that $\xi x\in\mathfrak{G}$.

(iv): Let $\Xi\in\mathcal{F}$ be such that $\mu(\Xi)=0$ and $(\forall\omega\in\complement\Xi)$ $x_n(\omega)\rightharpoonup x(\omega)$. Moreover, set

$$\big[\,(\forall n\in\mathbb{N})\ \ y_n=1_{\complement\Xi}x_n\,\big]\quad\text{and}\quad y=1_{\complement\Xi}x, \tag{2.14}$$

and let $z\in\mathfrak{G}$. For every $n\in\mathbb{N}$, it results from (iii) that $y_n\in\mathfrak{G}$ and, in turn, from (i) that the function $\Omega\to\mathbb{R}\colon\omega\mapsto\langle y_n(\omega)\mid z(\omega)\rangle_{\mathsf{H}_\omega}$ is $\mathcal{F}$-measurable. Additionally,

$$(\forall\omega\in\Xi)\qquad\lim\langle y_n(\omega)\mid z(\omega)\rangle_{\mathsf{H}_\omega}=0=\langle y(\omega)\mid z(\omega)\rangle_{\mathsf{H}_\omega} \tag{2.15}$$

and

$$(\forall\omega\in\complement\Xi)\quad \lim\langle y_n(\omega)\,|\,z(\omega)\rangle_{\mathsf{H}_\omega}=\lim\langle x_n(\omega)\,|\,z(\omega)\rangle_{\mathsf{H}_\omega}=\langle x(\omega)\,|\,z(\omega)\rangle_{\mathsf{H}_\omega}=\langle y(\omega)\,|\,z(\omega)\rangle_{\mathsf{H}_\omega}. \tag{2.16}$$

Hence, the function $\Omega\to\mathbb{R}\colon\omega\mapsto\langle y(\omega)\,|\,z(\omega)\rangle_{\mathsf{H}_\omega}$ is $\mathcal{F}$-measurable as the pointwise limit of a sequence of $\mathcal{F}$-measurable functions. Therefore, appealing to property [B] in Definition 1.1, we deduce that $y\in\mathfrak{G}$. Consequently, since $x=y$ μ-a.e., (ii) yields $x\in\mathfrak{G}$.

(v): Property [C] in Definition 1.1 guarantees the existence of a sequence $(e_n)_{n\in\mathbb{N}}$ in $\mathfrak{G}$ such that $(\forall\omega\in\Omega)$ $\overline{\operatorname{span}}\{e_n(\omega)\}_{n\in\mathbb{N}}=\mathsf{H}_\omega$. Now let $(r_n)_{n\in\mathbb{N}}$ be an enumeration of the set

$$\left\{\sum_{k=0}^{n}\alpha_k e_k\;\middle|\; n\in\mathbb{N}\text{ and }(\alpha_k)_{0\leqslant k\leqslant n}\in\mathbb{Q}^{n+1}\right\}. \tag{2.17}$$

Then

$$(\forall n\in\mathbb{N})\quad r_n\in\mathfrak{G} \tag{2.18}$$

and

$$(\forall\omega\in\Omega)\quad \overline{\{r_n(\omega)\}_{n\in\mathbb{N}}}=\mathsf{H}_\omega. \tag{2.19}$$

Since $(\Omega,\mathcal{F},\mu)$ is σ-finite, we obtain an increasing sequence $(\Omega_k)_{k\in\mathbb{N}}$ in $\mathcal{F}$ of finite μ-measure such that $\bigcup_{k\in\mathbb{N}}\Omega_k=\Omega$. Set

$$(\forall n\in\mathbb{N})(\forall m\in\mathbb{N})(\forall k\in\mathbb{N})\quad \Xi_{n,m,k}=\big\{\omega\in\Omega_k\;\big|\;\|r_n(\omega)\|_{\mathsf{H}_\omega}\leqslant m\big\}\quad\text{and}\quad s_{n,m,k}=1_{\Xi_{n,m,k}}r_n. \tag{2.20}$$

For every $n\in\mathbb{N}$, it results from (2.18) and property [A] in Definition 1.1 that the function $\Omega\to\mathbb{R}\colon\omega\mapsto\|r_n(\omega)\|_{\mathsf{H}_\omega}$ is $\mathcal{F}$-measurable. Therefore, for every $n\in\mathbb{N}$, every $m\in\mathbb{N}$, and every $k\in\mathbb{N}$, $\Xi_{n,m,k}\in\mathcal{F}$ and we thus infer from (iii) and (2.18) that $s_{n,m,k}\in\mathfrak{G}$ whereas, by (2.20),

$$\int_\Omega\|s_{n,m,k}(\omega)\|^2_{\mathsf{H}_\omega}\mu(d\omega)\leqslant\mu(\Xi_{n,m,k})m\leqslant\mu(\Omega_k)m<+\infty. \tag{2.21}$$

Next, take $\omega\in\Omega$, $\mathsf{x}\in\mathsf{H}_\omega$, and $\varepsilon\in\,]0,1[$. By (2.19), there exists $n\in\mathbb{N}$ such that $\|r_n(\omega)-\mathsf{x}\|_{\mathsf{H}_\omega}\leqslant\varepsilon$. In turn, the triangle inequality gives $\|r_n(\omega)\|_{\mathsf{H}_\omega}\leqslant\varepsilon+\|\mathsf{x}\|_{\mathsf{H}_\omega}$. However, since $\bigcup_{k\in\mathbb{N}}\Omega_k=\Omega$, there exists $k\in\mathbb{N}$ such that $\omega\in\Omega_k$. Therefore, upon choosing $m\in\mathbb{N}$ such that $m\geqslant\varepsilon+\|\mathsf{x}\|_{\mathsf{H}_\omega}$, we deduce that $\omega\in\Xi_{n,m,k}$. Thus, combining with (2.20) yields $\|s_{n,m,k}(\omega)-\mathsf{x}\|_{\mathsf{H}_\omega}=\|r_n(\omega)-\mathsf{x}\|_{\mathsf{H}_\omega}\leqslant\varepsilon$. ☐

Lemma 2.3 ([20, Proposition II.§1.5(ii)]). *Suppose that Assumption 1.2 is in force and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ which converges strongly to a point $x\in\mathcal{H}$. Then there exists a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $(\forall^\mu\omega\in\Omega)$ $x_{k_n}(\omega)\to x(\omega)$.*

§3. Hilbert direct integrals of set-valued operators

We study the properties of the Hilbert direct integrals of set-valued operators introduced in Definition 1.3. Let us first point out an important special case of Definition 1.3.

Definition 3.1. Suppose that Assumption 1.2 is in force and, for every $\omega \in \Omega$, let C_ω be a subset of H_ω. The *Hilbert direct integral of the sets* $(\mathsf{C}_\omega)_{\omega\in\Omega}$ *relative to* $\mathfrak{G}$ is

$$\overset{\mathfrak{G}}{\int_\Omega^{\oplus}} \mathsf{C}_\omega \mu(d\omega) = \big\{x \in \mathcal{H} \mid (\forall^\mu \omega \in \Omega)\ x(\omega) \in \mathsf{C}_\omega\big\}. \tag{3.1}$$

We first record the following facts, which are direct consequences of Definitions 1.3 and 3.1.

Proposition 3.2. *Suppose that Assumption 1.2 is in force and, for every $\omega \in \Omega$, let $\mathsf{A}_\omega \colon \mathsf{H}_\omega \to 2^{\mathsf{H}_\omega}$ be a set-valued operator. Set*

$$A = \overset{\mathfrak{G}}{\int_\Omega^{\oplus}} \mathsf{A}_\omega \mu(d\omega). \tag{3.2}$$

Then the following hold:

(i) $\operatorname{dom} A = \big\{x \in \mathcal{H} \mid (\exists\, x^* \in \mathfrak{H})(\forall^\mu \omega \in \Omega)\ x^*(\omega) \in \mathsf{A}_\omega(x(\omega))\big\}$.

(ii) $\operatorname{ran} A = \big\{x^* \in \mathcal{H} \mid (\exists\, x \in \mathfrak{H})(\forall^\mu \omega \in \Omega)\ x^*(\omega) \in \mathsf{A}_\omega(x(\omega))\big\}$.

(iii) $\operatorname{zer} A = \overset{\mathfrak{G}}{\int_\Omega^{\oplus}} \operatorname{zer} \mathsf{A}_\omega\, \mu(d\omega)$.

(iv) $A^{-1} = \overset{\mathfrak{G}}{\int_\Omega^{\oplus}} \mathsf{A}_\omega^{-1} \mu(d\omega)$.

(v) *Suppose that, for every $\omega \in \Omega$, A_ω is monotone. Then A is monotone.*

Remark 3.3. Regarding Proposition 3.2(i), consider the setting of Example 2.1(iii) and suppose that, in addition, $(\forall k \in \mathbb{N})\ \mathsf{H}_k = \mathbb{R}$. For every $k \in \mathbb{N}$, set $\mathsf{A}_k \colon \mathsf{H}_k \to \mathsf{H}_k \colon \mathsf{x} \mapsto k/\sqrt{\alpha_k}$. Then

$$\operatorname{dom}\left(\overset{\mathfrak{G}}{\int_\Omega^{\oplus}} \mathsf{A}_\omega \mu(d\omega)\right) = \varnothing. \tag{3.3}$$

The following result examines the interplay between the properties of the direct integral and those of its factor operators.

Proposition 3.4. *Suppose that Assumption 1.2 is in force and, for every $\omega \in \Omega$, let $\mathsf{T}_\omega \colon \mathsf{H}_\omega \to \mathsf{H}_\omega$ be sequentially strong-to-weak continuous. Set*

$$T = \overset{\mathfrak{G}}{\int_\Omega^{\oplus}} \mathsf{T}_\omega \mu(d\omega) \tag{3.4}$$

and suppose that the following are satisfied:

[A] *For every $x \in \mathfrak{H}$, the mapping $\omega \mapsto \mathsf{T}_\omega(x(\omega))$ lies in $\mathfrak{G}$.*

[B] *There exists $z \in \mathfrak{H}$ such that the mapping $\omega \mapsto \mathsf{T}_\omega(z(\omega))$ lies in $\mathfrak{H}$.*

Then the following hold:

(i) *Let $\beta \in [0, +\infty[$. Then the following are equivalent:*

  (a) *For μ-almost every $\omega \in \Omega$, T_ω is β-Lipschitzian.*

  (b) $\operatorname{dom} T = \mathcal{H}$ *and T is β-Lipschitzian.*

(ii) *Let $\tau \in \,]0, +\infty[$. Then the following are equivalent:*

(a) *For μ-almost every $\omega \in \Omega$, T_ω is τ-cocoercive.*

(b) $\operatorname{dom} T = \mathcal{H}$ *and T is τ-cocoercive.*

(iii) *Let $\alpha \in \left]0,1\right[$. Then the following are equivalent:*

(a) *For μ-almost every $\omega \in \Omega$, T_ω is α-averaged.*

(b) $\operatorname{dom} T = \mathcal{H}$ *and T is α-averaged.*

Proof. Observe that T is at most single-valued. On the other hand, Lemma 2.2(v) states that there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathfrak{H}$ such that

$$(\forall \omega \in \Omega) \quad \overline{\{u_n(\omega)\}_{n\in\mathbb{N}}} = \mathsf{H}_\omega. \tag{3.5}$$

(i)(a)⇒(i)(b): Let $\Xi \in \mathcal{F}$ be such that $\mu(\Xi) = 0$ and, for every $\omega \in \complement\Xi$, T_ω is β-Lipschitzian. Then

$$(\forall x \in \mathfrak{H})(\forall y \in \mathfrak{H})(\forall \omega \in \complement\Xi) \quad \big\|\mathsf{T}_\omega\big(x(\omega)\big) - \mathsf{T}_\omega\big(y(\omega)\big)\big\|_{\mathsf{H}_\omega} \leqslant \beta\|x(\omega) - y(\omega)\|_{\mathsf{H}_\omega}. \tag{3.6}$$

In turn, since $\mathfrak{G}$ is a vector subspace of $\prod_{\omega\in\Omega}\mathsf{H}_\omega$, we infer from [A] and (1.6) that, for every $x \in \mathfrak{H}$ and every $y \in \mathfrak{H}$, the mapping $\omega \mapsto \mathsf{T}_\omega(x(\omega)) - \mathsf{T}_\omega(y(\omega))$ lies in $\mathfrak{H}$. Thus, [B] implies that, for every $x \in \mathfrak{H}$, the mapping $\omega \mapsto \mathsf{T}_\omega(x(\omega))$ lies in $\mathfrak{H}$ as the sum of two mappings in $\mathfrak{H}$, namely $\omega \mapsto \mathsf{T}_\omega(x(\omega)) - \mathsf{T}_\omega(z(\omega))$ and $\omega \mapsto \mathsf{T}_\omega(z(\omega))$. Therefore $\operatorname{dom} T = \mathcal{H}$. Additionally, it results from (3.6) and (1.7) that T is β-Lipschitzian.

(i)(b)⇒(i)(a): Fix temporarily $n \in \mathbb{N}$ and $m \in \mathbb{N}$. For every $\Xi \in \mathcal{F}$ such that $\mu(\Xi) < +\infty$, since $1_\Xi u_n \in \mathfrak{H}$ and $1_\Xi u_m \in \mathfrak{H}$ thanks to Lemma 2.2(iii), we derive from (1.7) that

$$\begin{aligned}\int_\Xi \big\|\mathsf{T}_\omega\big(u_n(\omega)\big) - \mathsf{T}_\omega\big(u_m(\omega)\big)\big\|^2_{\mathsf{H}_\omega}\mu(d\omega) &= \|T(1_\Xi u_n) - T(1_\Xi u_m)\|^2_{\mathcal{H}}\\ &\leqslant \beta^2\|1_\Xi u_n - 1_\Xi u_m\|^2_{\mathcal{H}}\\ &= \int_\Xi \beta^2\|u_n(\omega) - u_m(\omega)\|^2_{\mathsf{H}_\omega}\mu(d\omega).\end{aligned} \tag{3.7}$$

Hence, since $(\Omega, \mathcal{F}, \mu)$ is σ-finite, there exists $\Xi_{n,m} \in \mathcal{F}$ such that

$$\mu(\Xi_{n,m}) = 0 \quad \text{and} \quad (\forall \omega \in \complement\Xi_{n,m}) \quad \big\|\mathsf{T}_\omega\big(u_n(\omega)\big) - \mathsf{T}_\omega\big(u_m(\omega)\big)\big\|_{\mathsf{H}_\omega} \leqslant \beta\|u_n(\omega) - u_m(\omega)\|_{\mathsf{H}_\omega}. \tag{3.8}$$

Now set $\Xi = \bigcup_{n\in\mathbb{N}, m\in\mathbb{N}} \Xi_{n,m}$, let $\omega \in \complement\Xi$, let $\mathsf{x} \in \mathsf{H}_\omega$, and let $\mathsf{y} \in \mathsf{H}_\omega$. Then, $\Xi \in \mathcal{F}$ with $\mu(\Xi) = 0$ and, in view of (3.5), there exist sequences $(k_n)_{n\in\mathbb{N}}$ and $(l_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $u_{k_n}(\omega) \to \mathsf{x}$ and $u_{l_n}(\omega) \to \mathsf{y}$. At the same time, by (3.8),

$$(\forall n \in \mathbb{N}) \quad \big\|\mathsf{T}_\omega\big(u_{k_n}(\omega)\big) - \mathsf{T}_\omega\big(u_{l_n}(\omega)\big)\big\|_{\mathsf{H}_\omega} \leqslant \beta\|u_{k_n}(\omega) - u_{l_n}(\omega)\|_{\mathsf{H}_\omega}. \tag{3.9}$$

Thus, since $\|\cdot\|_{\mathsf{H}_\omega}$ is weakly lower semicontinuous, letting $n \to +\infty$ and invoking the sequential strong-to-weak continuity of T_ω, we get $\|\mathsf{T}_\omega\mathsf{x} - \mathsf{T}_\omega\mathsf{y}\|_{\mathsf{H}_\omega} \leqslant \beta\|\mathsf{x} - \mathsf{y}\|_{\mathsf{H}_\omega}$.

