Variational Analysis of Proximal Compositions and Integral Proximal Mixtures*

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Abstract. This paper establishes various variational properties of parametrized versions of two convexity-preserving constructs that were recently introduced in the literature: the proximal composition of a function and a linear operator, and the integral proximal mixture of arbitrary families of functions and linear operators. We study in particular convexity, Legendre conjugacy, differentiability, Moreau envelopes, coercivity, minimizers, recession functions, and perspective functions of these constructs, as well as their asymptotic behavior as the parameter varies. The special case of the proximal expectation of a family of functions is also discussed.

Keywords. Convex function, integral proximal mixture, proximal composition, proximal expectation, variational analysis.

MSC classification. 28B20, 46N10, 47N10, 49J53, 49N15.

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§1. Introduction

Throughout, $\mathcal{H}$ is a real Hilbert space with power set $2^{\mathcal{H}}$, identity operator $\mathrm{Id}_{\mathcal{H}}$, scalar product $\langle \cdot \mid \cdot \rangle_{\mathcal{H}}$, associated norm $\|\cdot\|_{\mathcal{H}}$, and quadratic kernel $\mathcal{Q}_{\mathcal{H}} = \|\cdot\|_{\mathcal{H}}^2/2$. In addition, $\mathcal{G}$ is a real Hilbert space, the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{G}$ is denoted by $\mathcal{B}(\mathcal{H},\mathcal{G})$, and we set $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H},\mathcal{H})$. The Legendre conjugate of $f\colon \mathcal{H} \to [-\infty,+\infty]$ is

$$f^*\colon \mathcal{H} \to [-\infty,+\infty]\colon x^* \mapsto \sup_{x\in\mathcal{H}}\big(\langle x \mid x^*\rangle_{\mathcal{H}} - f(x)\big), \tag{1.1}$$

the Moreau envelope of index $\gamma \in \,]0,+\infty[$ of $f\colon \mathcal{H} \to [-\infty,+\infty]$ is

$$^{\gamma}\!f\colon \mathcal{H} \to [-\infty,+\infty]\colon x \mapsto \inf_{y\in\mathcal{H}}\Big(f(y) + \frac{1}{\gamma}\mathcal{Q}_{\mathcal{H}}(x-y)\Big), \tag{1.2}$$

and the adjoint of $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ is denoted by L^*.

In analysis, there are several ways to compose a function $g\colon \mathcal{G} \to [-\infty,+\infty]$ and an operator $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ in order to construct a function from $\mathcal{H}$ to $[-\infty,+\infty]$. The most common is the standard composition

$$g \circ L\colon \mathcal{H} \to [-\infty,+\infty]\colon x \mapsto g(Lx). \tag{1.3}$$

Another instance is the infimal postcomposition of g by L^*, that is (see [2, Section 12.5] and [16, Section I.5], and, for applications, [4, 5, 19]),

$$L^* \triangleright g\colon \mathcal{H} \to [-\infty,+\infty]\colon x \mapsto \inf_{\substack{y\in\mathcal{G}\\ L^*y=x}} g(y). \tag{1.4}$$

These two operations are dually related by the identities $(L^* \triangleright g)^* = g^* \circ L$ and, under certain qualification conditions, $(g \circ L)^* = L^* \triangleright g^*$ [2, Corollary 15.28]. The focus of the present paper is on the following alternative operations introduced in [9], where they were shown to manifest themselves in various variational models.

Definition 1.1. Let $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$, $g\colon \mathcal{G} \to [-\infty,+\infty]$, and $\gamma \in \,]0,+\infty[$. The *proximal composition* of g and L with parameter γ is the function $L \overset{\gamma}{\diamond} g\colon \mathcal{H} \to [-\infty,+\infty]$ given by

$$L \overset{\gamma}{\diamond} g = \Big(\,^{\frac{1}{\gamma}}(g^*) \circ L\Big)^* - \frac{1}{\gamma}\mathcal{Q}_{\mathcal{H}}, \tag{1.5}$$

and the *proximal cocomposition* of g and L with parameter γ is $L \overset{\gamma}{\blacklozenge} g = (L \overset{1/\gamma}{\diamond} g^*)^*$.

In [9], proximal compositions were studied only in the case when $\gamma = 1$ and few of their properties were explored. The goal of this paper is to carry out an in-depth analysis of these compositions, leading to results which are new even when $\gamma = 1$. We study in particular convexity, Legendre conjugacy, differentiability, subdifferentiability, Moreau envelopes, minimizers, recession functions, perspective functions, as well as the preservation of properties such as coercivity, supercoercivity, and Lipschitzianity. We also investigate the behavior of $L \overset{\gamma}{\diamond} g$ and $L \overset{\gamma}{\blacklozenge} g$ as γ varies. Another contribution of our work is to derive from these results a systematic analysis of the notions of integral proximal mixtures and comixtures. These operations, recently introduced in [7], combine arbitrary families of convex functions and linear operators acting in different spaces in such a way that the proximity operator of the mixture is explicitly computable in terms of those of the individual functions. In turn, this analysis leads to new results on the proximal expectation of a family of convex functions.

The remainder of the paper is organized as follows. In Section 2, we provide our notation and the necessary mathematical background. In Section 3, we investigate various variational properties of proximal compositions. Finally, Section 4 is devoted to applications to integral proximal mixtures and proximal expectations.

§2. Notation and background

We first present our notation, which follows [2] (see also the first paragraph of Section 1).

Let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$. The range of L is denoted by $\operatorname{ran} L$ and, if it is closed, the generalized inverse of L is denoted by $L^\dagger$. Further, L is called an isometry if $L^* \circ L = \mathrm{Id}_\mathcal{H}$ and a coisometry if $L \circ L^* = \mathrm{Id}_\mathcal{G}$. Let $f\colon \mathcal{H} \to [-\infty, +\infty]$. We set

$$\begin{cases} \operatorname{cam} f = \big\{h\colon \mathcal{H} \to \mathbb{R} \mid h \text{ is continuous, affine, and } h \leqslant f\big\} \\ \overline{f} = \sup\big\{h\colon \mathcal{H} \to [-\infty, +\infty] \mid h \text{ is lower semicontinuous and } h \leqslant f\big\} \\ \breve{f} = \sup\big\{h\colon \mathcal{H} \to [-\infty, +\infty] \mid h \text{ is lower semicontinuous, convex, and } h \leqslant f\big\}. \end{cases} \tag{2.1}$$

The infimal postcomposition of f by $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ (see (1.4)) is denoted by $L \triangleright f$ if, for every $y \in L(\operatorname{dom} f)$, there exists $x \in \mathcal{H}$ such that $Lx = y$ and $(L \triangleright f)(y) = f(x) \in \left]-\infty, +\infty\right]$. The function f is proper if $\operatorname{dom} f = \big\{x \in \mathcal{H} \mid f(x) < +\infty\big\} \neq \varnothing$ and $-\infty \notin f(\mathcal{H})$. If f is proper, its subdifferential is

$$\partial f\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{x^* \in \mathcal{H} \mid (\forall y \in \mathcal{H})\ \langle y - x \mid x^*\rangle_\mathcal{H} + f(x) \leqslant f(y)\big\} \tag{2.2}$$

and, if f is also convex, its recession function at $x \in \mathcal{H}$ is

$$(\operatorname{rec} f)(x) = \sup_{y \in \operatorname{dom} f} \big(f(x+y) - f(y)\big). \tag{2.3}$$

If f and $g\colon \mathcal{H} \to \left]-\infty, +\infty\right]$ are proper, their infimal convolution is

$$f \,\square\, g\colon \mathcal{H} \to [-\infty, +\infty]\colon x \mapsto \inf_{y \in \mathcal{H}} \big(f(y) + g(x-y)\big). \tag{2.4}$$

We denote by $\Gamma_0(\mathcal{H})$ the class of functions from $\mathcal{H}$ to $\left]-\infty, +\infty\right]$ which are proper, lower semicontinuous, and convex. If $f \in \Gamma_0(\mathcal{H})$, its proximity operator is

$$\operatorname{prox}_f\colon \mathcal{H} \to \mathcal{H}\colon x \mapsto \operatorname*{argmin}_{y \in \mathcal{H}} \big(f(y) + \mathcal{Q}_\mathcal{H}(x-y)\big). \tag{2.5}$$

Let $C \subset \mathcal{H}$. Then ι_C denotes the indicator function of C and σ_C the support function of C. If C is convex, its normal cone is denoted by N_C and its strong relative interior is the set $\operatorname{sri} C$ of points $x \in C$ such that the smallest cone containing $C - x$ is a closed vector subspace of $\mathcal{H}$. If C is nonempty, closed, and convex, its projection operator is denoted by proj_C. Finally, the closed ball with center $x \in \mathcal{H}$ and radius $\rho \in \left]0, +\infty\right[$ is denoted by $B(x; \rho)$.

The following facts will be frequently used in the paper.

Lemma 2.1. *Let f and g be functions from $\mathcal{H}$ to $[-\infty, +\infty]$. Then the following hold:*

(i) $f^{**} \leqslant f$.

(ii) $f \leqslant g \Rightarrow g^* \leqslant f^*$.
(iii) $f^{***} = f^*$.
(iv) $f^* \equiv +\infty \Leftrightarrow \operatorname{cam} f = \varnothing$.
(v) $f^* \in \Gamma_0(\mathcal{H}) \Leftrightarrow [\, f$ *is proper and* $\operatorname{cam} f \neq \varnothing\,]$.

Proof. (i)–(iii): [2, Proposition 13.16].
(iv): [2, Proposition 13.12(ii)].
(v): Combine [2, Proposition 13.10(ii)] and (iv). □

Lemma 2.2. [2, Propositions 13.10(ii) and 13.23(i)–(ii)] *Let* $f\colon \mathcal{H} \to [-\infty,+\infty]$ *and let* $\rho \in\,]0,+\infty[$. *Then the following hold:*

(i) $(\rho f)^* = \rho f^*(\cdot/\rho)$.
(ii) $(\rho f(\cdot/\rho))^* = \rho f^*$.
(iii) $(f(\rho\cdot))^* = f^*(\cdot/\rho)$.

The next lemma follows easily from (1.2).

Lemma 2.3. *Let* $f\colon \mathcal{H} \to [-\infty,+\infty]$, $\gamma \in\,]0,+\infty[$, *and* $\rho \in\,]0,+\infty[$. *Then the following hold:*

(i) $\rho(^{\gamma} f) = {}^{\frac{\gamma}{\rho}}(\rho f)$.
(ii) $(^{\gamma} f)(\rho\cdot) = {}^{\frac{\gamma}{\rho^2}}(f(\rho\cdot))$.

Lemma 2.4. *Let* $f \in \Gamma_0(\mathcal{H})$ *and* $\gamma \in\,]0,+\infty[$. *Then the following hold:*

(i) [2, Theorem 9.20] $\operatorname{cam} f \neq \varnothing$.
(ii) [2, Corollary 13.38] $f^* \in \Gamma_0(\mathcal{H})$ *and* $f^{**} = f$.
(iii) [2, Corollary 16.30] $\partial f^* = (\partial f)^{-1}$.
(iv) [2, Remark 14.4] $^1 f + {}^1(f^*) = \mathcal{Q}_{\mathcal{H}}$ *and* $\operatorname{prox}_f + \operatorname{prox}_{f^*} = \operatorname{Id}_{\mathcal{H}}$.
(v) [2, Theorem 13.49] $\operatorname{rec}(f^*) = \sigma_{\operatorname{dom} f}$ *and* $\operatorname{rec} f = \sigma_{\operatorname{dom} f^*}$.
(vi) [2, Propositions 12.15 and 12.30] $^{\gamma} f\colon \mathcal{H} \to \mathbb{R}$ *is convex and Fréchet differentiable.*
(vii) [2, Proposition 12.30] $\nabla(^{\gamma} f) = (\operatorname{Id}_{\mathcal{H}} - \operatorname{prox}_{\gamma f})/\gamma$.
(viii) [2, Proposition 14.1] $(f + \gamma\mathcal{Q}_{\mathcal{H}})^* = {}^{\gamma}(f^*)$.

Lemma 2.5. *Let* $f\colon \mathcal{H} \to\,]-\infty,+\infty]$, $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$, *and* $\gamma \in\,]0,+\infty[$. *Then the following hold:*

(i) [2, Proposition 13.24(iii)] $(^{\gamma} f)^* = f^* + \gamma\mathcal{Q}_{\mathcal{H}}$.
(ii) [2, Proposition 13.24(iv)] $(L \triangleright f)^* = f^* \circ L^*$.
(iii) [2, Corollary 15.28(i)] *Suppose that* $f \in \Gamma_0(\mathcal{H})$ *and* $0 \in \operatorname{sri}(\operatorname{dom} f - \operatorname{ran} L^*)$. *Then* $(f \circ L^*)^* = L \triangleright f^*$.

Lemma 2.6. *Let* $f \in \Gamma_0(\mathcal{H})$, $g \in \Gamma_0(\mathcal{H})$, *and* $\gamma \in\,]0,+\infty[$ *be such that* $^{\gamma} f = {}^{\gamma} g$. *Then* $f = g$.

Proof. By Lemma 2.5(i), $f^* = (^{\gamma} f)^* - \gamma\mathcal{Q}_{\mathcal{H}} = (^{\gamma} g)^* - \gamma\mathcal{Q}_{\mathcal{H}} = g^*$. Therefore, we deduce from Lemma 2.4(ii) that $f = f^{**} = g^{**} = g$. □

Lemma 2.7. *Let* $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ *and set* $\Phi = \mathcal{Q}_{\mathcal{G}} - \mathcal{Q}_{\mathcal{H}} \circ L^*$. *Then* Φ *is convex if and only if* $\|L\| \leqslant 1$.

Proof. Since $\operatorname{dom} \Phi = \mathcal{G}$ and $\nabla\Phi = \operatorname{Id}_{\mathcal{G}} - L \circ L^*$, we deduce from [2, Proposition 17.7] that Φ is convex $\Leftrightarrow \operatorname{Id}_{\mathcal{G}} - L \circ L^*$ is monotone $\Leftrightarrow \|L^* \cdot\|_{\mathcal{H}}^2 \leqslant \|\cdot\|_{\mathcal{G}}^2 \Leftrightarrow \|L^*\| \leqslant 1 \Leftrightarrow \|L\| \leqslant 1$. □

Lemma 2.8. [2, Proposition 17.36(iii)] *Let* $A \in \mathcal{B}(\mathcal{H})$ *be monotone and self-adjoint. Suppose that* $\operatorname{ran} A$ *is closed, set* $q_A\colon \mathcal{H} \to \mathbb{R}\colon x \mapsto \langle x \mid Ax\rangle_{\mathcal{H}}/2$, *and define* $q_{A^\dagger}$ *likewise. Then* $q_A^* = \iota_{\operatorname{ran} A} + q_{A^\dagger}$.

§3. Proximal compositions

3.1. General properties

We start with direct consequences of Definition 1.1.

Proposition 3.1. *Let $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$, $g\colon \mathcal{G} \to [-\infty,+\infty]$, $\gamma \in\,]0,+\infty[$, and $\rho \in\,]0,+\infty[$. Then the following hold:*

(i) *Let $h\colon \mathcal{G} \to [-\infty,+\infty]$ be such that $g^{**} \leqslant h \leqslant g$. Then $L \overset{\gamma}{\diamond} h = L \overset{\gamma}{\diamond} g$ and $L \overset{\gamma}{\blacklozenge} h = L \overset{\gamma}{\blacklozenge} g$.*

(ii) $(L \overset{\gamma}{\diamond} g)^* = L \overset{1/\gamma}{\blacklozenge} g^*$.

(iii) $(L \overset{\gamma}{\blacklozenge} g)^* = (L \overset{1/\gamma}{\diamond} g^*)^{**}$.

(iv) $(L \overset{\gamma}{\diamond} g)^{**} = (L \overset{1/\gamma}{\blacklozenge} g^*)^*$.

(v) $\rho(L \overset{\gamma}{\diamond} g) = L \overset{\gamma/\rho}{\diamond} (\rho g)$.

(vi) $(L \overset{\gamma}{\diamond} g)(\rho\cdot) = L \overset{\gamma/\rho^2}{\diamond} (g(\rho\cdot))$.

(vii) $\rho(L \overset{\gamma}{\blacklozenge} g) = L \overset{\gamma/\rho}{\blacklozenge} (\rho g)$.

(viii) $(L \overset{\gamma}{\blacklozenge} g)(\rho\cdot) = L \overset{\gamma/\rho^2}{\blacklozenge} (g(\rho\cdot))$.

Proof. (i): By Lemma 2.1(ii)–(iii), $g^* = g^{***} \geqslant h^* \geqslant g^*$. Therefore, $h^* = g^*$, and the claims follow from Definition 1.1.

(ii): It follows from Definition 1.1 and (i) that $L \overset{1/\gamma}{\blacklozenge} g^* = (L \overset{\gamma}{\diamond} g^{**})^* = (L \overset{\gamma}{\diamond} g)^*$.

(iii): An immediate consequence of Definition 1.1.

(iv): This follows from (ii).

(v): Combining Lemmas 2.2(ii), 2.3(i)–(ii), and 2.2(i), we obtain

$$\rho\Big({}^{\frac{1}{\gamma}}(g^*)\circ L\Big)^* = \Big(\rho\,{}^{\frac{1}{\gamma}}(g^*)\circ (L/\rho)\Big)^* = \Big({}^{\frac{\rho}{\gamma}}(\rho g^*(\cdot/\rho))\circ L\Big)^* = \Big({}^{\frac{\rho}{\gamma}}((\rho g)^*)\circ L\Big)^*. \tag{3.1}$$

The assertion therefore follows from Definition 1.1.

(vi): We deduce from Lemmas 2.2(iii) and 2.3(ii) that

$$\Big({}^{\frac{1}{\gamma}}(g^*)\circ L\Big)^*(\rho\cdot) = \Big({}^{\frac{1}{\gamma}}(g^*)\circ (L/\rho)\Big)^* = \Big({}^{\frac{\rho^2}{\gamma}}(g^*(\cdot/\rho))\circ L\Big)^* = \Big({}^{\frac{\rho^2}{\gamma}}\Big((g(\rho\cdot))^*\Big)\circ L\Big)^*. \tag{3.2}$$

In view of Definition 1.1, the assertion is established.

(vii): We invoke Definition 1.1, Lemma 2.2(ii), (v), (vi), and Lemma 2.2(i) to get

$$\rho(L \overset{\gamma}{\blacklozenge} g) = \rho(L \overset{1/\gamma}{\diamond} g^*)^* = \Big(\rho(L \overset{1/\gamma}{\diamond} g^*)(\cdot/\rho)\Big)^* = \Big(L \overset{\rho/\gamma}{\diamond} (\rho g)^*\Big)^* = L \overset{\gamma/\rho}{\blacklozenge} (\rho g). \tag{3.3}$$

(viii): By Definition 1.1, Lemma 2.2(iii), and (vi), we get

$$(L \overset{\gamma}{\blacklozenge} g)(\rho\cdot) = (L \overset{1/\gamma}{\diamond} g^*)^*(\rho\cdot) = \Big((L \overset{1/\gamma}{\diamond} g^*)(\cdot/\rho)\Big)^* = \Big(L \overset{\rho^2/\gamma}{\diamond} (g(\rho\cdot))^*\Big)^* = L \overset{\gamma/\rho^2}{\blacklozenge} (g(\rho\cdot)), \tag{3.4}$$

which completes the proof. ☐

Proposition 3.2. *Let $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$, let $g\colon \mathcal{G} \to\,]-\infty,+\infty]$ be a proper function such that $\operatorname{cam} g \neq \varnothing$, let $\gamma \in\,]0,+\infty[$, and set $\Phi = \mathcal{Q}_{\mathcal{G}} - \mathcal{Q}_{\mathcal{H}} \circ L^*$. Then the following hold:*

(i) $L \overset{\gamma}{\diamond} g = L^* \triangleright (g^{**} + \Phi/\gamma)$.