(ii) and (iii): Argue as in (i). ☐

Proposition 3.5. *Suppose that Assumption 1.2 is in force and, for every $\omega \in \Omega$, let $\mathsf{A}_\omega \colon \mathsf{H}_\omega \to 2^{\mathsf{H}_\omega}$ be a set-valued operator. Set*

$$A = {}^{\mathfrak{G}}\!\!\int_\Omega^\oplus \mathsf{A}_\omega \mu(d\omega) \tag{3.10}$$

and let $\gamma \in \left]0,+\infty\right[$. Then

$$J_{\gamma A} = {}^{\mathfrak{G}}\!\!\int_\Omega^\oplus J_{\gamma \mathsf{A}_\omega}\mu(d\omega) \quad \text{and} \quad {}^{\gamma}\!A = {}^{\mathfrak{G}}\!\!\int_\Omega^\oplus {}^{\gamma}\mathsf{A}_\omega\,\mu(d\omega). \tag{3.11}$$

Proof. Set $T = {}^{\mathfrak{G}}\!\int_{\Omega}^{\oplus} J_{\gamma \mathsf{A}_\omega}\mu(d\omega)$. We derive from Definition 1.3 and [3, Proposition 23.2(ii)] that

$$\begin{aligned}(\forall x \in \mathcal{H}) \quad Tx &= \big\{p \in \mathcal{H} \mid (\forall^{\mu}\omega \in \Omega)\ p(\omega) \in J_{\gamma \mathsf{A}_\omega}\big(x(\omega)\big)\big\} \\ &= \big\{p \in \mathcal{H} \mid (\forall^{\mu}\omega \in \Omega)\ \gamma^{-1}\big(x(\omega) - p(\omega)\big) \in \mathsf{A}_\omega\big(p(\omega)\big)\big\} \\ &= \big\{p \in \mathcal{H} \mid \gamma^{-1}(x-p) \in Ap\big\} \\ &= J_{\gamma A}x. \end{aligned} \tag{3.12}$$

Likewise, upon setting $R = {}^{\mathfrak{G}}\!\int_{\Omega}^{\oplus} {}^{\gamma}\mathsf{A}_\omega\,\mu(d\omega)$, we deduce from Definition 1.3 and [3, Proposition 23.2(iii)] that

$$\begin{aligned}(\forall x \in \mathcal{H}) \quad Rx &= \big\{p \in \mathcal{H} \mid (\forall^{\mu}\omega \in \Omega)\ p(\omega) \in {}^{\gamma}\mathsf{A}_\omega\big(x(\omega)\big)\big\} \\ &= \big\{p \in \mathcal{H} \mid (\forall^{\mu}\omega \in \Omega)\ p(\omega) \in \mathsf{A}_\omega\big(x(\omega) - \gamma p(\omega)\big)\big\} \\ &= \big\{p \in \mathcal{H} \mid p \in A(x - \gamma p)\big\} \\ &= {}^{\gamma}Ax, \end{aligned} \tag{3.13}$$

which completes the proof. ☐

Assumption 3.6. Assumption 1.2 and the following are in force:

[A] For every $\omega \in \Omega$, $\mathsf{A}_\omega \colon \mathsf{H}_\omega \to 2^{\mathsf{H}_\omega}$ is maximally monotone.

[B] For every $x \in \mathfrak{H}$, the mapping $\omega \mapsto J_{\mathsf{A}_\omega}(x(\omega))$ lies in $\mathfrak{G}$.

[C] $\operatorname{dom} {}^{\mathfrak{G}}\!\int_{\Omega}^{\oplus} \mathsf{A}_\omega \mu(d\omega) \neq \varnothing$.

Proposition 3.7. *Suppose that Assumption 3.6 is in force. Then the following hold:*

(i) *For every $\omega \in \Omega$, $\mathsf{A}_\omega^{-1} \colon \mathsf{H}_\omega \to 2^{\mathsf{H}_\omega}$ is maximally monotone.*

(ii) *For every $x \in \mathfrak{H}$, the mapping $\omega \mapsto J_{\mathsf{A}_\omega^{-1}}(x(\omega))$ lies in $\mathfrak{G}$.*

(iii) $\operatorname{dom} {}^{\mathfrak{G}}\!\int_{\Omega}^{\oplus} \mathsf{A}_\omega^{-1} \mu(d\omega) \neq \varnothing$.

Proof. We infer from Assumption 3.6[A] and [3, Propositions 20.22 and 23.20] that, for every $\omega \in \Omega$, A_ω^{-1} is maximally monotone and $J_{\mathsf{A}_\omega^{-1}} = \mathrm{Id}_{\mathsf{H}_\omega} - J_{\mathsf{A}_\omega}$. In turn, for every $x \in \mathfrak{H}$, since $\mathfrak{G}$ is a vector subspace of $\prod_{\omega \in \Omega} \mathsf{H}_\omega$, it follows from Assumption 3.6[B] that the mapping $\omega \mapsto J_{\mathsf{A}_\omega^{-1}}(x(\omega))$ lies in $\mathfrak{G}$ as the difference of the mappings x and $\omega \mapsto J_{\mathsf{A}_\omega}(x(\omega))$. Finally, Proposition 3.2(iv) and Assumption 3.6[C] yield $\operatorname{dom} {}^{\mathfrak{G}}\!\int_{\Omega}^{\oplus} \mathsf{A}_\omega^{-1} \mu(d\omega) = \operatorname{ran} {}^{\mathfrak{G}}\!\int_{\Omega}^{\oplus} \mathsf{A}_\omega \mu(d\omega) \neq \varnothing$. ☐

The main result of this section is the following theorem, which establishes the basic properties of Hilbert direct integrals of maximally monotone operators. Special cases of items (i) and (ii) corresponding to scenarios described in Example 2.1 can be found in [1, 3, 6, 18, 32].

Theorem 3.8. *Suppose that Assumption 3.6 is in force and set*

$$A = {}^{\mathfrak{G}}\!\int_{\Omega}^{\oplus} \mathsf{A}_\omega \mu(d\omega). \tag{3.14}$$

Then the following hold:

(i) *A is maximally monotone.*

(ii) *Let $\gamma \in \left]0, +\infty\right[$ and $x \in \mathfrak{H}$. Then the following are satisfied:*

(a) *The mapping* $\omega \mapsto J_{\gamma \mathsf{A}_\omega}(x(\omega))$ *lies in* $\mathfrak{H}$ *and* $J_{\gamma A} = \displaystyle{}^{\mathfrak{G}}\!\!\int_\Omega^\oplus J_{\gamma \mathsf{A}_\omega} \mu(d\omega)$.

(b) *The mapping* $\omega \mapsto {}^{\gamma}\mathsf{A}_\omega(x(\omega))$ *lies in* $\mathfrak{H}$ *and* ${}^{\gamma}A = \displaystyle{}^{\mathfrak{G}}\!\!\int_\Omega^\oplus {}^{\gamma}\mathsf{A}_\omega \mu(d\omega)$.

(iii) $\overline{\operatorname{dom}}\, A = \displaystyle{}^{\mathfrak{G}}\!\!\int_\Omega^\oplus \overline{\operatorname{dom}}\, \mathsf{A}_\omega\, \mu(d\omega) = \overline{{}^{\mathfrak{G}}\!\!\int_\Omega^\oplus \operatorname{dom} \mathsf{A}_\omega\, \mu(d\omega)}$.

(iv) $\overline{\operatorname{ran}}\, A = \displaystyle{}^{\mathfrak{G}}\!\!\int_\Omega^\oplus \overline{\operatorname{ran}}\, \mathsf{A}_\omega\, \mu(d\omega) = \overline{{}^{\mathfrak{G}}\!\!\int_\Omega^\oplus \operatorname{ran} \mathsf{A}_\omega\, \mu(d\omega)}$.

(v) *Let* $x \in \mathfrak{H}$ *be such that* $(\forall \omega \in \Omega)\ x(\omega) \in \operatorname{dom} \mathsf{A}_\omega$. *Then the following are satisfied:*

(a) *The mapping* $\omega \mapsto {}^{0}\mathsf{A}_\omega(x(\omega))$ *lies in* $\mathfrak{G}$.

(b) *Suppose that* $x \in \operatorname{dom} A$. *Then the mapping* $\omega \mapsto {}^{0}\mathsf{A}_\omega(x(\omega))$ *lies in* $\mathfrak{H}$ *and* ${}^{0}Ax = \displaystyle{}^{\mathfrak{G}}\!\!\int_\Omega^\oplus {}^{0}\mathsf{A}_\omega(x(\omega))\mu(d\omega)$.

Proof. (i): By [3, Proposition 23.2(i)] and Assumption 3.6[C], $\operatorname{ran} J_A = \operatorname{dom} A \neq \varnothing$ and there thus exist z and r in $\mathcal{H}$ such that $r \in J_A z$ or, equivalently, $z - r \in Ar$. Hence, for μ-almost every $\omega \in \Omega$, $z(\omega) - r(\omega) \in \mathsf{A}_\omega(r(\omega))$ and, therefore, the monotonicity of A_ω yields $r(\omega) = J_{\mathsf{A}_\omega}(z(\omega))$. Thus, because $r \in \mathfrak{H}$, we infer from Lemma 2.2(ii) that the mapping $\omega \mapsto J_{\mathsf{A}_\omega}(z(\omega))$ lies in $\mathfrak{H}$. In turn, appealing to Assumption 3.6[B], we deduce from Proposition 3.4(iii) (applied to the firmly nonexpansive operators $(J_{\mathsf{A}_\omega})_{\omega\in\Omega}$) and Proposition 3.5 that $J_A \colon \mathcal{H} \to \mathcal{H}$ is firmly nonexpansive. Consequently, [3, Proposition 23.8(iii)] guarantees that A is maximally monotone.

(ii): Use (i), Proposition 3.5, and Lemma 2.2(ii).

(iii): By (i) and [3, Corollary 21.14], $\overline{\operatorname{dom}}\, A$ is a nonempty closed convex subset of $\mathcal{H}$. Fix temporarily $x \in \mathfrak{H}$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $]0,1[$ such that $\gamma_n \downarrow 0$, and set

$$p = \operatorname{proj}_{\overline{\operatorname{dom}}\, A} x \quad \text{and} \quad (\forall n \in \mathbb{N})\ \ p_n \colon \omega \mapsto J_{\gamma_n \mathsf{A}_\omega}\big(x(\omega)\big). \tag{3.15}$$

We infer from (ii)(a) that, for every $n \in \mathbb{N}$, $p_n \in \mathfrak{H}$ and $p_n = J_{\gamma_n A} x$. Thus, it follows from (i) and [3, Theorem 23.48] that $p_n \to p$ in $\mathcal{H}$. In turn, Lemma 2.3 ensures that there exist a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ and a set $\Xi \in \mathcal{F}$ such that $\mu(\Xi) = 0$ and $(\forall \omega \in \complement\Xi)\ p_{k_n}(\omega) \to p(\omega)$. On the other hand, we deduce from Assumption 3.6[A] and [3, Theorem 23.48] that $(\forall \omega \in \complement\Xi)\ p_{k_n}(\omega) = J_{\gamma_{k_n}\mathsf{A}_\omega}(x(\omega)) \to \operatorname{proj}_{\overline{\operatorname{dom}}\, \mathsf{A}_\omega}(x(\omega))$. Therefore $(\forall \omega \in \complement\Xi)\ p(\omega) = \operatorname{proj}_{\overline{\operatorname{dom}}\, \mathsf{A}_\omega}(x(\omega))$. Hence, because $p \in \mathfrak{H}$, it results from Lemma 2.2(ii) that the mapping $\omega \mapsto \operatorname{proj}_{\overline{\operatorname{dom}}\, \mathsf{A}_\omega}(x(\omega))$ is a representative in $\mathfrak{H}$ of $\operatorname{proj}_{\overline{\operatorname{dom}}\, A} x$. This confirms that

$$\operatorname{proj}_{\overline{\operatorname{dom}}\, A} = {}^{\mathfrak{G}}\!\!\int_\Omega^\oplus \operatorname{proj}_{\overline{\operatorname{dom}}\, \mathsf{A}_\omega} \mu(d\omega). \tag{3.16}$$

Therefore, using Definition 3.1, we get

$$\begin{aligned} \overline{\operatorname{dom}}\, A &= \big\{x \in \mathcal{H} \mid x = \operatorname{proj}_{\overline{\operatorname{dom}}\, A} x\big\} \\ &= \Big\{x \in \mathcal{H} \;\Big|\; (\forall^\mu \omega \in \Omega)\ x(\omega) = \operatorname{proj}_{\overline{\operatorname{dom}}\, \mathsf{A}_\omega}\big(x(\omega)\big)\Big\} \\ &= \big\{x \in \mathcal{H} \mid (\forall^\mu \omega \in \Omega)\ x(\omega) \in \overline{\operatorname{dom}}\, \mathsf{A}_\omega\big\} \\ &= {}^{\mathfrak{G}}\!\!\int_\Omega^\oplus \overline{\operatorname{dom}}\, \mathsf{A}_\omega\, \mu(d\omega). \end{aligned} \tag{3.17}$$

Thus ${}^{\mathfrak{G}}\!\int_{\Omega}^{\oplus} \overline{\operatorname{dom} \mathsf{A}_\omega}\, \mu(d\omega)$ is a closed subset of $\mathcal{H}$. Consequently, we deduce from Proposition 3.2(i) and Definition 3.1 that

$$\overline{\operatorname{dom} A} \subset \overline{{}^{\mathfrak{G}}\!\int_{\Omega}^{\oplus} \operatorname{dom} \mathsf{A}_\omega\, \mu(d\omega)} \subset {}^{\mathfrak{G}}\!\int_{\Omega}^{\oplus} \overline{\operatorname{dom} \mathsf{A}_\omega}\, \mu(d\omega) = \overline{\operatorname{dom} A}, \tag{3.18}$$

which furnishes the desired identities.

(iv): In the light of Proposition 3.2(iv) and Proposition 3.7, the claim follows from (iii) applied to the family $(\mathsf{A}_\omega^{-1})_{\omega\in\Omega}$.

(v): Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $]0,1[$ such that $\gamma_n \downarrow 0$, and set

$$p\colon \omega \mapsto {}^{0}\mathsf{A}_\omega\big(x(\omega)\big) \quad \text{and} \quad (\forall n\in\mathbb{N})\ \ p_n\colon \omega \mapsto {}^{\gamma_n}\mathsf{A}_\omega\big(x(\omega)\big). \tag{3.19}$$

Then, on account of (ii)(b),

$$(\forall n\in\mathbb{N})\quad p_n \in \mathfrak{H} \quad \text{and} \quad p_n = {}^{\gamma_n}Ax. \tag{3.20}$$

(v)(a): For every $\omega\in\Omega$, since A_ω is maximally monotone and $x(\omega)\in\operatorname{dom}\mathsf{A}_\omega$, [3, Corollary 23.46(i)] yields $p_n(\omega)\to p(\omega)$. Hence, thanks to Lemma 2.2(iv), we obtain $p\in\mathfrak{G}$.