(ii) $L \overset{\gamma}{\blacklozenge} g = (g^* + \gamma\Phi)^* \circ L$.

(iii) $\operatorname{dom}(L \overset{\gamma}{\diamond} g) = L^*(\operatorname{dom} g^{**})$.

(iv) *Suppose that one of the following are satisfied:*

(a) $0 < \|L\| < 1$.

(b) $\operatorname{dom} g^{**} = \mathcal{G}$.

Then $\operatorname{dom}(L \overset{\gamma}{\blacklozenge} g) = \mathcal{H}$.

(v) $L \overset{\gamma}{\blacklozenge} g \geqslant {}^{\gamma}(g^{**}) \circ L$.

Proof. By Lemma 2.1(v), $g^* \in \Gamma_0(\mathcal{G})$. Therefore, Lemma 2.4(vi) implies that $\operatorname{dom} {}^{\frac{1}{\gamma}}(g^*) = \mathcal{G}$ and that ${}^{\frac{1}{\gamma}}(g^*) \in \Gamma_0(\mathcal{G})$.

(i): Let $x \in \mathcal{H}$. Because $\operatorname{dom} {}^{\frac{1}{\gamma}}(g^*) - \operatorname{ran} L = \mathcal{G}$, it follows from Definition 1.1 and items (iii) and (i) in Lemma 2.5 that

$$
\begin{aligned}
(L \overset{\gamma}{\diamond} g)(x) &= \left(\left({}^{\frac{1}{\gamma}}(g^*) \circ L\right)^* - \frac{1}{\gamma}\mathcal{Q}_{\mathcal{H}}\right)(x) \\
&= \left(L^* \triangleright \left({}^{\frac{1}{\gamma}}(g^*)\right)^*\right)(x) - \frac{1}{\gamma}\mathcal{Q}_{\mathcal{H}}(x) \\
&= \left(L^* \triangleright \left(g^{**} + \frac{1}{\gamma}\mathcal{Q}_{\mathcal{G}}\right)\right)(x) - \frac{1}{\gamma}\mathcal{Q}_{\mathcal{H}}(x) \\
&= \min_{\substack{y \in \mathcal{G} \\ L^* y = x}} \left(g^{**}(y) + \frac{1}{\gamma}\mathcal{Q}_{\mathcal{G}}(y)\right) - \frac{1}{\gamma}\mathcal{Q}_{\mathcal{H}}(x) \\
&= \min_{\substack{y \in \mathcal{G} \\ L^* y = x}} \left(g^{**}(y) + \frac{1}{\gamma}\mathcal{Q}_{\mathcal{G}}(y) - \frac{1}{\gamma}\mathcal{Q}_{\mathcal{H}}(L^* y)\right) \\
&= \min_{\substack{y \in \mathcal{G} \\ L^* y = x}} \left(g^{**}(y) + \frac{1}{\gamma}\Phi(y)\right). \qquad (3.5)
\end{aligned}
$$

(ii): By Definition 1.1, (i), and Lemmas 2.1(iii) and 2.5(ii),

$$
L \overset{\gamma}{\blacklozenge} g = \left(L \overset{1/\gamma}{\diamond} g^*\right)^* = \left(L^* \triangleright (g^{***} + \gamma\Phi)\right)^* = \left(L^* \triangleright (g^* + \gamma\Phi)\right)^* = (g^* + \gamma\Phi)^* \circ L. \qquad (3.6)
$$

(iii): Since $\operatorname{dom}\Phi = \mathcal{G}$, we deduce from [2, Proposition 12.36(i)] and (i) that $\operatorname{dom}(L \overset{\gamma}{\diamond} g) = L^*(\operatorname{dom}(g^{**} + \Phi/\gamma)) = L^*(\operatorname{dom} g^{**})$.

(iv): By Lemma 2.7, $\Phi \in \Gamma_0(\mathcal{G})$. Because $\operatorname{dom}\Phi = \mathcal{G}$, the identity $(\gamma\Phi)^* = \Phi^*/\gamma$ and [2, Proposition 15.2] imply that

$$
(g^* + \gamma\Phi)^* = g^{**} \,\Box\, (\gamma\Phi)^* = g^{**} \,\Box\, (\Phi^*/\gamma). \qquad (3.7)
$$

On the other hand, we have $(1 - \|L\|^2)\mathcal{Q}_{\mathcal{G}} \leqslant \Phi$. Hence, in view of property (iv)(a) and Lemma 2.1(ii), we have $\Phi^* \leqslant \mathcal{Q}_{\mathcal{G}}/(1 - \|L\|^2)$, which yields $\operatorname{dom}\Phi^* = \mathcal{G}$. We thus deduce from (3.7) that $\operatorname{dom}(g^* + \gamma\Phi)^* = \operatorname{dom} g^{**} + \operatorname{dom}\Phi^* = \mathcal{G}$ and obtain the assertion via (ii).

(v): Since $\Phi \leqslant \mathcal{Q}_{\mathcal{G}}$, $g^* + \gamma\Phi \leqslant g^* + \gamma\mathcal{Q}_{\mathcal{G}}$. In turn, Lemmas 2.4(viii) and 2.1(ii), and (ii) imply that

$$ {}^{\gamma}\!\left(g^{**}\right) \circ L = \left(g^* + \gamma\mathcal{Q}_{\mathcal{G}}\right)^* \circ L \leqslant \left(g^* + \gamma\Phi\right)^* \circ L = L \overset{\gamma}{\blacklozenge} g, \tag{3.8} $$

which completes the proof. ☐

Remark 3.3. Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ satisfies $\|L\| = 1$, set $\Phi = \mathcal{Q}_{\mathcal{G}} - \mathcal{Q}_{\mathcal{H}} \circ L^*$, and set $A = \mathrm{Id}_{\mathcal{G}} - L \circ L^*$. Then A is monotone and self-adjoint, $\Phi\colon y \mapsto \langle y \mid Ay\rangle_{\mathcal{G}}/2$, and Lemma 2.8 shows that $\operatorname{dom}\Phi^* = \operatorname{ran}A$ under the assumption that $\operatorname{ran}A$ is closed. In this case, arguing as in (3.7) and using Proposition 3.2(ii), we obtain $\operatorname{dom}(L \overset{\gamma}{\blacklozenge} g) = L^{-1}(\operatorname{dom}g^{**} + \operatorname{ran}A)$.

Proposition 3.4. *Let $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ be such that $\operatorname{ran}L$ is closed and $\ker L = \{0\}$, let $g\colon \mathcal{G} \to \left]-\infty,+\infty\right]$ be a proper function such that $\operatorname{cam}g \neq \varnothing$, and let $\gamma \in \left]0,+\infty\right[$. Then the following hold:*

(i) *Suppose that g^{**} is coercive. Then $L \overset{\gamma}{\blacklozenge} g$ is coercive.*

(ii) *Suppose that g^{**} is supercoercive. Then $L \overset{\gamma}{\blacklozenge} g$ is supercoercive.*

Proof. It follows from [2, Fact 2.26] that there exists $\alpha \in \left]0,+\infty\right[$ such that $\|L\cdot\|_{\mathcal{G}} \geqslant \alpha\|\cdot\|_{\mathcal{H}}$. Thus, $\|Lx\|_{\mathcal{G}} \to +\infty$ as $\|x\|_{\mathcal{H}} \to +\infty$. On the other hand, combining Lemmas 2.1(v) and 2.4(ii), we obtain $g^{**} \in \Gamma_0(\mathcal{G})$.

(i): By [2, Corollary 14.18(i)], ${}^{\gamma}(g^{**})$ is coercive. Therefore, Proposition 3.2(v) implies that $(L \overset{\gamma}{\blacklozenge} g)(x) \geqslant ({}^{\gamma}(g^{**}))(Lx) \to +\infty$ as $\|x\|_{\mathcal{H}} \to +\infty$.

(ii): By [2, Corollary 14.18(ii)], ${}^{\gamma}(g^{**})$ is supercoercive. Hence, Proposition 3.2(v) yields

$$ \frac{(L \overset{\gamma}{\blacklozenge} g)(x)}{\|x\|_{\mathcal{H}}} \geqslant \frac{{}^{\gamma}\!\left(g^{**}\right)(Lx)}{\|x\|_{\mathcal{H}}} \geqslant \alpha\frac{{}^{\gamma}\!\left(g^{**}\right)(Lx)}{\|Lx\|_{\mathcal{G}}} \to +\infty \quad \text{as} \quad \|x\|_{\mathcal{H}} \to +\infty, \tag{3.9} $$

which concludes the proof. ☐

The next proposition studies the effect of quadratic perturbations and translations.