(v)(b): Set $q = {}^{0}Ax$. It follows from (3.20), (i), and [3, Corollary 23.46(i)] that $p_n \to q$ in $\mathcal{H}$. Thus, we infer from Lemma 2.3 that there exists a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $(\forall^{\mu}\omega\in\Omega)$ $p_{k_n}(\omega)\to q(\omega)$. In turn, $p=q$ μ-a.e. and we conclude by invoking Lemma 2.2(ii). ☐

Example 3.9. Consider the setting of Example 2.1(iii) and suppose that, in addition, $(\forall k\in\mathbb{N})$ $\alpha_k = 1$ and $\mathsf{H}_k = \mathbb{R}$. Then $\mathcal{H} = \ell^2(\mathbb{N})$. Now define $(\forall k\in\mathbb{N})$ $\mathsf{A}_k\colon \mathsf{H}_k\to\mathsf{H}_k\colon \mathsf{x}\mapsto 2^k\mathsf{x}$. Then

$$\operatorname{dom}\bigg({}^{\mathfrak{G}}\!\int_{\Omega}^{\oplus} \mathsf{A}_\omega\mu(d\omega)\bigg) = \bigg\{(\mathsf{x}_k)_{k\in\mathbb{N}}\in\ell^2(\mathbb{N}) \ \bigg|\ \sum_{k\in\mathbb{N}} 4^k|\mathsf{x}_k|^2 < +\infty\bigg\} \neq \ell^2(\mathbb{N}) = {}^{\mathfrak{G}}\!\int_{\Omega}^{\oplus} \operatorname{dom}\mathsf{A}_\omega\,\mu(d\omega). \tag{3.21}$$

The closure operation in items (iii) and (iv) in Theorem 3.8 can therefore not be omitted.

Corollary 3.10. *Let $(\Omega,\mathcal{F},\mu)$ be a complete σ-finite measure space, let H be a separable real Hilbert space, and for every $\omega\in\Omega$, let $\mathsf{A}_\omega\colon\mathsf{H}\to 2^{\mathsf{H}}$ be maximally monotone. Set $\mathcal{H} = L^2(\Omega,\mathcal{F},\mu;\mathsf{H})$ and*

$$A\colon\mathcal{H}\to 2^{\mathcal{H}}\colon x\mapsto \big\{x^*\in\mathcal{H} \mid (\forall^{\mu}\omega\in\Omega)\ x^*(\omega)\in\mathsf{A}_\omega\big(x(\omega)\big)\big\}. \tag{3.22}$$

Suppose that $\operatorname{dom} A\neq\varnothing$. Then the following are equivalent:

(i) *A is maximally monotone.*

(ii) *For every $\mathsf{x}\in\mathsf{H}$, the mapping $\Omega\to\mathsf{H}\colon\omega\mapsto J_{\mathsf{A}_\omega}\mathsf{x}$ is $(\mathcal{F},\mathcal{B}_{\mathsf{H}})$-measurable.*

(iii) *For every open set $\mathbf{V}$ in $\mathsf{H}\oplus\mathsf{H}$, $\big\{\omega\in\Omega \mid \mathbf{V}\cap\operatorname{gra}\mathsf{A}_\omega\neq\varnothing\big\}\in\mathcal{F}$.*

Proof. In the light of Example 2.1(iv), $\mathcal{H}$ is the Hilbert direct integral of the $\mathcal{F}$-measurable vector field $((\mathsf{H}_\omega)_{\omega\in\Omega},\mathfrak{G})$ defined by

$$\big[\ (\forall\omega\in\Omega)\ \ \mathsf{H}_\omega = \mathsf{H}\ \big] \quad \text{and} \quad \mathfrak{G} = \big\{x\colon\Omega\to\mathsf{H} \mid x \text{ is } (\mathcal{F},\mathcal{B}_{\mathsf{H}})\text{-measurable}\big\}. \tag{3.23}$$

Additionally, by (3.22),

$$A = {}^{\mathfrak{G}}\!\int_{\Omega}^{\oplus} \mathsf{A}_\omega\mu(d\omega). \tag{3.24}$$

(i)⇒(ii): We have $\operatorname{dom} A \neq \varnothing$ and $J_A : \mathcal{H} \to \mathcal{H}$ [3, Corollary 23.11(i)]. Thus, invoking Proposition 3.5 and Lemma 2.2(ii), we deduce that

$$\big(\forall x \in \mathscr{L}^2(\Omega, \mathcal{F}, \mu; \mathsf{H})\big) \quad \text{the mapping } \Omega \to \mathsf{H} \colon \omega \mapsto J_{\mathsf{A}_\omega}\big(x(\omega)\big) \text{ lies in } \mathscr{L}^2\big(\Omega, \mathcal{F}, \mu; \mathsf{H}\big). \tag{3.25}$$

Next, take $\mathsf{x} \in \mathsf{H}$. Since $(\Omega, \mathcal{F}, \mu)$ is σ-finite, there exists an increasing sequence $(\Omega_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ of finite μ-measure sets such that $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$. In turn, $\{1_{\Omega_n}\mathsf{x}\}_{n\in\mathbb{N}} \subset \mathscr{L}^2(\Omega, \mathcal{F}, \mu; \mathsf{H})$ and $(\forall \omega \in \Omega)$ $1_{\Omega_n}(\omega)\mathsf{x} \to \mathsf{x}$. Hence, on account of (3.25), we deduce that, for every $n \in \mathbb{N}$, the mapping $\Omega \to \mathsf{H} \colon \omega \mapsto J_{\mathsf{A}_\omega}(1_{\Omega_n}(\omega)\mathsf{x})$ is $(\mathcal{F}, \mathcal{B}_{\mathsf{H}})$-measurable. In addition, the continuity of the operators $(J_{\mathsf{A}_\omega})_{\omega\in\Omega}$ yields $(\forall \omega \in \Omega)$ $J_{\mathsf{A}_\omega}(1_{\Omega_n}(\omega)\mathsf{x}) \to J_{\mathsf{A}_\omega}\mathsf{x}$. Altogether, it results from Lemma 2.2(iv) that the mapping $\Omega \to \mathsf{H} \colon \omega \mapsto J_{\mathsf{A}_\omega}\mathsf{x}$ is $(\mathcal{F}, \mathcal{B}_{\mathsf{H}})$-measurable.

(ii)⇒(i): Applying [14, Lemma III.14] to the mapping $\Omega \times \mathsf{H} \to \mathsf{H} \colon (\omega, \mathsf{x}) \mapsto J_{\mathsf{A}_\omega}\mathsf{x}$, we deduce that, for every $x \in \mathfrak{G}$, the mapping $\omega \mapsto J_{\mathsf{A}_\omega}(x(\omega))$ lies in $\mathfrak{G}$. Therefore, in the setting of (3.23), the family $(\mathsf{A}_\omega)_{\omega\in\Omega}$ satisfies Assumption 3.6. Consequently, we conclude via (3.24) and Theorem 3.8(i) that A is maximally monotone.

(ii)⇔(iii): Combine [1, Lemme 2.1] and [1, Théorème 2.1]. ☐

Remark 3.11. The implication (iii)⇒(i) in Corollary 3.10 is stated in [32, Theorem 5.1].

Proposition 3.12. *Suppose that Assumption 1.2 is in force. Let* G *be a separable real Hilbert space and, for every* $\omega \in \Omega$*, let* $\mathsf{L}_\omega \colon \mathsf{G} \to \mathsf{H}_\omega$ *be linear and bounded. Suppose that, for every* $\mathsf{z} \in \mathsf{G}$*, the mapping*

$$\mathfrak{e}_{\mathsf{L}}\mathsf{z} \colon \omega \mapsto \mathsf{L}_\omega \mathsf{z} \tag{3.26}$$

lies in $\mathfrak{G}$*. Then the following holds:*

(i) *The function* $\Omega \to \mathbb{R} \colon \omega \mapsto \|\mathsf{L}_\omega\|$ *is* $\mathcal{F}$*-measurable.*

Suppose additionally that $\int_\Omega \|\mathsf{L}_\omega\|^2 \mu(d\omega) < +\infty$ *and define*

$$L \colon \mathsf{G} \to \mathcal{H} \colon \mathsf{z} \mapsto \mathfrak{e}_{\mathsf{L}}\mathsf{z}. \tag{3.27}$$

Then the following hold:

(ii) *L is well defined, linear, and bounded with* $\|L\| \leqslant \sqrt{\int_\Omega \|\mathsf{L}_\omega\|^2 \mu(d\omega)}$.

(iii) *Let* $x^* \in \mathfrak{G}$*. Then the mapping* $\Omega \to \mathsf{G} \colon \omega \mapsto \mathsf{L}_\omega^*(x^*(\omega))$ *is* $(\mathcal{F}, \mathcal{B}_{\mathsf{G}})$*-measurable.*

(iv) *Let* $x^* \in \mathfrak{H}$*. Then the mapping* $\Omega \to \mathsf{G} \colon \omega \mapsto \mathsf{L}_\omega^*(x^*(\omega))$ *is Lebesgue* μ*-integrable.*

(v) $L^* \colon \mathcal{H} \to \mathsf{G} \colon x^* \mapsto \int_\Omega \mathsf{L}_\omega^*(x^*(\omega))\mu(d\omega)$.

Proof. (i): Let $\{\mathsf{z}_n\}_{n\in\mathbb{N}}$ be a dense subset of $\{\mathsf{z} \in \mathsf{G} \mid \|\mathsf{z}\|_{\mathsf{G}} \leqslant 1\}$. On the one hand, property [A] in Definition 1.1 ensures that, for every $n \in \mathbb{N}$, the function $\Omega \to \mathbb{R} \colon \omega \mapsto \|\mathsf{L}_\omega \mathsf{z}_n\|_{\mathsf{H}_\omega}$ is $\mathcal{F}$-measurable. On the other hand, thanks to the continuity of the operators $(\mathsf{L}_\omega)_{\omega\in\Omega}$,

$$(\forall \omega \in \Omega) \quad \|\mathsf{L}_\omega\| = \sup_{\substack{\mathsf{z}\in\mathsf{G} \\ \|\mathsf{z}\|_{\mathsf{G}}\leqslant 1}} \|\mathsf{L}_\omega \mathsf{z}\|_{\mathsf{H}_\omega} = \sup_{n\in\mathbb{N}} \|\mathsf{L}_\omega \mathsf{z}_n\|_{\mathsf{H}_\omega}. \tag{3.28}$$

Altogether, the function $\Omega \to \mathbb{R} \colon \omega \mapsto \|\mathsf{L}_\omega\|$ is $\mathcal{F}$-measurable.

(ii): For every $\mathsf{z} \in \mathsf{G}$, we deduce from (3.26) that

$$\int_\Omega \|(\mathfrak{e}_{\mathsf{L}}\mathsf{z})(\omega)\|_{\mathsf{H}_\omega}^2 \mu(d\omega) = \int_\Omega \|\mathsf{L}_\omega \mathsf{z}\|_{\mathsf{H}_\omega}^2 \mu(d\omega) \leqslant \|\mathsf{z}\|_{\mathsf{G}}^2 \int_\Omega \|\mathsf{L}_\omega\|^2 \mu(d\omega) < +\infty \tag{3.29}$$

and, in turn, from (1.6) that $\mathfrak{e}_{\mathsf{L}}\mathsf{z} \in \mathfrak{H}$. This confirms that L is well defined. In addition, the linearity of the operators $(\mathsf{L}_\omega)_{\omega\in\Omega}$ guarantees that of L. The last claims follow from (3.29) and (1.7).

(iii): For every $\mathsf{z} \in \mathsf{G}$, Lemma 2.2(i) implies that the function $\Omega \to \mathbb{R}\colon \omega \mapsto \langle \mathsf{z} \mid \mathsf{L}_\omega^*(x^*(\omega))\rangle_{\mathsf{G}} = \langle \mathsf{L}_\omega \mathsf{z} \mid x^*(\omega)\rangle_{\mathsf{H}_\omega}$ is $\mathcal{F}$-measurable. In turn, invoking the separability of G, as well as the fact that $(\Omega, \mathcal{F}, \mu)$ is complete and σ-finite, we derive from [36, Théorème 5.6.24] that the mapping $\Omega \to \mathsf{G}\colon \omega \mapsto \mathsf{L}_\omega^*(x^*(\omega))$ is $(\mathcal{F}, \mathcal{B}_{\mathsf{G}})$-measurable.

(iv): By the Cauchy–Schwarz inequality,

$$\begin{aligned}\int_\Omega \big\|\mathsf{L}_\omega^*(x^*(\omega))\big\|_{\mathsf{G}}\,\mu(d\omega) &\leqslant \int_\Omega \|\mathsf{L}_\omega\|\,\|x^*(\omega)\|_{\mathsf{H}_\omega}\mu(d\omega)\\ &\leqslant \sqrt{\int_\Omega \|\mathsf{L}_\omega\|^2\mu(d\omega)}\sqrt{\int_\Omega \|x^*(\omega)\|^2_{\mathsf{H}_\omega}\mu(d\omega)}\\ &< +\infty. \end{aligned}\tag{3.30}$$

Hence, the assertion follows from [36, Théorème 5.7.21].

(v): Take $x^* \in \mathcal{H}$. It results from (1.7), (3.27), (3.26), (iv), and [36, Théorème 5.8.16] that

$$\begin{aligned}(\forall \mathsf{z} \in \mathsf{G})\quad \langle \mathsf{z} \mid L^*x^*\rangle_{\mathsf{G}} &= \langle L\mathsf{z} \mid x^*\rangle_{\mathcal{H}}\\ &= \int_\Omega \langle \mathsf{L}_\omega \mathsf{z} \mid x^*(\omega)\rangle_{\mathsf{H}_\omega}\mu(d\omega)\\ &= \int_\Omega \big\langle \mathsf{z} \,\big|\, \mathsf{L}_\omega^*(x^*(\omega))\big\rangle_{\mathsf{G}}\,\mu(d\omega)\\ &= \left\langle \mathsf{z} \,\middle|\, \int_\Omega \mathsf{L}_\omega^*(x^*(\omega))\mu(d\omega)\right\rangle_{\mathsf{G}}, \end{aligned}\tag{3.31}$$

which completes the proof. ☐

§4. Hilbert direct integrals of functions

We study the Hilbert direct integrals of families of functions introduced in Definition 1.4.

Lemma 4.1. *Let $\mathcal{H}$ be a real Hilbert space and let $T\colon \mathcal{H} \to \mathcal{H}$. Then the following hold:*

(i) *There exists $f \in \Gamma_0(\mathcal{H})$ such that $T = \operatorname{prox}_f$ if and only if T is nonexpansive and cyclically monotone, that is, for every $2 \leqslant n \in \mathbb{N}$ and every $(x_1, \ldots, x_{n+1}) \in \mathcal{H}^{n+1}$ such that $x_{n+1} = x_1$,*

$$\sum_{k=1}^{n} \langle x_{k+1} - x_k \mid Tx_k\rangle_{\mathcal{H}} \leqslant 0. \tag{4.1}$$

(ii) *There exists a nonempty closed convex subset C of $\mathcal{H}$ such that $T = \operatorname{proj}_C$ if and only if*

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H})\quad \langle Ty - Tx \mid x - Tx\rangle_{\mathcal{H}} \leqslant 0. \tag{4.2}$$

Proof. (i): The core of our argument is implicitly in [28, Corollaire 10.c]. Suppose that there exists $f \in \Gamma_0(\mathcal{H})$ such that $T = \operatorname{prox}_f$. Then, on account of [28, Corollaire 10.c] and [3, Proposition 22.14], T is nonexpansive and cyclically monotone. Conversely, suppose that T is nonexpansive and cyclically monotone. Then T is monotone and it thus follows from [3, Corollary 20.28] that T is maximally monotone. Therefore, Rockafellar's cyclic monotonicity theorem [3, Theorem 22.18] guarantees the existence of a function $\varphi \in \Gamma_0(\mathcal{H})$ such that $T = \partial\varphi$. We conclude by invoking [28, Corollaire 10.c].

(ii): See [38, Theorem 1.1]. ☐

Remark 4.2. In connection with Lemma 4.1(i), a characterization of proximity operators based on firm nonexpansiveness and an alternative cyclic inequality is provided in [2, Theorem 6.6].

In [27, 28], Moreau showed that the convex combination of finitely many proximity operators acting on the same Hilbert space is a proximity operator. Here is a generalization of this result.

Theorem 4.3. *Suppose that Assumption 1.2 is in force. Let* G *be a separable real Hilbert space and, for every* $\omega \in \Omega$*, let* $\mathsf{f}_\omega \in \Gamma_0(\mathsf{H}_\omega)$ *and let* $\mathsf{L}_\omega \colon \mathsf{G} \to \mathsf{H}_\omega$ *be linear and bounded. Suppose that the following are satisfied:*

[A] *For every* $x \in \mathfrak{H}$*, the mapping* $\omega \mapsto \operatorname{prox}_{\mathsf{f}_\omega}(x(\omega))$ *lies in* $\mathfrak{G}$.
[B] *There exists* $z \in \mathfrak{H}$ *such that the mapping* $\omega \mapsto \operatorname{prox}_{\mathsf{f}_\omega}(z(\omega))$ *lies in* $\mathfrak{H}$.
[C] *For every* $\mathsf{z} \in \mathsf{G}$*, the mapping* $\mathfrak{e}_{\mathsf{L}}\mathsf{z} \colon \omega \mapsto \mathsf{L}_\omega \mathsf{z}$ *lies in* $\mathfrak{G}$.
[D] $\int_\Omega \|\mathsf{L}_\omega\|^2 \mu(d\omega) \leqslant 1$.

Then

$$\big(\exists\, \mathsf{g} \in \Gamma_0(\mathsf{G})\big)(\forall \mathsf{z} \in \mathsf{G}) \quad \operatorname{prox}_{\mathsf{g}} \mathsf{z} = \int_\Omega \mathsf{L}_\omega^*\big(\operatorname{prox}_{\mathsf{f}_\omega}(\mathsf{L}_\omega \mathsf{z})\big)\mu(d\omega). \tag{4.3}$$

Proof. Set $T = {}^{\mathfrak{G}}\!\!\int_\Omega^\oplus \operatorname{prox}_{\mathsf{f}_\omega} \mu(d\omega)$. Then, on account of Proposition 3.4(i), $T \colon \mathcal{H} \to \mathcal{H}$ is nonexpansive. Next, items (ii) and (v) of Proposition 3.12 ensure that the operator $L \colon \mathsf{G} \to \mathcal{H} \colon \mathsf{z} \mapsto \mathfrak{e}_{\mathsf{L}}\mathsf{z}$ is well defined, linear, and bounded, with $\|L\| \leqslant 1$, and its adjoint is given by

$$L^* \colon \mathcal{H} \to \mathsf{G} \colon x^* \mapsto \int_\Omega \mathsf{L}_\omega^*\big(x^*(\omega)\big)\mu(d\omega). \tag{4.4}$$

Hence, $L^* \circ T \circ L \colon \mathsf{G} \to \mathsf{G}$ is nonexpansive and

$$(\forall \mathsf{z} \in \mathsf{G}) \quad L^*\big(T(L\mathsf{z})\big) = \int_\Omega \mathsf{L}_\omega^*\big(\operatorname{prox}_{\mathsf{f}_\omega}(\mathsf{L}_\omega \mathsf{z})\big)\mu(d\omega). \tag{4.5}$$

Therefore, in the light of Lemma 4.1(i), it remains to show that $L^* \circ T \circ L$ is cyclically monotone. Towards this end, let $2 \leqslant n \in \mathbb{N}$ and let $(\mathsf{z}_1, \ldots, \mathsf{z}_{n+1}) \in \mathsf{G}^{n+1}$ be such that $\mathsf{z}_{n+1} = \mathsf{z}_1$. Then, appealing to the cyclic monotonicity of the operators $(\operatorname{prox}_{\mathsf{f}_\omega})_{\omega \in \Omega}$,

$$(\forall \omega \in \Omega) \quad \sum_{k=1}^n \big\langle \mathsf{L}_\omega \mathsf{z}_{k+1} - \mathsf{L}_\omega \mathsf{z}_k \,\big|\, \operatorname{prox}_{\mathsf{f}_\omega}(\mathsf{L}_\omega \mathsf{z}_k)\big\rangle_{\mathsf{H}_\omega} \leqslant 0. \tag{4.6}$$

Thus, it follows from (1.7) that

$$\begin{aligned}
\sum_{k=1}^n \big\langle \mathsf{z}_{k+1} - \mathsf{z}_k \,\big|\, L^*\big(T(L\mathsf{z}_k)\big)\big\rangle_{\mathsf{G}} &= \sum_{k=1}^n \langle L\mathsf{z}_{k+1} - L\mathsf{z}_k \mid T(L\mathsf{z}_k)\rangle_{\mathcal{H}} \\
&= \sum_{k=1}^n \int_\Omega \big\langle \mathsf{L}_\omega \mathsf{z}_{k+1} - \mathsf{L}_\omega \mathsf{z}_k \,\big|\, \operatorname{prox}_{\mathsf{f}_\omega}(\mathsf{L}_\omega \mathsf{z}_k)\big\rangle_{\mathsf{H}_\omega} \mu(d\omega) \\
&= \int_\Omega \sum_{k=1}^n \big\langle \mathsf{L}_\omega \mathsf{z}_{k+1} - \mathsf{L}_\omega \mathsf{z}_k \,\big|\, \operatorname{prox}_{\mathsf{f}_\omega}(\mathsf{L}_\omega \mathsf{z}_k)\big\rangle_{\mathsf{H}_\omega} \mu(d\omega) \\
&\leqslant 0,
\end{aligned} \tag{4.7}$$

which concludes the proof. ☐

Remark 4.4. Identifying the function g in (4.3) is a natural question, which led to the introduction of the notion of integral proximal mixtures in [12].

Proposition 4.5. *Suppose that Assumption 1.2 is in force and, for every $\omega \in \Omega$, let $\mathsf{A}_\omega \colon \mathsf{H}_\omega \to 2^{\mathsf{H}_\omega}$ be maximally monotone. Set*

$$A = \int_\Omega^{\mathfrak{G}\oplus} \mathsf{A}_\omega \mu(d\omega). \tag{4.8}$$

Then the following hold:

(i) *Suppose that there exists $f \in \Gamma_0(\mathcal{H})$ such that $A = \partial f$. Then, for μ-almost every $\omega \in \Omega$, there exists $\mathsf{f}_\omega \in \Gamma_0(\mathsf{H}_\omega)$ such that $\mathsf{A}_\omega = \partial \mathsf{f}_\omega$.*

(ii) *Suppose that there exists a nonempty closed convex subset C of $\mathcal{H}$ such that $A = N_C$. Then, for μ-almost every $\omega \in \Omega$, there exists a nonempty closed convex subset C_ω of H_ω such that $\mathsf{A}_\omega = N_{\mathsf{C}_\omega}$.*

Proof. Lemma 2.2(v) asserts that there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathfrak{H}$ such that

$$(\forall \omega \in \Omega) \quad \overline{\big\{u_n(\omega)\big\}_{n\in\mathbb{N}}} = \mathsf{H}_\omega. \tag{4.9}$$

(i): Set $\mathbb{I} = \big\{(i_k)_{1\leqslant k\leqslant n+1} \in \mathbb{N}^{n+1} \;\big|\; 2 \leqslant n \in \mathbb{N} \text{ and } i_{n+1} = i_1\big\}$, fix temporarily $\mathbf{i} = (i_k)_{1\leqslant k\leqslant n+1} \in \mathbb{I}$, and let $\Theta \in \mathcal{F}$ be such that $\mu(\Theta) < +\infty$. Then, by Lemma 2.2(iii), $\{1_\Theta u_{i_k}\}_{1\leqslant k\leqslant n} \subset \mathfrak{H}$. In turn, since $J_A \colon \mathcal{H} \to \mathcal{H}$, it follows from Proposition 3.5 and Lemma 2.2(ii) that, for every $k \in \{1,\ldots,n\}$, a representative in $\mathfrak{H}$ of $J_A(1_\Theta u_{i_k})$ is the mapping

$$\omega \mapsto \begin{cases} J_{\mathsf{A}_\omega}\big(u_{i_k}(\omega)\big), & \text{if } \omega \in \Theta;\\ J_{\mathsf{A}_\omega}0, & \text{if } \omega \in \complement\Theta. \end{cases} \tag{4.10}$$

At the same time, for every $k \in \{1,\ldots,n\}$, a representative in $\mathfrak{H}$ of $1_\Theta u_{i_k}$ is the mapping

$$\omega \mapsto \begin{cases} u_{i_k}(\omega), & \text{if } \omega \in \Theta;\\ 0, & \text{if } \omega \in \complement\Theta. \end{cases} \tag{4.11}$$

Hence, since $J_A = \operatorname{prox}_f$ is cyclically monotone by virtue of [3, Example 23.3] and Lemma 4.1, we derive from (1.7) that

$$\int_\Theta \sum_{k=1}^n \Big\langle u_{i_{k+1}}(\omega) - u_{i_k}(\omega) \,\Big|\, J_{\mathsf{A}_\omega}\big(u_{i_k}(\omega)\big)\Big\rangle_{\mathsf{H}_\omega} \mu(d\omega) = \sum_{k=1}^n \langle 1_\Theta u_{i_{k+1}} - 1_\Theta u_{i_k} \mid J_A(1_\Theta u_{i_k})\rangle_{\mathcal{H}} \leqslant 0. \tag{4.12}$$

Therefore, thanks to the fact that $(\Omega, \mathcal{F}, \mu)$ is σ-finite, there exists $\Xi_{\mathbf{i}} \in \mathcal{F}$ such that

$$\mu(\Xi_{\mathbf{i}}) = 0 \quad \text{and} \quad (\forall \omega \in \complement\Xi_{\mathbf{i}}) \;\; \sum_{k=1}^n \Big\langle u_{i_{k+1}}(\omega) - u_{i_k}(\omega) \,\Big|\, J_{\mathsf{A}_\omega}\big(u_{i_k}(\omega)\big)\Big\rangle_{\mathsf{H}_\omega} \leqslant 0. \tag{4.13}$$

Now set $\Xi = \bigcup_{\mathbf{i}\in\mathbb{I}} \Xi_{\mathbf{i}}$. Since $\mathbb{I}$ is countable, $\Xi \in \mathcal{F}$ and $\mu(\Xi) = 0$. Additionally, (4.13) implies that

$$\big(\forall \mathbf{i} = (i_k)_{1\leqslant k\leqslant n+1} \in \mathbb{I}\big)\big(\forall \omega \in \complement\Xi\big) \;\; \sum_{k=1}^n \Big\langle u_{i_{k+1}}(\omega) - u_{i_k}(\omega) \,\Big|\, J_{\mathsf{A}_\omega}\big(u_{i_k}(\omega)\big)\Big\rangle_{\mathsf{H}_\omega} \leqslant 0. \tag{4.14}$$

To proceed further, take $\omega \in \complement\Xi$, let $2 \leqslant n \in \mathbb{N}$, and let $(\mathsf{x}_1, \ldots, \mathsf{x}_{n+1})$ be a family in H_ω such that $\mathsf{x}_{n+1} = \mathsf{x}_1$. For every $k \in \{1, \ldots, n\}$, we infer from (4.9) that there exists a sequence $(i_{k,m})_{m\in\mathbb{N}}$ in $\mathbb{N}$ such that $u_{i_{k,m}}(\omega) \to \mathsf{x}_k$. Set $(\forall m \in \mathbb{N})\ i_{n+1,m} = i_{1,m}$. Then, for every $m \in \mathbb{N}$, because $(i_{k,m})_{1\leqslant k\leqslant n+1} \in \mathbb{I}$, it results from (4.14) that

$$\sum_{k=1}^{n} \Big\langle u_{i_{k+1,m}}(\omega) - u_{i_{k,m}}(\omega) \,\Big|\, J_{\mathsf{A}_\omega}\big(u_{i_{k,m}}(\omega)\big)\Big\rangle_{\mathsf{H}_\omega} \leqslant 0. \tag{4.15}$$

Therefore, the continuity of J_{A_ω} forces $\sum_{k=1}^{n} \langle \mathsf{x}_{k+1} - \mathsf{x}_k \mid J_{\mathsf{A}_\omega}\mathsf{x}_k\rangle_{\mathsf{H}_\omega} \leqslant 0$. Consequently, since J_{A_ω} is nonexpansive, we conclude via Lemma 4.1(i) that there exists $\mathsf{f}_\omega \in \Gamma_0(\mathsf{H}_\omega)$ such that $J_{\mathsf{A}_\omega} = \operatorname{prox}_{\mathsf{f}_\omega}$.

(ii): Argue as in (i). ☐

Let us collect the main properties of Hilbert direct integral functions under the umbrella of the following assumption.

Assumption 4.6. Assumption 1.2 and the following are in force:

[A] For every $\omega \in \Omega$, $\mathsf{f}_\omega \in \Gamma_0(\mathsf{H}_\omega)$.

[B] For every $x \in \mathfrak{H}$, the mapping $\omega \mapsto \operatorname{prox}_{\mathsf{f}_\omega}(x(\omega))$ lies in $\mathfrak{G}$.

[C] There exists $r \in \mathfrak{H}$ such that the function $\omega \mapsto \mathsf{f}_\omega(r(\omega))$ lies in $\mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$.

[D] There exist $s^* \in \mathfrak{H}$ and $\vartheta \in \mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ such that

$$(\forall^\mu \omega \in \Omega) \quad \mathsf{f}_\omega \geqslant \langle \cdot \mid s^*(\omega)\rangle_{\mathsf{H}_\omega} + \vartheta(\omega). \tag{4.16}$$

The following theorem presents the main properties of Hilbert direct integrals of convex functions. In the literature, such properties are available only in the setting of Examples 2.1(i) and 2.1(iv); see [3, 11, 14, 35], where different techniques are employed which are not applicable in our general context.

Theorem 4.7. *Suppose that Assumption 4.6 is in force and define*

$$f = {}^{\mathfrak{G}}\!\!\int_{\Omega}^{\oplus} \mathsf{f}_\omega \mu(d\omega). \tag{4.17}$$

Then the following hold:

(i) *f is well defined.*

(ii) *$f \in \Gamma_0(\mathcal{H})$.*

(iii) *$\partial f = {}^{\mathfrak{G}}\!\!\int_{\Omega}^{\oplus} \partial\mathsf{f}_\omega \mu(d\omega)$.*

(iv) *Let $\gamma \in \left]0, +\infty\right[$ and $x \in \mathfrak{H}$. Then the mapping $\omega \mapsto \operatorname{prox}_{\gamma\mathsf{f}_\omega}(x(\omega))$ lies in $\mathfrak{H}$ and $\operatorname{prox}_{\gamma f} = {}^{\mathfrak{G}}\!\!\int_{\Omega}^{\oplus} \operatorname{prox}_{\gamma\mathsf{f}_\omega} \mu(d\omega)$.*

(v) $\overline{\operatorname{dom} f} = {}^{\mathfrak{G}}\!\!\int_{\Omega}^{\oplus} \overline{\operatorname{dom} \mathsf{f}_\omega}\, \mu(d\omega) = \overline{{}^{\mathfrak{G}}\!\!\int_{\Omega}^{\oplus} \operatorname{dom} \mathsf{f}_\omega\, \mu(d\omega)}$.

(vi) $\operatorname{Argmin} f = {}^{\mathfrak{G}}\!\!\int_{\Omega}^{\oplus} \operatorname{Argmin} \mathsf{f}_\omega\, \mu(d\omega)$.

(vii) *Let $\beta \in \left[0, +\infty\right[$ and suppose that, for every $\omega \in \Omega$, $\operatorname{dom} \mathsf{f}_\omega = \mathsf{H}_\omega$ and f_ω is Gâteaux differentiable on H_ω. Then the following are equivalent:*

(a) *For μ-almost every $\omega \in \Omega$, $\nabla \mathsf{f}_\omega$ is β-Lipschitzian.*

(b) $\operatorname{dom} f = \mathcal{H}$*, f is Fréchet differentiable, and ∇f is β-Lipschitzian.*

(viii) *Let $\gamma \in \left]0, +\infty\right[$. Then* $^{\gamma} f = \displaystyle{}^{\mathfrak{G}}\!\!\int_\Omega^{\oplus} {}^{\gamma}\mathsf{f}_\omega \mu(d\omega)$.