Proposition 3.5. *Let $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$, $g \in \Gamma_0(\mathcal{G})$, $\alpha \in \mathbb{R}$, $\gamma \in \left]0,+\infty\right[$, $\rho \in \left[0,+\infty\right[$, and $u \in \mathcal{H}$. Given $w \in \mathcal{G}$, set $\tau_w g\colon y \mapsto g(y-w)$. Then the following hold:*

(i) *Set $\beta = \gamma/(1+\rho\gamma)$. Then $L \overset{\gamma}{\diamond} (g + \rho\mathcal{Q}_{\mathcal{G}} + \langle\cdot \mid Lu\rangle_{\mathcal{G}} + \alpha) = (L \overset{\beta}{\diamond} g) + \rho\mathcal{Q}_{\mathcal{H}} + \langle\cdot \mid u\rangle_{\mathcal{H}} + \alpha$.*

(ii) *$L \overset{\gamma}{\blacklozenge} (\tau_{Lu}\, g + \alpha) = \tau_u(L \overset{\gamma}{\blacklozenge} g) + \alpha$.*

Proof. (i): Let $x \in \mathcal{H}$, set $h = g + \rho\mathcal{Q}_{\mathcal{G}} + \langle\cdot \mid Lu\rangle_{\mathcal{G}} + \alpha$, and set $\Phi = \mathcal{Q}_{\mathcal{G}} - \mathcal{Q}_{\mathcal{H}} \circ L^*$. Since $g \in \Gamma_0(\mathcal{G})$ and $\rho \geqslant 0$, we have $h \in \Gamma_0(\mathcal{G})$. In turn, Lemma 2.4(ii) yields $h^* \in \Gamma_0(\mathcal{G})$, $h^{**} = h$, and $g^{**} = g$. Therefore, it follows from Proposition 3.2(i) that

$$
\begin{aligned}
(L \overset{\gamma}{\diamond} h)(x) &= \min_{\substack{y\in\mathcal{G}\\ L^*y=x}} \Big(h(y) + \frac{1}{\gamma}\Phi(y)\Big) \\
&= \min_{\substack{y\in\mathcal{G}\\ L^*y=x}} \Big(g(y) + \rho\mathcal{Q}_{\mathcal{G}}(y) + \langle y \mid Lu\rangle_{\mathcal{G}} + \alpha + \frac{1}{\gamma}\Phi(y)\Big) \\
&= \min_{\substack{y\in\mathcal{G}\\ L^*y=x}} \Big(g(y) + \rho\Phi(y) + \rho\mathcal{Q}_{\mathcal{H}}(L^*y) + \langle L^*y \mid u\rangle_{\mathcal{H}} + \frac{1}{\gamma}\Phi(y)\Big) + \alpha \\
&= \min_{\substack{y\in\mathcal{G}\\ L^*y=x}} \Big(g(y) + \Big(\rho + \frac{1}{\gamma}\Big)\Phi(y)\Big) + \rho\mathcal{Q}_{\mathcal{H}}(x) + \langle x \mid u\rangle_{\mathcal{H}} + \alpha \\
&= \min_{\substack{y\in\mathcal{G}\\ L^*y=x}} \Big(g(y) + \frac{1}{\beta}\Phi(y)\Big) + \rho\mathcal{Q}_{\mathcal{H}}(x) + \langle x \mid u\rangle_{\mathcal{H}} + \alpha \\
&= (L \overset{\beta}{\diamond} g)(x) + \rho\mathcal{Q}_{\mathcal{H}}(x) + \langle x \mid u\rangle_{\mathcal{H}} + \alpha. \qquad (3.10)
\end{aligned}
$$

(ii): Set $h = \tau_{Lu}\, g + \alpha$. We recall from [2, Proposition 13.23(iii)] that $h^* = g^* + \langle\cdot \mid Lu\rangle_{\mathcal{G}} - \alpha$. Hence, using Definition 1.1 and (i), we get

$$
\begin{aligned}
L \overset{\gamma}{\blacklozenge} h &= \Big(L \overset{1/\gamma}{\diamond} \big(g^* + \langle\cdot \mid Lu\rangle_{\mathcal{G}} - \alpha\big)\Big)^* \\
&= \Big(\big(L \overset{1/\gamma}{\diamond} g^*\big) + \langle\cdot \mid u\rangle_{\mathcal{H}} - \alpha\Big)^* \\
&= \tau_u\big(L \overset{1/\gamma}{\diamond} g^*\big)^* + \alpha \\
&= \tau_u\big(L \overset{\gamma}{\blacklozenge} g\big) + \alpha, \qquad (3.11)
\end{aligned}
$$

as claimed. ☐

3.2. Convex-analytical properties

We first study the convexity, Legendre conjugacy, and differentiability properties of proximal compositions. We then turn our attention to the evaluation of their proximity operators, subdifferentials, Moreau envelopes, recession functions, and perspective functions.

Proposition 3.6. *Suppose that $0 \neq L \in \mathcal{B}(\mathcal{H},\mathcal{G})$, let $g\colon \mathcal{G} \to \,]-\infty,+\infty]$ be a proper function such that* $\operatorname{cam} g \neq \varnothing$*, let $\gamma \in \,]0,+\infty[$, and let $\alpha \in [-1/\gamma,+\infty[$. Suppose that $g^{**} - \alpha\mathcal{Q}_{\mathcal{G}}$ is convex and set $\beta = (\alpha + 1/\gamma)/\|L\|^2 - 1/\gamma$. Then $L \overset{\gamma}{\diamond} g - \beta\mathcal{Q}_{\mathcal{H}} \in \Gamma_0(\mathcal{H})$.*

Proof. By Lemma 2.1(v), $g^* \in \Gamma_0(\mathcal{G})$. Thus, Lemma 2.4(vi) implies that $\overset{1}{\gamma}(g^*) \circ L \in \Gamma_0(\mathcal{H})$. In turn, we deduce from Lemma 2.4(ii) and Definition 1.1 that $L \overset{\gamma}{\diamond} g + \mathcal{Q}_{\mathcal{H}}/\gamma = (\overset{1}{\gamma}(g^*) \circ L)^* \in \Gamma_0(\mathcal{H})$. Since $(-\beta - 1/\gamma)\mathcal{Q}_{\mathcal{H}}$ is continuous with domain $\mathcal{G}$, by [2, Lemma 1.27], $L \overset{\gamma}{\diamond} g - \beta\mathcal{Q}_{\mathcal{H}} = L \overset{\gamma}{\diamond} g + \mathcal{Q}_{\mathcal{H}}/\gamma + (-\beta - 1/\gamma)\mathcal{Q}_{\mathcal{H}}$ is proper and lower semicontinuous. It remains to show that $L \overset{\gamma}{\diamond} g - \beta\mathcal{Q}_{\mathcal{H}}$ is convex. Let $x \in \mathcal{H}$, set

$\psi = \|L\|^2 \mathscr{Q}_{\mathcal{G}} - \mathscr{Q}_{\mathcal{H}} \circ L^*$, and set $\Phi = \mathscr{Q}_{\mathcal{G}} - \mathscr{Q}_{\mathcal{H}} \circ L^*$. By Proposition 3.2(i),

$$\begin{aligned}
(L \overset{\gamma}{\diamond} g)(x) - \beta\mathscr{Q}_{\mathcal{H}}(x) &= \min_{\substack{y\in\mathcal{G}\\ L^*y=x}} \Big(g^{**}(y) + \frac{1}{\gamma}\Phi(y)\Big) - \beta\mathscr{Q}_{\mathcal{H}}(x)\\
&= \min_{\substack{y\in\mathcal{G}\\ L^*y=x}} \Big(g^{**}(y) + \frac{1}{\gamma}\Phi(y) - \beta\mathscr{Q}_{\mathcal{H}}(L^*y)\Big)\\
&= \min_{\substack{y\in\mathcal{G}\\ L^*y=x}} \Big(g^{**}(y) + \frac{1}{\gamma}\mathscr{Q}_{\mathcal{G}}(y) - \frac{1}{\|L\|^2}\Big(\alpha + \frac{1}{\gamma}\Big)\mathscr{Q}_{\mathcal{H}}(L^*y)\Big)\\
&= \min_{\substack{y\in\mathcal{G}\\ L^*y=x}} \Big(\big(g^{**}(y) - \alpha\mathscr{Q}_{\mathcal{G}}(y)\big) + \Big(\beta + \frac{1}{\gamma}\Big)\psi(y)\Big).
\end{aligned} \tag{3.12}$$

Since $\nabla\psi = \|L\|^2\mathrm{Id}_{\mathcal{G}} - L\circ L^*$, for every $y\in\mathcal{G}$, $\langle\nabla\psi(y)\mid y\rangle_{\mathcal{G}} = \|L\|^2\|y\|^2_{\mathcal{G}} - \|L^*y\|^2_{\mathcal{H}} \geqslant 0$. Therefore, we infer from [2, Proposition 17.7] that ψ is convex. Further, since $\alpha + 1/\gamma \geqslant 0$, $(\beta+1/\gamma)\psi$ is convex with domain $\mathcal{G}$. By assumption, $g^{**} - \alpha\mathscr{Q}_{\mathcal{G}} \in \Gamma_0(\mathcal{G})$. Hence, the function $(g^{**} - \alpha\mathscr{Q}_{\mathcal{G}}) + (\beta+1/\gamma)\psi$ is proper and convex. Altogether, in view of (3.12) and [2, Proposition 12.36(ii)], we conclude that $L \overset{\gamma}{\diamond} g - \beta\mathscr{Q}_{\mathcal{H}}$ is convex. ☐

Proposition 3.7. *Suppose that* $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ *satisfies* $0 < \|L\| \leqslant 1$, *let* $g\colon \mathcal{G} \to\,]-\infty,+\infty]$ *be a proper function such that* $\operatorname{cam} g \neq \varnothing$, *and let* $\gamma \in\,]0,+\infty[$. *Then the following hold:*

(i) $L \overset{\gamma}{\diamond} g \in \Gamma_0(\mathcal{H})$ *and* $L \overset{\gamma}{\blacklozenge} g \in \Gamma_0(\mathcal{H})$.

(ii) $(L \overset{\gamma}{\blacklozenge} g)^* = L \overset{1/\gamma}{\diamond} g^*$

(iii) $L \overset{\gamma}{\diamond} g = (L \overset{1/\gamma}{\blacklozenge} g^*)^*$.

Proof. Recall that Lemmas 2.1(v) and 2.4(i) assert that $g^* \in \Gamma_0(\mathcal{G})$ and $\operatorname{cam} g^* \neq \varnothing$.

(i): Lemma 2.4(ii) yields $g^{**} \in \Gamma_0(\mathcal{G})$. Now set $\beta = (1/\|L\|^2 - 1)/\gamma$. Then $\beta \geqslant 0$ and, by applying Proposition 3.6 with $\alpha = 0$, we see that $L \overset{\gamma}{\diamond} g - \beta\mathscr{Q}_{\mathcal{H}} \in \Gamma_0(\mathcal{H})$ and hence that $L \overset{\gamma}{\diamond} g \in \Gamma_0(\mathcal{H})$. Likewise, applying Proposition 3.6 with $\alpha = 0$ to $g^* \in \Gamma_0(\mathcal{G})$ and using Lemma 2.1(iii) we get $L \overset{1/\gamma}{\diamond} g^* \in \Gamma_0(\mathcal{H})$. In view of Definition 1.1 and Lemma 2.4(ii), we conclude that $L \overset{\gamma}{\blacklozenge} g \in \Gamma_0(\mathcal{H})$.

(ii): We derive from Definition 1.1, (i), and Lemma 2.4(ii) that $(L \overset{\gamma}{\blacklozenge} g)^* = (L \overset{1/\gamma}{\diamond} g^*)^{**} = L \overset{1/\gamma}{\diamond} g^*$.

(iii): By Proposition 3.1(iv), (i), and Lemma 2.4(ii), $(L \overset{1/\gamma}{\blacklozenge} g^*)^* = (L \overset{\gamma}{\diamond} g)^{**} = L \overset{\gamma}{\diamond} g$. ☐

The next result examines differentiability.

Proposition 3.8. *Suppose that* $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ *satisfies* $0 < \|L\| \leqslant 1$, *let* $g\colon \mathcal{G} \to\,]-\infty,+\infty]$ *be a proper function such that* $\operatorname{cam} g \neq \varnothing$, *and let* $\gamma \in\,]0,+\infty[$. *Then the following hold:*

(i) *Suppose that* $\|L\| < 1$ *and set* $\beta = \gamma(1/\|L\|^2 - 1)$. *Then* $L \overset{\gamma}{\blacklozenge} g$ *is differentiable with a* $(1/\beta)$*-Lipschitzian gradient.*

(ii) *Let* $\theta \in\,]0,+\infty[$, *suppose that* g *is real-valued, convex, and differentiable with a* θ*-Lipschitzian gradient, and set* $\beta = (1/\theta + \gamma)/\|L\|^2 - \gamma$. *Then* $L \overset{\gamma}{\blacklozenge} g$ *is differentiable with a* $(1/\beta)$*-Lipschitzian gradient.*

Proof. We recall that a continuous convex function $f\colon \mathcal{H} \to \mathbb{R}$ is differentiable with a $(1/\beta)$-Lipschitzian gradient if and only if $f^* - \beta\mathcal{Q}_{\mathcal{H}}$ is convex [2, Theorem 18.15]. Further, by Proposition 3.7(ii), $(L \overset{\gamma}{\blacklozenge} g)^* = L \overset{1/\gamma}{\diamond} g^*$.

(i): By Proposition 3.2(iv)(a), $\operatorname{dom}(L \overset{\gamma}{\blacklozenge} g) = \mathcal{H}$. Now set $\alpha = 0$. Since $\alpha > -\gamma$, we deduce from Proposition 3.6 that $L \overset{1/\gamma}{\diamond} g^* - \beta\mathcal{Q}_{\mathcal{H}}$ is convex, i.e., that $(L \overset{\gamma}{\blacklozenge} g)^* - \beta\mathcal{Q}_{\mathcal{H}}$ is convex.

(ii): Since $g \in \Gamma_0(\mathcal{G})$, Lemma 2.4(ii) yields $\operatorname{dom} g^{**} = \operatorname{dom} g = \mathcal{G}$. Thus, it results from Proposition 3.2(iv)(b) that $\operatorname{dom}(L \overset{\gamma}{\blacklozenge} g) = \mathcal{H}$. Now set $\alpha = 1/\theta$. Since $g^* - \alpha\mathcal{Q}_{\mathcal{G}}$is convex and $\alpha > -\gamma$, Proposition 3.6 implies that $(L \overset{\gamma}{\blacklozenge} g)^* - \beta\mathcal{Q}_{\mathcal{H}} = L \overset{1/\gamma}{\diamond} g^* - \beta\mathcal{Q}_{\mathcal{H}}$ is convex. ☐

Remark 3.9. Proposition 3.8(i) guarantees the smoothness of the proximal cocomposition when $0 < \|L\| < 1$. Proposition 3.8(ii) shows that the Lipschitz constant of the gradient of the cocomposition is improved when the original function is itself smooth.

The following proposition motivates calling $L \overset{\gamma}{\diamond} g$ a proximal composition.

Proposition 3.10. *Suppose that* $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ *satisfies* $0 < \|L\| \leqslant 1$, *let* $g\colon \mathcal{G} \to \left]-\infty,+\infty\right]$ *be a proper function such that* $\operatorname{cam} g \neq \varnothing$, *and let* $\gamma \in \left]0,+\infty\right[$. *Then the following hold:*

(i) $\operatorname{prox}_{\gamma(L \overset{\gamma}{\diamond} g)} = L^* \circ \operatorname{prox}_{\gamma g^{**}} \circ L$.

(ii) $\operatorname{prox}_{\gamma(L \overset{\gamma}{\blacklozenge} g)} = \operatorname{Id}_{\mathcal{H}} - L^* \circ \big(\operatorname{Id}_{\mathcal{G}} - \operatorname{prox}_{\gamma g^{**}}\big) \circ L$.

Proof. As previously noted, $g^* \in \Gamma_0(\mathcal{G})$ and $g^{**} \in \Gamma_0(\mathcal{G})$.

(i): It follows from Proposition 3.1(v) and Definition 1.1 that

$$\mathcal{Q}_{\mathcal{H}} + \gamma\big(L \overset{\gamma}{\diamond} g\big) = \mathcal{Q}_{\mathcal{H}} + L \overset{1}{\diamond} (\gamma g) = \Big({}^{1}\big((\gamma g)^*\big) \circ L\Big)^*. \tag{3.13}$$

Since Proposition 3.7(i) yields $L \overset{\gamma}{\diamond} g \in \Gamma_0(\mathcal{H})$, we deduce from [2, Corollary 16.48(iii)], (3.13), and items (iii) and (vii) in Lemma 2.4 that

$$\operatorname{Id}_{\mathcal{H}} + \gamma\partial\big(L \overset{\gamma}{\diamond} g\big) = \partial\Big(\mathcal{Q}_{\mathcal{H}} + \gamma\big(L \overset{\gamma}{\diamond} g\big)\Big) = \bigg(\nabla\Big({}^{1}\big((\gamma g)^*\big) \circ L\Big)\bigg)^{-1} = \Big(L^* \circ \big(\operatorname{Id}_{\mathcal{G}} - \operatorname{prox}_{(\gamma g)^*}\big) \circ L\Big)^{-1}. \tag{3.14}$$

Hence, by [2, Proposition 16.44] and Lemma 2.4(iv), we obtain $\operatorname{prox}_{\gamma(L \overset{\gamma}{\diamond} g)} = (\operatorname{Id}_{\mathcal{H}} + \gamma\partial(L \overset{\gamma}{\diamond} g))^{-1} = L^* \circ \operatorname{prox}_{(\gamma g)^{**}} \circ L = L^* \circ \operatorname{prox}_{\gamma g^{**}} \circ L$.

(ii): By Proposition 3.1(vii) and Definition 1.1, $\gamma(L \overset{\gamma}{\blacklozenge} g) = L \overset{1}{\blacklozenge} (\gamma g) = (L \overset{1}{\diamond} (\gamma g)^*)^*$. Therefore, Proposition 3.7(i) and Lemma 2.4(ii) entail that $(\gamma(L \overset{\gamma}{\blacklozenge} g))^* = L \overset{1}{\diamond} (\gamma g)^*$. In turn, Lemma 2.4(iv) and (i) yield $\operatorname{prox}_{\gamma(L \overset{\gamma}{\blacklozenge} g)} = \operatorname{Id}_{\mathcal{H}} - \operatorname{prox}_{L \overset{1}{\diamond} (\gamma g)^*} = \operatorname{Id}_{\mathcal{H}} - L^* \circ (\operatorname{Id}_{\mathcal{G}} - \operatorname{prox}_{\gamma g^{**}}) \circ L$. ☐

Our next result concerns the subdifferential of proximal compositions. We recall that the parallel composition of $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ by $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ is $L \triangleright A = (L \circ A^{-1} \circ L^*)^{-1}$ [2, Section 25.6].

Proposition 3.11. *Suppose that* $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ *satisfies* $0 < \|L\| \leqslant 1$, *let* $g\colon \mathcal{G} \to \left]-\infty,+\infty\right]$ *be a proper function such that* $\operatorname{cam} g \neq \varnothing$, *and let* $\gamma \in \left]0,+\infty\right[$. *Then the following hold:*

(i) $\partial(L \overset{\gamma}{\diamond} g) = L^* \triangleright (\partial g^{**} + (\operatorname{Id}_{\mathcal{G}} - L \circ L^*)/\gamma)$.