(ix) $f^* = \displaystyle{}^{\mathfrak{G}}\!\!\int_\Omega^{\oplus} \mathsf{f}_\omega^* \mu(d\omega)$.

(x) $\operatorname{rec} f = \displaystyle{}^{\mathfrak{G}}\!\!\int_\Omega^{\oplus} \operatorname{rec} \mathsf{f}_\omega \, \mu(d\omega)$.

Proof. According to (4.16), there exists $\Xi \in \mathcal{F}$ such that

$$\mu(\Xi) = 0 \tag{4.18}$$

and

$$(\forall x \in \mathfrak{G})(\forall \omega \in \complement\Xi) \quad \mathsf{f}_\omega\big(x(\omega)\big) \geqslant \langle x(\omega) \mid s^*(\omega)\rangle_{\mathsf{H}_\omega} + \vartheta(\omega). \tag{4.19}$$

Let us define

$$p \colon \omega \mapsto \operatorname{prox}_{\mathsf{f}_\omega}\big(r(\omega) + s^*(\omega)\big). \tag{4.20}$$

Since $r + s^* \in \mathfrak{H}$, Assumption 4.6[B] ensures that $p \in \mathfrak{G}$. In addition, we deduce from [3, Proposition 16.44] that

$$(\forall \omega \in \Omega) \quad r(\omega) + s^*(\omega) - p(\omega) \in \partial \mathsf{f}_\omega\big(p(\omega)\big) \tag{4.21}$$

and, in turn, from (2.2) and (4.19) that

$$\begin{aligned}(\forall \omega \in \complement\Xi) \quad & \mathsf{f}_\omega\big(r(\omega)\big) - \langle r(\omega) \mid s^*(\omega)\rangle_{\mathsf{H}_\omega} \\ & \geqslant \mathsf{f}_\omega\big(p(\omega)\big) + \big\langle r(\omega) - p(\omega) \,\big|\, r(\omega) + s^*(\omega) - p(\omega)\big\rangle_{\mathsf{H}_\omega} - \langle r(\omega) \mid s^*(\omega)\rangle_{\mathsf{H}_\omega} \\ & = \mathsf{f}_\omega\big(p(\omega)\big) - \langle p(\omega) \mid s^*(\omega)\rangle_{\mathsf{H}_\omega} + \|r(\omega) - p(\omega)\|_{\mathsf{H}_\omega}^2 \\ & \geqslant \vartheta(\omega) + \|r(\omega) - p(\omega)\|_{\mathsf{H}_\omega}^2.\end{aligned} \tag{4.22}$$

On the other hand, thanks to items [C] and [D] in Assumption 4.6, the function $\omega \mapsto \mathsf{f}_\omega(r(\omega)) - \langle r(\omega) \mid s^*(\omega)\rangle_{\mathsf{H}_\omega} - \vartheta(\omega)$ lies in $\mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$. Therefore, it results from (4.22) that $r - p \in \mathfrak{H}$ and, since $r \in \mathfrak{H}$ by Assumption 4.6[C], we get

$$p \in \mathfrak{H}. \tag{4.23}$$

Now set

$$A = {}^{\mathfrak{G}}\!\!\int_\Omega^{\oplus} \partial \mathsf{f}_\omega \mu(d\omega). \tag{4.24}$$

Assumption 4.6[A] and [28, Proposition 12.b] imply that the operators $(\partial \mathsf{f}_\omega)_{\omega \in \Omega}$ are maximally monotone. Moreover, since $r + s^* \in \mathfrak{H}$, we infer from (4.21) and (4.23) that $p \in \operatorname{dom} A$. Moreover, Assumption 4.6[B] and [3, Example 23.3] guarantee that, for every $x \in \mathfrak{H}$, the mapping $\omega \mapsto J_{\partial \mathsf{f}_\omega}(x(\omega))$ lies in $\mathfrak{G}$. Altogether,

$$\text{the family } (\partial \mathsf{f}_\omega)_{\omega \in \Omega} \text{ satisfies the assumption of Theorem 3.8.} \tag{4.25}$$

Hence, it follows from Theorem 3.8(i) that

$$A \text{ is maximally monotone} \tag{4.26}$$

and from Theorem 3.8(ii)(a) and [3, Example 23.3] that

$$(\forall \gamma \in \left]0,+\infty\right[)(\forall x \in \mathfrak{H}) \quad \text{the mapping } \omega \mapsto \operatorname{prox}_{\gamma \mathsf{f}_\omega}\big(x(\omega)\big) \text{ lies in } \mathfrak{H}. \tag{4.27}$$

(i): We must show that, for every $x \in \mathfrak{H}$, the function $\Omega \to \left]-\infty,+\infty\right]\colon \omega \mapsto \mathsf{f}_\omega(x(\omega))$ is $\mathcal{F}$-measurable. To do so, we employ a Moreau envelope approximation technique inspired by [1]. Take $x \in \mathfrak{H}$. For every $\gamma \in \left]0,+\infty\right[$, let Ψ_γ be the mapping defined on $[0,1] \times \Omega$ by

$$\big(\forall (t,\omega) \in [0,1] \times \Omega\big) \quad \Psi_\gamma(t,\omega) = r(\omega) + t\big(x(\omega) - r(\omega)\big) - \operatorname{prox}_{\gamma \mathsf{f}_\omega}\Big(r(\omega) + t\big(x(\omega) - r(\omega)\big)\Big) \tag{4.28}$$

and define

$$\phi_\gamma\colon [0,1] \times \Omega \to \mathbb{R}\colon (t,\omega) \mapsto \big\langle x(\omega) - r(\omega) \,\big|\, \Psi_\gamma(t,\omega)\big\rangle_{\mathsf{H}_\omega}. \tag{4.29}$$

Then, for every $\gamma \in \left]0,+\infty\right[$, the continuity of the mappings $(\Psi_\gamma(\cdot,\omega))_{\omega\in\Omega}$ ensures that the functions $(\phi_\gamma(\cdot,\omega))_{\omega\in\Omega}$ are continuous, while (4.27) and Lemma 2.2(i) ensure that the functions $(\phi_\gamma(t,\cdot))_{t\in[0,1]}$ are $\mathcal{F}$-measurable. Hence, the functions $(\phi_\gamma)_{\gamma\in\left]0,+\infty\right[}$ are $\mathcal{B}_{[0,1]} \otimes \mathcal{F}$-measurable [14, Lemma III.14]. In turn, invoking the fact that $(\Omega,\mathcal{F},\mu)$ is σ-finite, we deduce that, for every $\gamma \in \left]0,+\infty\right[$, the function $\Omega \to \mathbb{R}\colon \omega \mapsto \int_0^1 \phi_\gamma(t,\omega)dt$ is $\mathcal{F}$-measurable. Therefore, for every $\gamma \in \left]0,+\infty\right[$, since [3, Proposition 12.30] implies that

$$(\forall \omega \in \Omega) \quad {}^{\gamma}\mathsf{f}_\omega\big(x(\omega)\big) - {}^{\gamma}\mathsf{f}_\omega\big(r(\omega)\big) = \gamma^{-1}\int_0^1 \big\langle x(\omega) - r(\omega) \,\big|\, \Psi_\gamma(t,\omega)\big\rangle_{\mathsf{H}_\omega} dt = \gamma^{-1}\int_0^1 \phi_\gamma(t,\omega)dt, \tag{4.30}$$

we infer that the function $\Omega \to \mathbb{R}\colon \omega \mapsto {}^{\gamma}\mathsf{f}_\omega(x(\omega)) - {}^{\gamma}\mathsf{f}_\omega(r(\omega))$ is $\mathcal{F}$-measurable. However, [3, Proposition 12.33(ii)] and Assumption 4.6[C] give

$$(\forall \omega \in \Omega) \quad \mathsf{f}_\omega\big(x(\omega)\big) - \mathsf{f}_\omega\big(r(\omega)\big) = \lim_{\gamma\downarrow 0}\Big({}^{\gamma}\mathsf{f}_\omega\big(x(\omega)\big) - {}^{\gamma}\mathsf{f}_\omega\big(r(\omega)\big)\Big). \tag{4.31}$$

Hence, the function $\Omega \to \left]-\infty,+\infty\right]\colon \omega \mapsto \mathsf{f}_\omega(x(\omega)) - \mathsf{f}_\omega(r(\omega))$ is $\mathcal{F}$-measurable. Consequently, invoking Assumption 4.6[C] once more, we conclude that the function $\Omega \to \left]-\infty,+\infty\right]\colon \omega \mapsto \mathsf{f}_\omega(x(\omega))$ is $\mathcal{F}$-measurable.

(ii): By (4.19), (4.18), and Assumption 4.6[D],

$$(\forall x \in \mathcal{H}) \quad f(x) = \int_\Omega \mathsf{f}_\omega\big(x(\omega)\big)\mu(d\omega) \geqslant \int_\Omega \langle x(\omega) \mid s^*(\omega)\rangle_{\mathsf{H}_\omega}\mu(d\omega) + \int_\Omega \vartheta(\omega)\mu(d\omega) > -\infty, \tag{4.32}$$

which yields

$$-\infty \notin f(\mathcal{H}). \tag{4.33}$$

At the same time, since the functions $(\mathsf{f}_\omega)_{\omega\in\Omega}$ are convex by Assumption 4.6[A], so is f. Moreover, Assumption 4.6[C] implies that $\operatorname{dom} f \neq \varnothing$. Therefore, it remains to show that f is lower semicontinuous. Take $\xi \in \mathbb{R}$, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$, let $x \in \mathcal{H}$, and suppose that

$$\sup_{n\in\mathbb{N}} f(x_n) \leqslant \xi \quad \text{and} \quad x_n \to x. \tag{4.34}$$

Then Lemma 2.3 asserts that there exists a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $(\forall^\mu \omega \in \Omega)$ $x_{k_n}(\omega) \to x(\omega)$. Let us define

$$(\forall n \in \mathbb{N}) \quad \varrho_n \colon \Omega \to \left]-\infty,+\infty\right] \colon \omega \mapsto \mathsf{f}_\omega\big(x_{k_n}(\omega)\big) - \langle x_{k_n}(\omega) \,|\, s^*(\omega)\rangle_{\mathsf{H}_\omega}. \tag{4.35}$$

By (i) and Lemma 2.2(i), the functions $(\varrho_n)_{n\in\mathbb{N}}$ are $\mathcal{F}$-measurable. Additionally,

$$(\forall n \in \mathbb{N}) \quad \varrho_n \geqslant \vartheta \ \ \mu\text{-a.e.} \quad \text{and} \quad \int_\Omega \varrho_n(\omega)\mu(d\omega) = f(x_{k_n}) - \langle x_{k_n} \,|\, s^*\rangle_{\mathcal{H}}, \tag{4.36}$$

and, since the functions $(\mathsf{f}_\omega)_{\omega\in\Omega}$ are lower semicontinuous, $(\forall^\mu \omega \in \Omega)$ $\mathsf{f}_\omega(x(\omega)) - \langle x(\omega) \,|\, s^*(\omega)\rangle_{\mathsf{H}_\omega} \leqslant \underline{\lim}\, \varrho_n(\omega)$. Thus, we derive from Fatou's lemma and (4.34) that

$$\begin{aligned} f(x) - \langle x \,|\, s^*\rangle_{\mathcal{H}} &= \int_\Omega \Big(\mathsf{f}_\omega\big(x(\omega)\big) - \langle x(\omega) \,|\, s^*(\omega)\rangle_{\mathsf{H}_\omega}\Big)\mu(d\omega) \\ &\leqslant \int_\Omega \underline{\lim}\, \varrho_n(\omega)\,\mu(d\omega) \\ &\leqslant \underline{\lim} \int_\Omega \varrho_n(\omega)\mu(d\omega) \\ &= \underline{\lim}\big(f(x_{k_n}) - \langle x_{k_n} \,|\, s^*\rangle_{\mathcal{H}}\big) \\ &\leqslant \xi - \langle x \,|\, s^*\rangle_{\mathcal{H}}. \end{aligned} \tag{4.37}$$

Hence $f(x) \leqslant \xi$ and we conclude via [3, Lemma 1.24] that f is lower semicontinuous.

(iii): Let $(x, x^*) \in \operatorname{gra} A$ and let $\Theta \in \mathcal{F}$ be such that $\mu(\Theta) = 0$ and $(\forall \omega \in \complement\Theta)$ $x^*(\omega) \in \partial\mathsf{f}_\omega(x(\omega))$. For every $y \in \mathcal{H}$, thanks to the inequalities

$$(\forall \omega \in \complement\Theta) \quad \big\langle y(\omega) - x(\omega) \,\big|\, x^*(\omega)\big\rangle_{\mathsf{H}_\omega} + \mathsf{f}_\omega\big(x(\omega)\big) \leqslant \mathsf{f}_\omega\big(y(\omega)\big), \tag{4.38}$$

we obtain $\langle y - x \,|\, x^*\rangle_{\mathcal{H}} + f(x) \leqslant f(y)$. Hence, $(x, x^*) \in \operatorname{gra} \partial f$ and we thus have $\operatorname{gra} A \subset \operatorname{gra} \partial f$. Consequently, the monotonicity of ∂f and (4.26) force $\partial f = A$.

(iv): Combine (ii), [3, Example 23.3], (iii), (4.25), and Theorem 3.8(ii)(a).

(v): We derive from (ii), [3, Proposition 16.38], (iii), (4.25), and Theorem 3.8(iii) that

$$\overline{\operatorname{dom}}\, f = \overline{\operatorname{dom}}\, \partial f = {}^{\mathfrak{G}}\!\!\int_\Omega^{\oplus} \overline{\operatorname{dom}}\, \partial\mathsf{f}_\omega \,\mu(d\omega) = {}^{\mathfrak{G}}\!\!\int_\Omega^{\oplus} \overline{\operatorname{dom}}\, \mathsf{f}_\omega \,\mu(d\omega). \tag{4.39}$$

This shows that ${}^{\mathfrak{G}}\!\int_\Omega^{\oplus} \overline{\operatorname{dom}}\, \mathsf{f}_\omega \,\mu(d\omega)$ is a closed subset of $\mathcal{H}$. On the other hand, for every $x \in \operatorname{dom} f$, it results from Definition 1.4 that, for μ-almost every $\omega \in \Omega$, $x(\omega) \in \operatorname{dom} \mathsf{f}_\omega$ and, therefore, that $x \in {}^{\mathfrak{G}}\!\int_\Omega^{\oplus} \operatorname{dom} \mathsf{f}_\omega \,\mu(d\omega)$. Consequently,

$$\overline{\operatorname{dom}}\, f \subset \overline{{}^{\mathfrak{G}}\!\!\int_\Omega^{\oplus} \operatorname{dom} \mathsf{f}_\omega \,\mu(d\omega)} \subset {}^{\mathfrak{G}}\!\!\int_\Omega^{\oplus} \overline{\operatorname{dom}}\, \mathsf{f}_\omega \,\mu(d\omega) = \overline{\operatorname{dom}}\, f, \tag{4.40}$$

which yields the desired identities.

(vi): This follows from Fermat's rule, (iii), and Proposition 3.2(iii).

(vii): Appealing to (4.25), we deduce from Theorem 3.8(v)(a) and [3, Proposition 17.31(i)] that, for every $x \in \mathfrak{H}$, the mapping $\omega \mapsto {}^0(\partial\mathsf{f}_\omega)(x(\omega)) = \nabla\mathsf{f}_\omega(x(\omega))$ lies in $\mathfrak{G}$. In addition, by (iii),

$$\partial f = {}^{\mathfrak{G}}\!\!\int_\Omega^{\oplus} \nabla\mathsf{f}_\omega \,\mu(d\omega). \tag{4.41}$$

Furthermore, for every $\omega \in \Omega$, [3, Corollary 17.40] asserts that $\nabla \mathsf{f}_\omega \colon \mathsf{H}_\omega \to \mathsf{H}_\omega$ is strong-to-weak continuous. Consequently, in the light of [3, Proposition 17.41], the assertion follows from Proposition 3.4(i) applied to the operators $(\nabla \mathsf{f}_\omega)_{\omega\in\Omega}$.

(viii): Take $x \in \mathfrak{H}$ and define $q \colon \omega \mapsto \operatorname{prox}_{\gamma \mathsf{f}_\omega}(x(\omega))$. Then (iv) asserts that $q \in \mathfrak{H}$ and $q = \operatorname{prox}_{\gamma f} x$. Hence, we derive from (ii), [3, Remark 12.24], and Definition 1.4 that

$$\begin{aligned}
{}^{\gamma} f(x) &= f\big(\operatorname{prox}_{\gamma f} x\big) + (2\gamma)^{-1}\|x - \operatorname{prox}_{\gamma f} x\|_{\mathcal{H}}^2 \\
&= \int_\Omega \Big(\mathsf{f}_\omega\big(q(\omega)\big) + (2\gamma)^{-1}\|x(\omega) - q(\omega)\|_{\mathsf{H}_\omega}^2\Big)\mu(d\omega) \\
&= \int_\Omega {}^{\gamma}\mathsf{f}_\omega\big(x(\omega)\big)\mu(d\omega),
\end{aligned} \tag{4.42}$$

as claimed.

(ix): Let $x^* \in \mathfrak{H}$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $]0,1[$ such that $\gamma_n \downarrow 0$, and define

$$(\forall n \in \mathbb{N}) \quad \vartheta_n \colon \Omega \to \mathbb{R} \colon \omega \mapsto {}^{\gamma_n}\big(\mathsf{f}_\omega^*\big)\big(x^*(\omega)\big). \tag{4.43}$$

For every $n \in \mathbb{N}$, since Moreau's decomposition theorem [3, Theorem 14.3(i)] gives

$$(\forall \omega \in \Omega) \quad \vartheta_n(\omega) = \gamma_n^{-1}\|x^*(\omega)\|_{\mathsf{H}_\omega}^2 - {}^{\gamma_n^{-1}}\mathsf{f}_\omega\big(\gamma_n^{-1}x^*(\omega)\big), \tag{4.44}$$

it follows from (viii) that $\vartheta_n \in \mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$. Further, we deduce from [3, Proposition 12.33(ii)] that

$$(\forall \omega \in \Omega) \quad \big(\vartheta_n(\omega)\big)_{n\in\mathbb{N}} \text{ is increasing and } \vartheta_n(\omega) \uparrow \mathsf{f}_\omega^*\big(x^*(\omega)\big) \tag{4.45}$$

and, therefore, that the function $\Omega \to]-\infty,+\infty] \colon \omega \mapsto \mathsf{f}_\omega^*(x^*(\omega))$ is $\mathcal{F}$-measurable. On the other hand, invoking (4.44), (viii), Moreau's decomposition theorem, and [3, Proposition 12.33(ii)], we obtain

$$\begin{aligned}
\int_\Omega \vartheta_n(\omega)\mu(d\omega) &= \int_\Omega \Big(\gamma_n^{-1}\|x^*(\omega)\|_{\mathsf{H}_\omega}^2 - {}^{\gamma_n^{-1}}\mathsf{f}_\omega\big(\gamma_n^{-1}x^*(\omega)\big)\Big)\mu(d\omega) \\
&= \gamma_n^{-1}\|x^*\|_{\mathcal{H}}^2 - {}^{\gamma_n^{-1}}f\big(\gamma_n^{-1}x^*\big) \\
&= {}^{\gamma_n}(f^*)(x^*) \\
&\to f^*(x^*) \text{ as } n \to +\infty.
\end{aligned} \tag{4.46}$$

Thus, in view of (4.45), we infer from the Beppo Levi monotone convergence theorem [5, Theorem 2.8.2 and Corollary 2.8.6] that

$$f^*(x^*) = \lim \int_\Omega \vartheta_n(\omega)\mu(d\omega) = \int_\Omega \lim \vartheta_n(\omega)\,\mu(d\omega) = \int_\Omega \mathsf{f}_\omega^*\big(x^*(\omega)\big)\mu(d\omega). \tag{4.47}$$

(x): Assumption 4.6[C] ensures that $(\forall \omega \in \Omega)\ r(\omega) \in \operatorname{dom}\mathsf{f}_\omega$. Now take $x \in \mathcal{H}$ and set

$$(\forall \alpha \in]0,+\infty[) \quad \theta_\alpha \colon \Omega \to]-\infty,+\infty] \colon \omega \mapsto \frac{\mathsf{f}_\omega\big(r(\omega) + \alpha x(\omega)\big) - \mathsf{f}_\omega\big(r(\omega)\big)}{\alpha}. \tag{4.48}$$

Then, for every $\alpha \in]0,+\infty[$, since $r + \alpha x \in \mathfrak{H}$ and $r \in \mathfrak{H}$, it results from (i) that θ_α is $\mathcal{F}$-measurable. On the other hand, by Assumption 4.6[A] and [3, Propositions 9.27 and 9.30(ii)], we obtain

$$(\forall \omega \in \Omega) \quad \text{the net } \big(\theta_\alpha(\omega)\big)_{\alpha\in]0,+\infty[} \text{ is increasing and } (\operatorname{rec}\mathsf{f}_\omega)\big(x(\omega)\big) = \lim_{\alpha\uparrow+\infty} \theta_\alpha(\omega). \tag{4.49}$$

Altogether, we infer from the Beppo Levi monotone convergence theorem, Assumption 4.6[C], (ii), and [3, Proposition 9.30(ii)] that

$$
\begin{aligned}
\int_\Omega (\operatorname{rec} \mathsf{f}_\omega)\big(x(\omega)\big)\mu(d\omega) &= \int_\Omega \lim_{\alpha\uparrow+\infty} \theta_\alpha(\omega)\,\mu(d\omega) \\
&= \lim_{\alpha\uparrow+\infty} \int_\Omega \theta_\alpha(\omega)\mu(d\omega) \\
&= \lim_{\alpha\uparrow+\infty} \frac{1}{\alpha}\bigg(\int_\Omega \mathsf{f}_\omega\big(r(\omega)+\alpha x(\omega)\big)\mu(d\omega) - \int_\Omega \mathsf{f}_\omega\big(r(\omega)\big)\mu(d\omega)\bigg) \\
&= \lim_{\alpha\uparrow+\infty} \frac{f(r+\alpha x)-f(r)}{\alpha} \\
&= (\operatorname{rec} f)(x),
\end{aligned}
\tag{4.50}
$$

which completes the proof. ☐

Remark 4.8. Consider Theorem 4.7 in the special case of Example 2.1(iv). Then (ii), (iii), (iv), (ix), and (x) were obtained, respectively, in [34, Corollary, p. 227], [34, Equation (25)], [3, Proposition 24.13], [34, Theorem 2], and [4, Proposition II.1]. On the other hand, in the special case of Example 2.1(iii), (iv) was obtained in [18, Corollary 2.2].

Example 4.9. Consider the setting of Example 2.1(iii) and suppose, in addition, that $(\forall k \in \mathbb{N})\ \alpha_k = 1$ and $\mathsf{H}_k = \mathbb{R}$. Then $\mathcal{H} = \ell^2(\mathbb{N})$. Now set $(\forall k \in \mathbb{N})\ \mathsf{f}_k = |\cdot|$. Then

$$
\operatorname{dom}\bigg({}^{\mathfrak{G}}\!\!\int_\Omega^\oplus \mathsf{f}_\omega \mu(d\omega)\bigg) = \ell^1(\mathbb{N}) \neq \ell^2(\mathbb{N}) = {}^{\mathfrak{G}}\!\!\int_\Omega^\oplus \operatorname{dom} \mathsf{f}_\omega\, \mu(d\omega).
\tag{4.51}
$$

Thus, the closure operation in Theorem 4.7(v) must not be omitted.

Every maximally monotone operator on $\mathbb{R}$ is the subdifferential of a function in $\Gamma_0(\mathbb{R})$ [3, Corollary 22.23]. The following result is an extension of this fact.

Corollary 4.10. *Let $(\Omega, \mathcal{F}, \mu)$ be a complete σ-finite measure space and, for every $\omega \in \Omega$, let $\mathsf{A}_\omega \colon \mathbb{R} \to 2^{\mathbb{R}}$ be maximally monotone. Set $\mathcal{H} = L^2(\Omega, \mathcal{F}, \mu; \mathbb{R})$ and*

$$
A \colon \mathcal{H} \to 2^{\mathcal{H}} \colon x \mapsto \big\{x^* \in \mathcal{H} \mid (\forall^\mu \omega \in \Omega)\ x^*(\omega) \in \mathsf{A}_\omega\big(x(\omega)\big)\big\}.
\tag{4.52}
$$

Then the following are equivalent:

(i) *A is maximally monotone.*

(ii) *There exists $f \in \Gamma_0(\mathcal{H})$ such that $A = \partial f$.*

(iii) *$\operatorname{dom} A \neq \varnothing$ and, for every $\mathsf{x} \in \mathbb{R}$, the function $\Omega \to \mathbb{R} \colon \omega \mapsto J_{\mathsf{A}_\omega}\mathsf{x}$ is $\mathcal{F}$-measurable.*

Proof. (ii)⇒(i): Use Moreau's theorem [28, Proposition 12.b].

(i)⇒(iii): This is a special case of Corollary 3.10.

(iii)⇒(ii): Set $\mathfrak{G} = \big\{x \colon \Omega \to \mathbb{R} \mid x \text{ is } \mathcal{F}\text{-measurable}\big\}$. Then, as seen in Example 2.1(iv), $\mathcal{H} = {}^{\mathfrak{G}}\!\int_\Omega^\oplus \mathbb{R}\,\mu(d\omega)$. For every $\omega \in \Omega$, [3, Corollary 22.23] asserts that there exists $\mathsf{g}_\omega \in \Gamma_0(\mathbb{R})$ such that $\mathsf{A}_\omega = \partial \mathsf{g}_\omega$. Next, since $\operatorname{dom} A \neq \varnothing$ and $(\Omega, \mathcal{F}, \mu)$ is complete, there exist r and s^* in $\mathscr{L}^2(\Omega, \mathcal{F}, \mu; \mathbb{R})$ such that

$$
(\forall^\mu \omega \in \Omega) \quad s^*(\omega) \in \mathsf{A}_\omega\big(r(\omega)\big) = \partial \mathsf{g}_\omega\big(r(\omega)\big)
\tag{4.53}
$$

and

$$(\forall\omega\in\Omega)\quad r(\omega)\in\operatorname{dom}\mathsf{A}_\omega\subset\operatorname{dom}\mathsf{g}_\omega. \tag{4.54}$$

Now set

$$(\forall\omega\in\Omega)\quad \mathsf{f}_\omega=\mathsf{g}_\omega-\mathsf{g}_\omega\big(r(\omega)\big). \tag{4.55}$$

Then the functions $(\mathsf{f}_\omega)_{\omega\in\Omega}$ lie in $\Gamma_0(\mathbb{R})$ and, by [3, Proposition 24.8(i) and Example 23.3], $(\forall\omega\in\Omega)$ $\operatorname{prox}_{\mathsf{f}_\omega}=\operatorname{prox}_{\mathsf{g}_\omega}=J_{\mathsf{A}_\omega}$. In turn, appealing to the continuity of the operators $(J_{\mathsf{A}_\omega})_{\omega\in\Omega}$, we deduce from [14, Lemma III.14] that the mapping $\Omega\times\mathbb{R}\to\mathbb{R}\colon(\omega,\mathsf{x})\mapsto\operatorname{prox}_{\mathsf{f}_\omega}\mathsf{x}$ is $\mathcal{F}\otimes\mathcal{B}_{\mathbb{R}}$-measurable. Therefore, for every $x\in\mathscr{L}^2(\Omega,\mathcal{F},\mu;\mathbb{R})$, the mapping $\Omega\to\mathbb{R}\colon\omega\mapsto\operatorname{prox}_{\mathsf{f}_\omega}(x(\omega))$ lies in $\mathfrak{G}$. Next, we get from (4.55) and (4.54) that $(\forall\omega\in\Omega)$ $\mathsf{f}_\omega(r(\omega))=0$. Moreover, by (4.53),

$$(\forall^\mu\omega\in\Omega)(\forall\mathsf{x}\in\mathbb{R})\quad \mathsf{f}_\omega(\mathsf{x})=\mathsf{g}_\omega(\mathsf{x})-\mathsf{g}_\omega\big(r(\omega)\big)\geqslant\big(\mathsf{x}-r(\omega)\big)s^*(\omega)=\mathsf{x}s^*(\omega)-r(\omega)s^*(\omega). \tag{4.56}$$

Hence, since $\omega\mapsto r(\omega)s^*(\omega)$ lies in $\mathscr{L}^1(\Omega,\mathcal{F},\mu;\mathbb{R})$, the family $(\mathsf{f}_\omega)_{\omega\in\Omega}$ satisfies the assumption of Theorem 4.7. Altogether, we conclude via Theorem 4.7(ii) that

$$\int_\Omega^{\mathfrak{G}\oplus}\mathsf{f}_\omega\mu(d\omega)\in\Gamma_0(\mathcal{H}) \tag{4.57}$$

and via Theorem 4.7(iii) and (4.52) that

$$\partial\bigg(\int_\Omega^{\mathfrak{G}\oplus}\mathsf{f}_\omega\mu(d\omega)\bigg)=\int_\Omega^{\mathfrak{G}\oplus}\partial\mathsf{f}_\omega\mu(d\omega)=\int_\Omega^{\mathfrak{G}\oplus}\mathsf{A}_\omega\mu(d\omega)=A, \tag{4.58}$$

as desired. ☐

Corollary 4.11. *Let $(\mathsf{A}_k)_{k\in\mathbb{N}}$ be a family of maximally monotone operators from $\mathbb{R}$ to $2^{\mathbb{R}}$, and define*

$$A\colon\ell^2(\mathbb{N})\to 2^{\ell^2(\mathbb{N})}\colon(\mathsf{x}_k)_{k\in\mathbb{N}}\mapsto\big\{(\mathsf{x}_k^*)_{k\in\mathbb{N}}\in\ell^2(\mathbb{N})\;\big|\;(\forall k\in\mathbb{N})\;\mathsf{x}_k^*\in\mathsf{A}_k\mathsf{x}_k\big\}. \tag{4.59}$$

Suppose that $\operatorname{dom}A\neq\varnothing$. Then A is maximally monotone and there exists $f\in\Gamma_0(\ell^2(\mathbb{N}))$ such that $A=\partial f$.

Proof. Apply Corollary 4.10 to the case where $\Omega=\mathbb{N}$, $\mathcal{F}=2^{\mathbb{N}}$, and μ is the counting measure. ☐

Corollary 4.12. *Suppose that Assumption 1.2 is in force and, for every $\omega\in\Omega$, let C_ω be a nonempty closed convex subset of H_ω. Set*

$$C=\int_\Omega^{\mathfrak{G}\oplus}\mathsf{C}_\omega\mu(d\omega). \tag{4.60}$$

Suppose that $C\neq\varnothing$ and that, for every $x\in\mathfrak{H}$, the mapping $\omega\mapsto\operatorname{proj}_{\mathsf{C}_\omega}(x(\omega))$ lies in $\mathfrak{G}$. Then the following hold:

(i) *C is a closed convex subset of $\mathcal{H}$.*

(ii) $N_C=\displaystyle\int_\Omega^{\mathfrak{G}\oplus}N_{\mathsf{C}_\omega}\mu(d\omega)$.