(ii) $\partial(L \overset{\gamma}{\blacklozenge} g) = L^* \circ (\partial g^* + \gamma(\mathrm{Id}_{\mathcal{G}} - L \circ L^*))^{-1} \circ L$.

Proof. As seen in Proposition 3.7(i), $L \overset{\gamma}{\diamond} g \in \Gamma_0(\mathcal{H})$ and $L \overset{\gamma}{\blacklozenge} g \in \Gamma_0(\mathcal{H})$. Now set $\Phi = \mathcal{Q}_{\mathcal{G}} - \mathcal{Q}_{\mathcal{H}} \circ L^*$ and $h = g^{**} + \Phi/\gamma$. We deduce from Lemmas 2.1(v), 2.4(ii), and 2.7 that $g^* \in \Gamma_0(\mathcal{G})$, $g^{**} \in \Gamma_0(\mathcal{G})$, and $\Phi \in \Gamma_0(\mathcal{G})$. Therefore, since $\mathrm{dom}\,\Phi = \mathcal{G}$, we have $h \in \Gamma_0(\mathcal{G})$ and, by Lemma 2.4(ii), $h^{**} = h$. On the other hand, $\mathrm{dom}\,h^* \cap \mathrm{ran}\,L \neq \varnothing$ since Propositions 3.2(ii) and 3.7(i) yield $h^* \circ L = L \overset{1/\gamma}{\blacklozenge} g^* \in \Gamma_0(\mathcal{G})$. Upon invoking Propositions 3.2(i) and 3.7(iii), we get

$$L^* \triangleright h = L \overset{\gamma}{\diamond} g = \Big(L \overset{1/\gamma}{\blacklozenge} g^*\Big)^* = \big(h^* \circ L\big)^*. \tag{3.15}$$

Therefore, [2, Proposition 16.42], Lemma 2.4(iii), and [2, Corollary 16.48(iii)] imply that

$$\partial(h^* \circ L) = L^* \circ \partial h^* \circ L = L^* \circ \big(\partial h\big)^{-1} \circ L = L^* \circ \big(\partial g^{**} + \nabla\Phi/\gamma\big)^{-1} \circ L. \tag{3.16}$$

(i): Combining (3.15), Lemma 2.4(iii), and (3.16), we obtain

$$\partial\big(L \overset{\gamma}{\diamond} g\big) = \partial\big(h^* \circ L\big)^* = \big(\partial(h^* \circ L)\big)^{-1} = \Big(L^* \circ \big(\partial g^{**} + \nabla\Phi/\gamma\big)^{-1} \circ L\Big)^{-1} = L^* \triangleright \big(\partial g^{**} + \nabla\Phi/\gamma\big). \tag{3.17}$$

(ii): By Definition 1.1, Lemma 2.4(iii), (i), and Lemma 2.1(iii),

$$\partial\big(L \overset{\gamma}{\blacklozenge} g\big) = \partial\big(L \overset{1/\gamma}{\diamond} g^*\big)^* = \Big(\partial\big(L \overset{1/\gamma}{\diamond} g^*\big)\Big)^{-1} = \Big(L^* \triangleright \big(\partial g^{***} + \gamma\nabla\Phi\big)\Big)^{-1} = L^* \circ \big(\partial g^* + \gamma\nabla\Phi\big)^{-1} \circ L, \tag{3.18}$$

which completes the proof. ☐

Corollary 3.12. *Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$, let $\beta \in \left]0,+\infty\right[$, let $\gamma \in \left]0,+\infty\right[$, and let $g\colon \mathcal{G} \to \mathbb{R}$ be convex and β-Lipschitzian. Then $L \overset{\gamma}{\blacklozenge} g$ is $(\beta\|L\|)$-Lipschitzian.*

Proof. We recall that a lower semicontinuous convex function $f\colon \mathcal{H} \to \mathbb{R}$ is β-Lipschitzian if and only if $\mathrm{ran}\,\partial f = \mathrm{dom}\,\partial f^* \subset B(0;\beta)$ [2, Corollary 17.19]. Since $g \in \Gamma_0(\mathcal{G})$, Lemma 2.4(ii) yields $g^* \in \Gamma_0(\mathcal{G})$. We therefore invoke Proposition 3.11(ii) to get

$$\begin{aligned} \mathrm{ran}\,\partial\big(L \overset{\gamma}{\blacklozenge} g\big) &\subset L^*\Big(\mathrm{ran}\big(\partial g^* + \gamma(\mathrm{Id}_{\mathcal{G}} - L \circ L^*)\big)^{-1}\Big) \\ &= L^*\Big(\mathrm{dom}\big(\partial g^* + \gamma(\mathrm{Id}_{\mathcal{G}} - L \circ L^*)\big)\Big) \\ &= L^*\big(\mathrm{dom}\,\partial g^*\big) \\ &\subset L^*\big(B(0;\beta)\big) \\ &\subset B(0;\beta\|L\|), \end{aligned} \tag{3.19}$$

where $L \overset{\gamma}{\blacklozenge} g\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ is a real-valued lower semicontinuous convex function by Propositions 3.2(iv)(b) and 3.7(i). ☐

Let us now evaluate Moreau envelopes of proximal cocompositions.

Proposition 3.13. *Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$, let $g\colon \mathcal{G} \to \left]-\infty,+\infty\right]$ be a proper function such that $\mathrm{cam}\,g \neq \varnothing$, let $\gamma \in \left]0,+\infty\right[$, and let $\rho \in \left]0,+\infty\right[$. Then the following hold:*

(i) ${}^{\rho}\big(L \overset{\gamma+\rho}{\blacklozenge} g\big) = L \overset{\gamma}{\blacklozenge} ({}^{\rho}g)$.

(ii) ${}^{\gamma}(L \overset{\gamma}{\blacklozenge} g) = {}^{\gamma}(g^{**}) \circ L$.

Proof. By Lemma 2.1(v) and Proposition 3.7(i), $L \overset{1/\gamma}{\diamond} g^* \in \Gamma_0(\mathcal{H})$. Therefore, Lemma 2.4(viii) and Definition 1.1 yield

$$\Big(\big(L \overset{1/\gamma}{\diamond} g^*\big) + \rho\mathcal{Q}_{\mathcal{H}}\Big)^* = {}^{\rho}\Big(\big(L \overset{1/\gamma}{\diamond} g^*\big)^*\Big) = {}^{\rho}\big(L \overset{\gamma}{\blacklozenge} g\big). \tag{3.20}$$

(i): We combine Definition 1.1, Lemma 2.5(i), Proposition 3.5(i), and (3.20) to arrive at

$$L \overset{\gamma}{\blacklozenge} ({}^{\rho}g) = \Big(L \overset{1/\gamma}{\diamond} ({}^{\rho}g)^*\Big)^* = \Big(L \overset{1/\gamma}{\diamond} (g^* + \rho\mathcal{Q}_{\mathcal{G}})\Big)^* = \Big(\big(L \overset{1/(\gamma+\rho)}{\diamond} g^*\big) + \rho\mathcal{Q}_{\mathcal{H}}\Big)^* = {}^{\rho}\big(L \overset{\gamma+\rho}{\blacklozenge} g\big). \tag{3.21}$$

(ii): Since $g^* \in \Gamma_0(\mathcal{G})$, items (ii) and (vi) in Lemma 2.4 imply that ${}^{\gamma}(g^{**}) \in \Gamma_0(\mathcal{G})$ and that $\operatorname{dom}{}^{\gamma}(g^{**}) = \mathcal{G}$. Hence, ${}^{\gamma}(g^{**}) \circ L \in \Gamma_0(\mathcal{H})$ and it follows from Lemma 2.4(ii), Definition 1.1, and (3.20) that

$${}^{\gamma}(g^{**}) \circ L = \Big({}^{\gamma}(g^{**}) \circ L\Big)^{**} = \Big(\big(L \overset{1/\gamma}{\diamond} g^*\big) + \gamma\mathcal{Q}_{\mathcal{G}}\Big)^* = {}^{\gamma}\big(L \overset{\gamma}{\blacklozenge} g\big), \tag{3.22}$$

as announced. ☐

Corollary 3.14. *Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$, let $g\colon \mathcal{G} \to \left]-\infty,+\infty\right]$ be a proper function such that $\operatorname{cam} g \neq \varnothing$, and let $\gamma \in \left]0,+\infty\right[$. Then $\operatorname{Argmin}(L \overset{\gamma}{\blacklozenge} g) = \operatorname{Argmin}({}^{\gamma}(g^{**}) \circ L)$.*

Proof. Since the set of minimizers of a function in $\Gamma_0(\mathcal{H})$ coincides with that of its Moreau envelope [2, Propositions 17.5], the assertion follows from Proposition 3.13(ii). ☐

Corollary 3.15. *Let $\mathcal{K}$ be a real Hilbert space, suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ and $S \in \mathcal{B}(\mathcal{K},\mathcal{H})$ satisfy $\|L\| \leqslant 1$, $\|S\| \leqslant 1$, and $L \circ S \neq 0$, let $g\colon \mathcal{G} \to \left]-\infty,+\infty\right]$ be a proper function such that $\operatorname{cam} g \neq \varnothing$, and let $\gamma \in \left]0,+\infty\right[$. Then the following hold:*

(i) $S \overset{\gamma}{\blacklozenge} (L \overset{\gamma}{\blacklozenge} g) = (L \circ S) \overset{\gamma}{\blacklozenge} g$.

(ii) $S \overset{\gamma}{\diamond} (L \overset{\gamma}{\diamond} g) = (L \circ S) \overset{\gamma}{\diamond} g$.