(iii) $\operatorname{proj}_C=\displaystyle\int_\Omega^{\mathfrak{G}\oplus}\operatorname{proj}_{\mathsf{C}_\omega}\mu(d\omega)$.

(iv) $d_C^2 = {}^{\mathfrak{G}}\!\!\int_\Omega^\oplus d_{\mathsf{C}_\omega}^2 \mu(d\omega)$.

(v) $\sigma_C = {}^{\mathfrak{G}}\!\!\int_\Omega^\oplus \sigma_{\mathsf{C}_\omega} \mu(d\omega)$.

(vi) *Suppose that, for every* $\omega \in \Omega$, C_ω *is a cone in* H_ω. *Then* $C^\ominus = {}^{\mathfrak{G}}\!\!\int_\Omega^\oplus \mathsf{C}_\omega^\ominus \mu(d\omega)$.

(vii) *Suppose that, for every* $\omega \in \Omega$, C_ω *is a vector subspace of* H_ω. *Then* $C^\perp = {}^{\mathfrak{G}}\!\!\int_\Omega^\oplus \mathsf{C}_\omega^\perp \mu(d\omega)$.

Proof. Set $(\forall \omega \in \Omega)$ $\mathsf{f}_\omega = \iota_{\mathsf{C}_\omega}$. Then, for every $\omega \in \Omega$, $\mathsf{f}_\omega \in \Gamma_0(\mathsf{H}_\omega)$, $\mathsf{f}_\omega \geqslant 0$, and $\operatorname{prox}_{\mathsf{f}_\omega} = \operatorname{proj}_{\mathsf{C}_\omega}$. Moreover, since $C \neq \varnothing$ and $(\Omega, \mathcal{F}, \mu)$ is complete, there exists $r \in \mathfrak{H}$ such that, for every $\omega \in \Omega$, $r(\omega) \in \mathsf{C}_\omega$ or, equivalently, $\mathsf{f}_\omega(r(\omega)) = 0$. Altogether, the family $(\mathsf{f}_\omega)_{\omega\in\Omega}$ satisfies the assumption of Theorem 4.7. Therefore, in view of items (i) and (ii) in Theorem 4.7,

$$f = {}^{\mathfrak{G}}\!\!\int_\Omega^\oplus \mathsf{f}_\omega \mu(d\omega) \text{ is well defined and lies in } \Gamma_0(\mathcal{H}). \tag{4.61}$$

(i): Using Definitions 1.4 and 3.1, together with (4.60), we obtain

$$\begin{aligned}(\forall x \in \mathcal{H}) \quad f(x) &= \int_\Omega \iota_{\mathsf{C}_\omega}\big(x(\omega)\big)\mu(d\omega)\\ &= \begin{cases} 0, & \text{if } (\forall^\mu \omega \in \Omega)\ x(\omega) \in \mathsf{C}_\omega;\\ +\infty, & \text{otherwise}\end{cases}\\ &= \begin{cases} 0, & \text{if } x \in C;\\ +\infty, & \text{otherwise}\end{cases}\\ &= \iota_C(x),\end{aligned} \tag{4.62}$$

and the claim thus follows from (4.61).

(ii)–(v): In the light of (4.61) and (4.62), these follow from items (iii), (iv), (viii), and (ix) in Theorem 4.7, respectively.

(vi): We deduce from [3, Example 6.40] and (ii) that

$$C^\ominus = N_C 0 = {}^{\mathfrak{G}}\!\!\int_\Omega^\oplus \big(N_{\mathsf{C}_\omega} 0\big)\mu(d\omega) = {}^{\mathfrak{G}}\!\!\int_\Omega^\oplus \mathsf{C}_\omega^\ominus \mu(d\omega). \tag{4.63}$$

(vii): Use (vi) and [3, Proposition 6.23]. ☐

Proposition 4.13. *Suppose that Assumption 4.6 is in force. Let* G *be a separable real Hilbert space and, for every* $\omega \in \Omega$, *let* $\mathsf{L}_\omega \colon \mathsf{G} \to \mathsf{H}_\omega$ *be linear and bounded. Suppose that, for every* $\mathsf{z} \in \mathsf{G}$, *the mapping* $\mathfrak{e}_{\mathsf{L}}\mathsf{z} \colon \omega \mapsto \mathsf{L}_\omega \mathsf{z}$ *lies in* $\mathfrak{G}$. *Additionally, suppose that* $\int_\Omega \|\mathsf{L}_\omega\|^2 \mu(d\omega) < +\infty$ *and that there exists* $\mathsf{w} \in \mathsf{G}$ *such that* $\int_\Omega \mathsf{f}_\omega(\mathsf{L}_\omega \mathsf{w})\mu(d\omega) < +\infty$. *Define*

$$\mathsf{g} \colon \mathsf{G} \to \left]-\infty, +\infty\right] \colon \mathsf{z} \mapsto \int_\Omega \mathsf{f}_\omega\big(\mathsf{L}_\omega \mathsf{z}\big)\mu(d\omega). \tag{4.64}$$

Then the following hold:

(i) g *is well defined and lies in* $\Gamma_0(\mathsf{G})$.

(ii) *Let* $(\mathsf{z},\mathsf{z}^*) \in \mathsf{G}\times\mathsf{G}$. *Then* $\mathsf{z}^* \in \partial\mathsf{g}(\mathsf{z})$ *if and only if there exist sequences* $(\gamma_n)_{n\in\mathbb{N}}$ *in* $]0,+\infty[$ *and* $(\mathsf{z}_n)_{n\in\mathbb{N}}$ *in* G *such that*

$$\gamma_n \downarrow 0, \quad \mathsf{z}_n \to \mathsf{z}, \quad \text{and} \quad \int_\Omega \mathsf{L}_\omega^*\Big(\mathrm{prox}_{\gamma_n^{-1}\mathsf{f}_\omega^*}\big(\gamma_n^{-1}\mathsf{L}_\omega\mathsf{z}_n\big)\Big)\mu(d\omega) \to \mathsf{z}^*. \tag{4.65}$$

Proof. Theorem 4.7(i)–(ii) state that

$$f = \int_\Omega^{\mathfrak{G}\oplus} \mathsf{f}_\omega\mu(d\omega) \text{ is well defined and lies in } \Gamma_0(\mathcal{H}). \tag{4.66}$$

On the other hand, according to Proposition 3.12(ii),

$$L\colon \mathsf{G}\to\mathcal{H}\colon \mathsf{z}\mapsto \mathfrak{e}_{\mathsf{L}}\mathsf{z} \text{ is well defined, linear, and bounded.} \tag{4.67}$$

(i): Because $L\mathsf{w} \in \mathrm{dom}\, f$, it follows from (4.64), (4.66), and (4.67) that

$$\mathsf{g} = f\circ L \in \Gamma_0(\mathsf{G}). \tag{4.68}$$

(ii): It results from Theorem 4.7(iii), Proposition 3.5, and Moreau's decomposition [3, Theorem 14.3(ii)] that

$$(\forall\gamma\in\,]0,+\infty[)\quad {}^{\gamma}(\partial f) = \int_\Omega^{\mathfrak{G}\oplus} {}^{\gamma}(\partial\mathsf{f}_\omega)\mu(d\omega) = \int_\Omega^{\mathfrak{G}\oplus} \mathrm{prox}_{\gamma^{-1}\mathsf{f}_\omega}\circ\big(\gamma^{-1}\mathrm{Id}_{\mathsf{H}_\omega}\big)\mu(d\omega). \tag{4.69}$$

Hence, for every $\gamma\in\,]0,+\infty[$ and every $\mathsf{w}\in\mathsf{G}$, since $\mathfrak{e}_{\mathsf{L}}\mathsf{w}\colon\omega\mapsto\mathsf{L}_\omega\mathsf{w}$ is a representative in $\mathfrak{H}$ of $L\mathsf{w}$, Proposition 3.12(v) implies that

$$L^*\big({}^{\gamma}(\partial f)(L\mathsf{w})\big) = \int_\Omega \mathsf{L}_\omega^*\Big(\mathrm{prox}_{\gamma^{-1}\mathsf{f}_\omega^*}\big(\gamma^{-1}\mathsf{L}_\omega\mathsf{w}\big)\Big)\mu(d\omega). \tag{4.70}$$

In addition, appealing to (4.66), (4.67), and (4.68), we derive from [32, Theorem 4.1] and a remark on [32, p. 88] that $\mathrm{gra}\,\partial\mathsf{g}$ is the set of points $(\mathsf{w},\mathsf{w}^*)\in\mathsf{G}\times\mathsf{G}$ for which there exist sequences $(\gamma_n)_{n\in\mathbb{N}}$ in $]0,+\infty[$ and $(\mathsf{w}_n)_{n\in\mathbb{N}}$ in G such that $\gamma_n\downarrow 0$, $\mathsf{w}_n\to\mathsf{w}$, and $L^*({}^{\gamma_n}(\partial f)(L\mathsf{w}_n)) \to \mathsf{w}_n^*$. Altogether, the proof is complete. ☐

§5. Application to integral composite inclusion problems

Let G and $(\mathsf{H}_k)_{1\leqslant k\leqslant p}$ be real Hilbert spaces. For every $k\in\{1,\ldots,p\}$, let $\mathsf{A}_k\colon\mathsf{H}_k\to 2^{\mathsf{H}_k}$ be monotone and let $\mathsf{L}_k\colon\mathsf{G}\to\mathsf{H}_k$ be linear and bounded. Finite compositions of the form $\sum_{k=1}^p \mathsf{L}_k^*\circ\mathsf{A}_k\circ\mathsf{L}_k$ arise in many theoretical and modeling aspects of monotone operator theory [3, 8, 16, 21, 22]. The main object of this section is to extend this construction to arbitrary families of monotone and linear operators. More precisely, our focus is on the following monotonicity-preserving operation, which involves the Aumann integral of (2.4).

Proposition 5.1. *Suppose that Assumption 1.2 is in force. Let* G *be a separable real Hilbert space and, for every* $\omega\in\Omega$, *let* $\mathsf{A}_\omega\colon\mathsf{H}_\omega\to 2^{\mathsf{H}_\omega}$ *be monotone and let* $\mathsf{L}_\omega\colon\mathsf{G}\to\mathsf{H}_\omega$ *be linear and bounded. Then*

$$\mathsf{M}\colon\mathsf{G}\to 2^{\mathsf{G}}\colon\mathsf{z}\mapsto\int_\Omega \mathsf{L}_\omega^*\big(\mathsf{A}_\omega(\mathsf{L}_\omega\mathsf{z})\big)\mu(d\omega) \tag{5.1}$$

is monotone.

Proof. Suppose that $(\mathsf{z},\mathsf{z}^*)$ and $(\mathsf{w},\mathsf{w}^*)$ are in $\operatorname{gra}\mathsf{M}$. Then, by (2.4), there exist x^* and y^* in $\prod_{\omega\in\Omega}\mathsf{H}_\omega$ such that

$$\begin{cases}(\forall^\mu\omega\in\Omega)\;\; x^*(\omega)\in\mathsf{A}_\omega(\mathsf{L}_\omega\mathsf{z}) \text{ and } y^*(\omega)\in\mathsf{A}_\omega(\mathsf{L}_\omega\mathsf{w})\\ \text{the mappings } \omega\mapsto\mathsf{L}_\omega^*(x^*(\omega)) \text{ and } \omega\mapsto\mathsf{L}_\omega^*(y^*(\omega)) \text{ lie in } \mathscr{L}^1(\Omega,\mathcal{F},\mu;\mathsf{G})\\ \int_\Omega\mathsf{L}_\omega^*(x^*(\omega))\mu(d\omega)=\mathsf{z}^* \text{ and } \int_\Omega\mathsf{L}_\omega^*(y^*(\omega))\mu(d\omega)=\mathsf{w}^*.\end{cases} \tag{5.2}$$

The monotonicity of the operators $(\mathsf{A}_\omega)_{\omega\in\Omega}$ ensures that

$$(\forall^\mu\omega\in\Omega)\quad \big\langle\mathsf{z}-\mathsf{w}\,\big|\,\mathsf{L}_\omega^*\big(x^*(\omega)\big)-\mathsf{L}_\omega^*\big(y^*(\omega)\big)\big\rangle_{\mathsf{G}}=\langle\mathsf{L}_\omega\mathsf{z}-\mathsf{L}_\omega\mathsf{w}\,|\,x^*(\omega)-y^*(\omega)\rangle_{\mathsf{H}_\omega}\geqslant 0. \tag{5.3}$$

Therefore, using [36, Théorème 5.8.16], we obtain

$$\begin{aligned}\langle\mathsf{z}-\mathsf{w}\,|\,\mathsf{z}^*-\mathsf{w}^*\rangle_{\mathsf{G}}&=\left\langle\mathsf{z}-\mathsf{w}\,\middle|\,\int_\Omega\mathsf{L}_\omega^*\big(x^*(\omega)\big)\mu(d\omega)-\int_\Omega\mathsf{L}_\omega^*\big(y^*(\omega)\big)\mu(d\omega)\right\rangle_{\mathsf{G}}\\&=\int_\Omega\big\langle\mathsf{z}-\mathsf{w}\,\big|\,\mathsf{L}_\omega^*\big(x^*(\omega)\big)-\mathsf{L}_\omega^*\big(y^*(\omega)\big)\big\rangle_{\mathsf{G}}\,\mu(d\omega)\\&\geqslant 0,\end{aligned} \tag{5.4}$$

which yields the assertion. ☐

The inclusion problem under investigation involves the integral composite operator (5.1) and is placed in the following environment.

Assumption 5.2. Assumption 1.2 and the following are in force:

[A] G is a separable real Hilbert space.

[B] For every $\omega\in\Omega$, $\mathsf{L}_\omega\colon\mathsf{G}\to\mathsf{H}_\omega$ is linear and bounded.

[C] For every $\mathsf{z}\in\mathsf{G}$, the mapping $\mathfrak{e}_{\mathsf{L}}\mathsf{z}\colon\omega\mapsto\mathsf{L}_\omega\mathsf{z}$ lies in $\mathfrak{G}$.

[D] $\int_\Omega\|\mathsf{L}_\omega\|^2\mu(d\omega)<+\infty$.

Problem 5.3. Suppose that Assumptions 3.6 and 5.2 are in force, and let $\mathsf{W}\colon\mathsf{G}\to 2^{\mathsf{G}}$ be maximally monotone. The objective is to

$$\text{find } \mathsf{z}\in\mathsf{G} \text{ such that } 0\in\mathsf{W}\mathsf{z}+\int_\Omega\mathsf{L}_\omega^*\big(\mathsf{A}_\omega(\mathsf{L}_\omega\mathsf{z})\big)\mu(d\omega). \tag{5.5}$$

In traditional variational methods, duality provides a powerful framework to analyze and solve minimization problems [3, 21, 35]. More generally, for inclusion problems, notions of duality have been proposed at various levels of generality [10, 17, 31, 33] in the context of Example 2.1(i), which corresponds to the inclusion problem

$$\text{find } \mathsf{z}\in\mathsf{G} \text{ such that } 0\in\mathsf{W}\mathsf{z}+\sum_{k=1}^{p}\mathsf{L}_k^*\big(\mathsf{A}_k(\mathsf{L}_k\mathsf{z})\big). \tag{5.6}$$

The next theorem extends duality concepts to the general setting of Problem 5.3.