Proof. (i): Set $f = L \overset{\gamma}{\blacklozenge} g$. Since $\|L \circ S\| \leqslant \|L\|\,\|S\| \leqslant 1$, we deduce from Proposition 3.7(i) that $f \in \Gamma_0(\mathcal{H})$, $S \overset{\gamma}{\blacklozenge} f \in \Gamma_0(\mathcal{K})$, and $(L \circ S) \overset{\gamma}{\blacklozenge} g \in \Gamma_0(\mathcal{K})$. By Lemma 2.4(ii), $f^{**} = f$. Hence, Proposition 3.13(ii) yields

$${}^{\gamma}(S \overset{\gamma}{\blacklozenge} f) = {}^{\gamma}(f^{**}) \circ S = {}^{\gamma}f \circ S = \Big({}^{\gamma}(g^{**}) \circ L\Big) \circ S = {}^{\gamma}\Big((L \circ S) \overset{\gamma}{\blacklozenge} g\Big). \tag{3.23}$$

Therefore, the assertion follows from Lemma 2.6.

(ii): By Proposition 3.7(i), $L \overset{\gamma}{\diamond} g \in \Gamma_0(\mathcal{H})$, $S \overset{\gamma}{\diamond} (L \overset{\gamma}{\diamond} g) \in \Gamma_0(\mathcal{K})$, and $(L \circ S) \overset{\gamma}{\diamond} g \in \Gamma_0(\mathcal{K})$. Therefore, using Propositions 3.7(iii) and 3.1(ii), together with (i), we get

$$S \overset{\gamma}{\diamond} (L \overset{\gamma}{\diamond} g) = \Big(S \overset{1/\gamma}{\blacklozenge} (L \overset{\gamma}{\diamond} g)^*\Big)^* = \Big(S \overset{1/\gamma}{\blacklozenge} \big(L \overset{1/\gamma}{\blacklozenge} g^*\big)\Big)^* = \Big((L \circ S) \overset{1/\gamma}{\blacklozenge} g^*\Big)^* = (L \circ S) \overset{\gamma}{\diamond} g, \tag{3.24}$$

which completes the proof. ☐

Proposition 3.16. *Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$, let $g\colon \mathcal{G} \to \left]-\infty,+\infty\right]$ be a proper function such that $\operatorname{cam} g \neq \varnothing$, and let $\gamma \in \left]0,+\infty\right[$. Then $\operatorname{rec}(L \overset{\gamma}{\blacklozenge} g) = (\operatorname{rec}(g^{**})) \circ L$.*

Proof. By Lemmas 2.1(v) and 2.4(ii), $g^* \in \Gamma_0(\mathcal{G})$ and $g^{**} \in \Gamma_0(\mathcal{G})$. Therefore, Lemma 2.4(v), Propositions 3.7(ii) and 3.2(iii), and Lemma 2.1(iii) imply that

$$\operatorname{rec}(L \overset{\gamma}{\blacklozenge} g) = \sigma_{\operatorname{dom}(L \overset{\gamma}{\blacklozenge} g)^*} = \sigma_{\operatorname{dom}(L \overset{1/\gamma}{\diamond} g^*)} = \sigma_{L^*(\operatorname{dom} g^{***})} = \sigma_{\operatorname{dom} g^*} \circ L = \big(\operatorname{rec}(g^{**})\big) \circ L, \tag{3.25}$$

as claimed. ☐

Proposition 3.17. *Suppose that* $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ *satisfies* $0 < \|L\| \leqslant 1$, *let* $g \in \Gamma_0(\mathcal{G})$, *let*

$$\widetilde{g} \colon \mathcal{G} \oplus \mathbb{R} \to \left]-\infty, +\infty\right] \colon (y, \eta) \mapsto \begin{cases} \eta g(y/\eta), & \text{if } \eta > 0; \\ (\operatorname{rec} g)(y), & \text{if } \eta = 0; \\ +\infty, & \text{otherwise} \end{cases} \tag{3.26}$$

be its perspective, let $\gamma \in \left]0, +\infty\right[$, *and set* $\widetilde{L} \colon \mathcal{H} \oplus \mathbb{R} \to \mathcal{G} \oplus \mathbb{R} \colon (x, \xi) \mapsto (Lx, \xi)$. *Then*

$$\widetilde{L \overset{\gamma}{\blacklozenge} g} \colon \mathcal{H} \oplus \mathbb{R} \to \left]-\infty, +\infty\right] \colon (x, \xi) \mapsto \begin{cases} \Big(\widetilde{L} \overset{\xi\gamma}{\blacklozenge} \widetilde{g}\Big)(x, \xi), & \text{if } \xi > 0; \\ (\operatorname{rec} g)(Lx), & \text{if } \xi = 0; \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.27}$$

Proof. Let $(x, \xi) \in \mathcal{H} \oplus \mathbb{R}$, set $\Phi = \mathcal{Q}_{\mathcal{G}} - \mathcal{Q}_{\mathcal{H}} \circ L^*$, and set $\Psi = \mathcal{Q}_{\mathcal{G} \oplus \mathbb{R}} - \mathcal{Q}_{\mathcal{H} \oplus \mathbb{R}} \circ \widetilde{L}^*$. We consider two cases.

- $\xi = 0$: It follows from Proposition 3.16 and Lemma 2.4(i)–(ii) that $(\widetilde{L \overset{\gamma}{\blacklozenge} g})(x, 0) = (\operatorname{rec}(L \overset{\gamma}{\blacklozenge} g))(x) = (\operatorname{rec} g)(Lx)$.
- $\xi > 0$: Set $C = \big\{(y^*, \eta) \in \mathcal{G} \oplus \mathbb{R} \mid \eta + g^*(y^*) \leqslant 0\big\}$. Then [8, Items (ii) and (iv) in Proposition 2.3] assert that $\widetilde{g} \in \Gamma_0(\mathcal{G} \oplus \mathbb{R})$ and $(\widetilde{g})^* = \iota_C$. Therefore, by Lemma 2.2(ii),

$$\begin{aligned} (\forall y^* \in \mathcal{G}) \quad \sup_{\eta \in \mathbb{R}} \big(\eta\xi - (\widetilde{g})^*(y^*, \eta)\big) &= \sup_{\eta \in \mathbb{R}} \big(\eta\xi - \iota_C(y^*, \eta)\big) \\ &= \sup_{\eta \in \left]-\infty, -g^*(y^*)\right]} \eta\xi \\ &= -\xi g^*(y^*) \\ &= -\big(\xi g(\cdot/\xi)\big)^*(y^*). \end{aligned} \tag{3.28}$$

  On the other hand, for every $\eta \in \mathbb{R}$, $\Psi(\cdot, \eta) = \Phi$ and, since $0 < \|L\| \leqslant 1$, we have $0 < \|\widetilde{L}\| \leqslant 1$.

Hence, appealing to Proposition 3.2(ii), (3.28), and Proposition 3.1(vii)–(viii),

$$
\begin{aligned}
\Big(\widetilde{L} \overset{\xi\gamma}{\blacklozenge} \widetilde{g}\Big)(x,\xi) &= \big((\widetilde{g})^* + \xi\gamma\Psi\big)^*\big(\widetilde{L}(x,\xi)\big)\\
&= \big((\widetilde{g})^* + \xi\gamma\Psi\big)^*(Lx,\xi)\\
&= \sup_{(y^*,\eta)\in\mathcal{G}\oplus\mathbb{R}} \big(\langle (Lx,\xi) \mid (y^*,\eta)\rangle_{\mathcal{G}\oplus\mathbb{R}} - (\widetilde{g})^*(y^*,\eta) - \xi\gamma\Psi(y^*,\eta)\big)\\
&= \sup_{(y^*,\eta)\in\mathcal{G}\oplus\mathbb{R}} \big(\eta\xi + \langle Lx \mid y^*\rangle_{\mathcal{G}} - (\widetilde{g})^*(y^*,\eta) - \xi\gamma\Phi(y^*)\big)\\
&= \sup_{y^*\in\mathcal{G}} \Big(\langle Lx \mid y^*\rangle_{\mathcal{G}} - \xi\gamma\Phi(y^*) + \sup_{\eta\in\mathbb{R}}\big(\eta\xi - (\widetilde{g})^*(y^*,\eta)\big)\Big)\\
&= \sup_{y^*\in\mathcal{G}} \big(\langle Lx \mid y^*\rangle_{\mathcal{G}} - \xi\gamma\Phi(y^*) - \big(\xi g(\cdot/\xi)\big)^*(y^*)\big)\\
&= \Big(\big(\xi g(\cdot/\xi)\big)^* + \xi\gamma\Phi\Big)^*(Lx)\\
&= \Big(L \overset{\xi\gamma}{\blacklozenge} \big(\xi g(\cdot/\xi)\big)\Big)(x)\\
&= \xi\big(L \overset{\gamma}{\blacklozenge} g\big)(x/\xi)\\
&= \Big(\widetilde{L \overset{\gamma}{\blacklozenge} g}\Big)(x,\xi).
\end{aligned}
\tag{3.29}
$$

We have thus proved (3.27). ☐

3.3. Comparison with standard compositions and infimal postcompositions

As mentioned in Section 1, our discussion involves several ways to compose a function defined on $\mathcal{G}$ with a linear operator from $\mathcal{H}$ to $\mathcal{G}$ in order to obtain a function defined on $\mathcal{H}$: the standard composition (1.3), the infimal postcomposition (1.4), and the proximal composition and cocomposition of Definition 1.1. We saw in Proposition 3.10 that a numerical advantage of the proximal compositions is that their proximity operators are easily decomposable in terms of that of the underlying function. Our purpose here is to compare these various compositions.