Theorem 5.4. *Consider the setting of Problem 5.3, as well as the* dual problem

$$\textit{find } x^* \in \mathcal{H} \textit{ such that } \left(\exists\, \mathsf{z} \in \mathsf{W}^{-1}\left(-\int_{\Omega} \mathsf{L}_{\omega}^*\big(x^*(\omega)\big)\mu(d\omega)\right)\right)(\forall^{\mu}\omega \in \Omega)\ \mathsf{L}_{\omega}\mathsf{z} \in \mathsf{A}_{\omega}^{-1}\big(x^*(\omega)\big), \tag{5.7}$$

and denote by Z *and* Z^* *the sets of solutions to* (5.5) *and* (5.7)*, respectively. Let* $\mathfrak{K}$ *be the* Kuhn–Tucker operator *associated to Problem* 5.3*, that is,*

$$\begin{aligned} \mathfrak{K}\colon\ \ \mathsf{G} \oplus \mathcal{H} &\to 2^{\mathsf{G}\oplus\mathcal{H}} \\ (\mathsf{z}, x^*) &\mapsto \left(\mathsf{W}\mathsf{z} + \int_{\Omega} \mathsf{L}_{\omega}^*\big(x^*(\omega)\big)\mu(d\omega)\right) \times \left(-\mathfrak{e}_{\mathsf{L}}\mathsf{z} + {}^{\mathfrak{G}}\!\!\int_{\Omega}^{\oplus} \mathsf{A}_{\omega}^{-1}\big(x^*(\omega)\big)\mu(d\omega)\right), \end{aligned} \tag{5.8}$$

and let $\mathfrak{S}$ *be the* saddle operator *associated to Problem* 5.3*, that is,*

$$\begin{aligned} \mathfrak{S}\colon\ \ \mathsf{G} \oplus \mathcal{H} \oplus \mathcal{H} &\to 2^{\mathsf{G}\oplus\mathcal{H}\oplus\mathcal{H}} \\ (\mathsf{z}, x, u^*) &\mapsto \left(\mathsf{W}\mathsf{z} + \int_{\Omega} \mathsf{L}_{\omega}^*\big(u^*(\omega)\big)\mu(d\omega)\right) \times \left({}^{\mathfrak{G}}\!\!\int_{\Omega}^{\oplus} \mathsf{A}_{\omega}\big(x(\omega)\big)\mu(d\omega) - u^*\right) \times \big(-\mathfrak{e}_{\mathsf{L}}\mathsf{z} + x\big). \end{aligned} \tag{5.9}$$

Then the following hold:

(i) $\mathfrak{K}$ *and* $\mathfrak{S}$ *are maximally monotone.*

(ii) $\operatorname{zer}\mathfrak{K}$ *and* $\operatorname{zer}\mathfrak{S}$ *are closed and convex.*

(iii) *Let* $(\mathsf{z}, x^*) \in \mathsf{G} \times \mathcal{H}$*. Then* $(\mathsf{z}, x^*) \in \operatorname{zer}\mathfrak{K} \Rightarrow (\mathsf{z}, x^*) \in \mathsf{Z} \times Z^*$.

(iv) *Let* $(\mathsf{z}, x, u^*) \in \mathsf{G} \times \mathcal{H} \times \mathcal{H}$*. Then* $(\mathsf{z}, x, u^*) \in \operatorname{zer}\mathfrak{S} \Rightarrow (\mathsf{z}, u^*) \in \mathsf{Z} \times Z^*$.

(v) $\operatorname{zer}\mathfrak{S} \neq \varnothing \Leftrightarrow \operatorname{zer}\mathfrak{K} \neq \varnothing \Leftrightarrow Z^* \neq \varnothing \Rightarrow \mathsf{Z} \neq \varnothing$.

Proof. Set

$$A = {}^{\mathfrak{G}}\!\!\int_{\Omega}^{\oplus} \mathsf{A}_{\omega}\mu(d\omega). \tag{5.10}$$

Theorem 3.8(i) states that

$$A \text{ is maximally monotone,} \tag{5.11}$$

while Proposition 3.2(iv) states that

$$A^{-1} = {}^{\mathfrak{G}}\!\!\int_{\Omega}^{\oplus} \mathsf{A}_{\omega}^{-1}\mu(d\omega). \tag{5.12}$$

Moreover, in view of Assumption 5.2, items (ii) and (v) of Proposition 3.12 imply that the operator

$$L\colon \mathsf{G} \to \mathcal{H}\colon \mathsf{z} \mapsto \mathfrak{e}_{\mathsf{L}}\mathsf{z} \tag{5.13}$$

is well defined, linear, and bounded, with adjoint

$$L^*\colon \mathcal{H} \to \mathsf{G}\colon x^* \mapsto \int_{\Omega} \mathsf{L}_{\omega}^*\big(x^*(\omega)\big)\mu(d\omega). \tag{5.14}$$

Hence, we deduce from (5.8) that

$$\mathfrak{K}\colon \mathsf{G} \oplus \mathcal{H} \to 2^{\mathsf{G}\oplus\mathcal{H}}\colon (\mathsf{z}, x^*) \mapsto \big(\mathsf{W}\mathsf{z} + L^*x^*\big) \times \big(-L\mathsf{z} + A^{-1}x^*\big) \tag{5.15}$$

and from (5.9) that

$$\mathcal{S}\colon \mathsf{G}\oplus\mathcal{H}\oplus\mathcal{H}\to 2^{\mathsf{G}\oplus\mathcal{H}\oplus\mathcal{H}}\colon (\mathsf{z},x,u^*)\mapsto \big(\mathsf{W}\mathsf{z}+L^*u^*\big)\times\big(Ax-u^*\big)\times\big\{-L\mathsf{z}+x\big\}. \tag{5.16}$$

Additionally, the dual problem (5.7) can be rewritten as

$$\text{find } x^*\in\mathcal{H} \text{ such that } 0\in -L\big(\mathsf{W}^{-1}(-L^*x^*)\big)+A^{-1}x^*. \tag{5.17}$$

(i): In view of (5.11) and the maximal monotonicity of W, it follows from (5.15) and [3, Proposition 26.32(iii)] that $\mathcal{K}$ is maximally monotone, and from (5.16) and [9, Lemma 2.2(ii)] that $\mathcal{S}$ is maximally monotone.

(ii): Combine (i) and [3, Proposition 23.39].

(iii): Suppose that $(\mathsf{z},x^*)\in\operatorname{zer}\mathcal{K}$. Then, by (5.15), $L\mathsf{z}\in A^{-1}x^*$ or, equivalently, $x^*\in A(L\mathsf{z})$. Therefore, it follows from (5.13), Assumption 5.2[C], and (5.10) that, for μ-almost every $\omega\in\Omega$, $x^*(\omega)\in\mathsf{A}_\omega(\mathsf{L}_\omega\mathsf{z})$ and, in turn, that $\mathsf{L}^*_\omega(x^*(\omega))\in\mathsf{L}^*_\omega(\mathsf{A}_\omega(\mathsf{L}_\omega\mathsf{z}))$. Hence, because Proposition 3.12(iv) asserts that the mapping $\Omega\to\mathsf{G}\colon\omega\mapsto\mathsf{L}^*_\omega(x^*(\omega))$ is μ-integrable, we infer from (5.8) and (2.4) that

$$0\in\mathsf{W}\mathsf{z}+\int_\Omega \mathsf{L}^*_\omega\big(x^*(\omega)\big)\mu(d\omega)\subset\mathsf{W}\mathsf{z}+\int_\Omega\mathsf{L}^*_\omega\big(\mathsf{A}_\omega(\mathsf{L}_\omega\mathsf{z})\big)\mu(d\omega). \tag{5.18}$$

Finally, since $(\mathsf{z},x^*)\in\operatorname{zer}\mathcal{K}$, it follows from [3, Proposition 26.33(ii)] that x^* solves (5.17) and, therefore, (5.7).

(iv): Argue as in (iii).

(v): By virtue of (5.15), (5.16), and (5.17), the equivalences $\operatorname{zer}\mathcal{S}\neq\varnothing\Leftrightarrow\operatorname{zer}\mathcal{K}\neq\varnothing\Leftrightarrow Z^*\neq\varnothing$ follow from [9, Lemma 2.2(iv)], while the implication $\operatorname{zer}\mathcal{K}\neq\varnothing\Rightarrow\mathsf{Z}\neq\varnothing$ follows from (iii). ☐

Remark 5.5. Consider the setting of Theorem 5.4, and define A as in (5.10) and L as in (5.13).

(i) $\operatorname{zer}(\mathsf{W}+L^*\circ A\circ L)$ is a subset of Z which, in general, is proper.

(ii) According to Theorem 5.4(iii)–(iv), to solve (5.5) and its dual (5.7), it is enough to find a zero of the operator $\mathcal{K}$ of (5.8) or of the operator $\mathcal{S}$ of (5.9). This can be achieved by using splitting algorithms [16]. For instance, to find a zero of $\mathcal{S}$, each operator A_ω is decomposed as $\mathsf{A}_\omega=\mathsf{A}^m_\omega+\mathsf{A}^c_\omega+\mathsf{A}^\ell_\omega$, where $\mathsf{A}^m_\omega\colon\mathsf{H}_\omega\to 2^{\mathsf{H}_\omega}$ is maximally monotone, $\mathsf{A}^c_\omega\colon\mathsf{H}_\omega\to\mathsf{H}_\omega$ is cocoercive, and $\mathsf{A}^\ell_\omega\colon\mathsf{H}_\omega\to\mathsf{H}_\omega$ is monotone and Lipschitzian. Thus, A is decomposed as

$$A={}^{\mathfrak{G}}\!\!\int_\Omega^\oplus\mathsf{A}^m_\omega\,\mu(d\omega)+{}^{\mathfrak{G}}\!\!\int_\Omega^\oplus\mathsf{A}^c_\omega\mu(d\omega)+{}^{\mathfrak{G}}\!\!\int_\Omega^\oplus\mathsf{A}^\ell_\omega\mu(d\omega). \tag{5.19}$$

One can then employ the algorithm of [16, Section 8.5]. It requires the resolvent of ${}^{\mathfrak{G}}\!\int_\Omega^\oplus\mathsf{A}^m_\omega\,\mu(d\omega)$, which can be implemented via Theorem 3.8(ii)(a), as well as Euler steps on ${}^{\mathfrak{G}}\!\int_\Omega^\oplus\mathsf{A}^c_\omega\mu(d\omega)$ and ${}^{\mathfrak{G}}\!\int_\Omega^\oplus\mathsf{A}^\ell_\omega\mu(d\omega)$, which can be implemented via Proposition 3.4(i)–(ii).

We conclude the paper by providing a few illustrations of Problem 5.3 and the proposed duality framework; see [12] for further applications.

Example 5.6. In the setting of Example 2.1(i), the primal inclusion (5.5) reduces to (5.6) and Theorem 5.4 specializes to results found in [10, Proposition 1].

Example 5.7. Suppose that Assumptions 4.6 and 5.2 are in force, let $\mathsf{g} \in \Gamma_0(\mathsf{G})$, and suppose that there exists $z^* \in \mathfrak{H}$ such that

$$\left(\exists\, \mathsf{w} \in \partial \mathsf{g}^*\left(-\int_\Omega \mathsf{L}_\omega^*\big(z^*(\omega)\big)\mu(d\omega)\right)\right)(\forall^\mu \omega \in \Omega) \quad \mathsf{L}_\omega \mathsf{w} \in \partial \mathsf{f}_\omega^*\big(z^*(\omega)\big). \tag{5.20}$$

Now set $\mathsf{W} = \partial \mathsf{g}$ and $(\forall \omega \in \Omega)$ $\mathsf{A}_\omega = \partial \mathsf{f}_\omega$. Then it follows from Theorem 4.7, Proposition 3.12, and standard convex calculus that every solution to the primal problem (5.5) solves

$$\underset{\mathsf{z} \in \mathsf{G}}{\text{minimize}} \;\; \mathsf{g}(\mathsf{z}) + \int_\Omega \mathsf{f}_\omega\big(\mathsf{L}_\omega \mathsf{z}\big)\mu(d\omega), \tag{5.21}$$

and every solution to the dual problem (5.7) solves

$$\underset{x^* \in \mathcal{H}}{\text{minimize}} \;\; \mathsf{g}^*\left(-\int_\Omega \mathsf{L}_\omega^*\big(x^*(\omega)\big)\mu(d\omega)\right) + \int_\Omega \mathsf{f}_\omega^*\big(x^*(\omega)\big)\mu(d\omega). \tag{5.22}$$

A noteworthy instance is when μ is a probability measure and, for every $\omega \in \Omega$, $\mathsf{H}_\omega = \mathsf{G}$ and $\mathsf{L}_\omega = \mathrm{Id}_\mathsf{G}$. In this setting, (5.21) describes a standard stochastic optimization problem [29]. Our setting makes it possible to extend such stochastic problems to composite ones involving functions acting on different spaces $(\mathsf{H}_\omega)_{\omega \in \Omega}$.

Example 5.8. Suppose that Assumption 5.2 is in force, let $\mathsf{W}\colon \mathsf{G} \to 2^\mathsf{G}$ be maximally monotone, and, for every $\omega \in \Omega$, let $\mathsf{B}_\omega\colon \mathsf{H}_\omega \to 2^{\mathsf{H}_\omega}$ be maximally monotone. Additionally, suppose that $\operatorname{dom} {}^{\mathfrak{G}}\!\int_\Omega^\oplus \mathsf{B}_\omega \mu(d\omega) \neq \varnothing$ and that, for every $x \in \mathfrak{H}$, the mapping $\omega \mapsto J_{\mathsf{B}_\omega}(x(\omega))$ lies in $\mathfrak{G}$. Now let $\gamma \in \left]0, +\infty\right[$ and set $(\forall \omega \in \Omega)$ $\mathsf{A}_\omega = {}^\gamma\mathsf{B}_\omega$. Then, by Theorem 3.8(ii)(b) and Proposition 3.4(ii), the family $(\mathsf{A}_\omega)_{\omega \in \Omega}$ satisfies Assumption 3.6. Further, the primal problem (5.5) becomes

$$\text{find } \mathsf{z} \in \mathsf{G} \text{ such that } 0 \in \mathsf{W}\mathsf{z} + \int_\Omega \mathsf{L}_\omega^*\big({}^\gamma\mathsf{B}_\omega(\mathsf{L}_\omega \mathsf{z})\big)\mu(d\omega), \tag{5.23}$$

and the dual problem (5.7) reads

$$\begin{aligned}&\text{find } x^* \in \mathcal{H} \text{ such that}\\ &\quad \left(\exists\, \mathsf{z} \in \mathsf{W}^{-1}\left(-\int_\Omega \mathsf{L}_\omega^*\big(x^*(\omega)\big)\mu(d\omega)\right)\right)(\forall^\mu \omega \in \Omega)\;\; \mathsf{L}_\omega \mathsf{z} \in \mathsf{B}_\omega^{-1}\big(x^*(\omega)\big) + \gamma x^*(\omega).\end{aligned} \tag{5.24}$$

As in the special case discussed in [19, Proposition 4.1], which is set in the context of Example 2.1(i), the inclusion (5.23) can be shown to be an exact relaxation of the inclusion problem

$$\text{find } \mathsf{z} \in \operatorname{zer} \mathsf{W} \text{ such that } (\forall^\mu \omega \in \Omega)\;\; {}^\gamma\mathsf{B}_\omega(\mathsf{L}_\omega \mathsf{z}) = 0 \tag{5.25}$$

or, equivalently, of the so-called split common zero problem

$$\text{find } \mathsf{z} \in \operatorname{zer} \mathsf{W} \text{ such that } (\forall^\mu \omega \in \Omega)\;\; 0 \in \mathsf{B}_\omega(\mathsf{L}_\omega \mathsf{z}) \tag{5.26}$$

in the sense that, if (5.26) has solutions, they are the same as those of (5.23). If we further specialize to the case when μ is a probability measure, $\mathsf{W} = 0$, and for every $\omega \in \Omega$, $\mathsf{H}_\omega = \mathsf{G}$, $\mathsf{L}_\omega = \mathrm{Id}_\mathsf{G}$, and $\mathsf{B}_\omega = N_{\mathsf{C}_\omega}$, where C_ω is a nonempty closed convex subset of G, then (5.26) collapses to the stochastic convex feasibility problem of [13].

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