Example 3.18. Let

$$
\begin{cases}
L\colon \mathbb{R}^2 \to \mathbb{R}^5\colon (\xi_1,\xi_2) \mapsto \big(0.5\xi_2, -0.5\xi_1, -0.5\xi_2, 0.3\xi_1 + 0.4\xi_2, 0.1\xi_1 - 0.3\xi_2\big)\\
g\colon \mathbb{R}^5 \to \mathbb{R}\colon (\eta_1,\eta_2,\eta_3,\eta_4,\eta_5) \mapsto \|(\eta_1,\eta_2,\eta_3)\|_1 + \|(\eta_4 - 1, \eta_5 + 2)\|.
\end{cases}
\tag{3.30}
$$

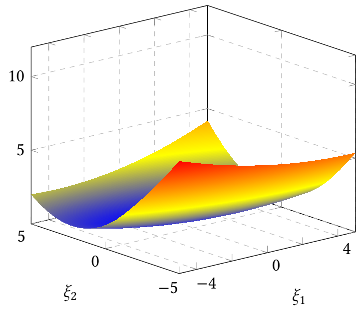
Figure 1 shows the graphs of both the standard composition and proximal cocomposition for various values of γ.

Example 3.19. Let $C = B(0;2)$ and

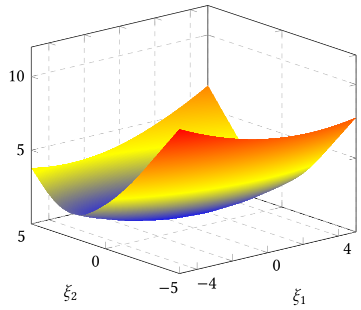
$$
\begin{cases}
L\colon \mathbb{R}^2 \to \mathbb{R}^3\colon (\xi_1,\xi_2) \mapsto \big(0.7\xi_1 + 0.1\xi_2, -0.3\xi_1 + 0.4\xi_2, 0.5\xi_1 - 0.3\xi_2\big)\\
g\colon \mathbb{R}^3 \to \mathbb{R}\colon (\eta_1,\eta_2,\eta_3) \mapsto d_C(\eta_1,\eta_2,\eta_3).
\end{cases}
\tag{3.31}
$$

Figure 2 shows the graphs of both the standard composition and proximal cocomposition for various values of γ.

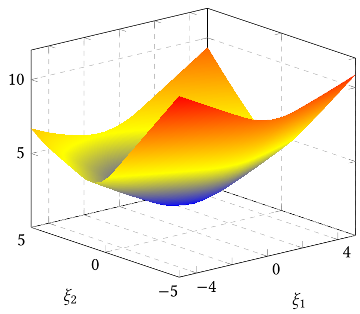
As we now show, the pointwise orderings suggested by Figures 1 and 2 are generally true.



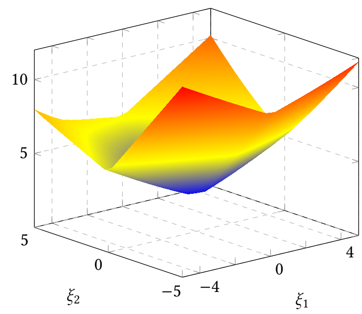
(a) Graph of $L \overset{10}{\diamond} g$.



(b) Graph of $L \overset{5}{\diamond} g$.

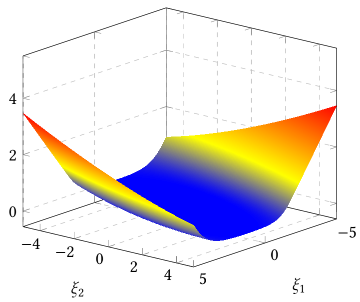


(c) Graph of $L \overset{1}{\diamond} g$.

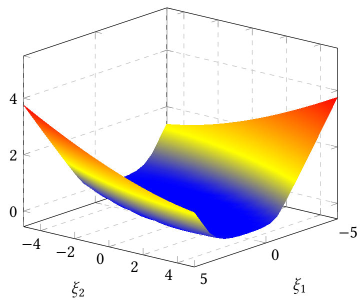


(d) Graph of $g \circ L$.

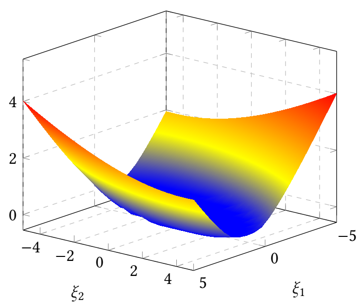
Figure 1: Graphs of the proximal cocomposition and of the standard composition in Example 3.18.



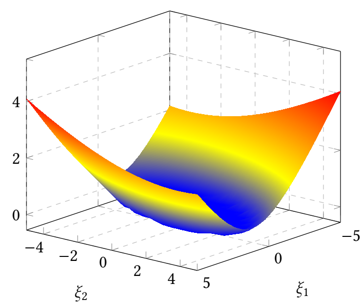
(a) Graph of $L \overset{10}{\diamond} g$.



(b) Graph of $L \overset{5}{\diamond} g$.



(c) Graph of $L \overset{1}{\diamond} g$.



(d) Graph of $g \circ L$.

Figure 2: Graphs of the proximal cocomposition and of the standard composition in Example 3.19.

Proposition 3.20. *Suppose that* $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ *satisfies* $0 < \|L\| \leqslant 1$, *let* $g\colon \mathcal{G} \to \left]-\infty,+\infty\right]$ *be a proper function such that* $\operatorname{cam} g \neq \varnothing$, *and let* $\gamma \in \left]0,+\infty\right[$. *Then the following hold:*

(i) $L^* \blacktriangleright g^{**} \leqslant L \overset{\gamma}{\diamond} g$.

(ii) ${}^{\gamma}(g^{**}) \circ L \leqslant L \overset{\gamma}{\blacklozenge} g \leqslant g^{**} \circ L$.

(iii) $L \overset{\gamma}{\blacklozenge} g \leqslant L \overset{\gamma}{\diamond} g$.

(iv) *Suppose that* L *is an isometry. Then* $L \overset{\gamma}{\diamond} g = L \overset{\gamma}{\blacklozenge} g$.

(v) *Suppose that* L *is a coisometry. Then* $L \overset{\gamma}{\diamond} g = L^* \blacktriangleright g^{**}$ *and* $L \overset{\gamma}{\blacklozenge} g = g^{**} \circ L$.

(vi) *Suppose that* L *is invertible with* $L^{-1} = L^*$. *Then* $L \overset{\gamma}{\diamond} g = L^* \blacktriangleright g^{**} = g^{**} \circ L = L \overset{\gamma}{\blacklozenge} g$.

Proof. Set $\Phi = \mathcal{Q}_{\mathcal{G}} - \mathcal{Q}_{\mathcal{H}} \circ L^*$ and observe that $0 \leqslant \Phi \leqslant \mathcal{Q}_{\mathcal{G}}$.

(i): Let $x \in \mathcal{H}$. By Proposition 3.2(i),

$$\big(L \overset{\gamma}{\diamond} g\big)(x) = \min_{\substack{y\in\mathcal{G}\\ L^*y=x}} \left(g^{**}(y) + \frac{1}{\gamma}\Phi(y)\right) \geqslant \inf_{\substack{y\in\mathcal{G}\\ L^*y=x}} g^{**}(y) = \big(L^* \blacktriangleright g^{**}\big)(x). \tag{3.32}$$

(ii): The leftmost inequality is established in Proposition 3.2(v). Let us prove rightmost inequality. By Lemma 2.1(ii) and (i), $(L \overset{1/\gamma}{\diamond} g^*)^* \leqslant (L^* \blacktriangleright g^{***})^*$. It therefore follows from Definition 1.1 and Lemmas 2.1(iii) and 2.5(ii) that

$$L \overset{\gamma}{\blacklozenge} g = \big(L \overset{1/\gamma}{\diamond} g^*\big)^* \leqslant (L^* \blacktriangleright g^*)^* = g^{**} \circ L. \tag{3.33}$$

(iii): Set $f = {}^{1}(g^{**}) \circ L$. Since $\|L\| \leqslant 1$, $\mathcal{Q}_{\mathcal{G}} \circ L \leqslant \mathcal{Q}_{\mathcal{H}}$, and we deduce from Lemma 2.1(ii) that $(\mathcal{Q}_{\mathcal{H}} - f)^* \leqslant (\mathcal{Q}_{\mathcal{G}} \circ L - f)^*$. However, it results from Lemma 2.4(iv) that $\mathcal{Q}_{\mathcal{G}} \circ L - f = (\mathcal{Q}_{\mathcal{G}} - {}^{1}(g^{**})) \circ L = {}^{1}(g^*) \circ L$. Altogether, it follows from Definition 1.1 and [2, Proposition 13.29] that

$$L \overset{1}{\blacklozenge} g = \big(f^* - \mathcal{Q}_{\mathcal{H}}\big)^* = \big(\mathcal{Q}_{\mathcal{H}} - f\big)^* - \mathcal{Q}_{\mathcal{H}} \leqslant \big({}^{1}(g^*) \circ L\big)^* - \mathcal{Q}_{\mathcal{H}} = L \overset{1}{\diamond} g. \tag{3.34}$$

Hence, by Proposition 3.1(vii), (3.34), and Proposition 3.1(v), we get

$$L \overset{\gamma}{\blacklozenge} g = \frac{1}{\gamma}\big(L \overset{1}{\blacklozenge} (\gamma g)\big) \leqslant \frac{1}{\gamma}\big(L \overset{1}{\diamond} (\gamma g)\big) = L \overset{\gamma}{\diamond} g. \tag{3.35}$$

(iv): Here $\mathcal{Q}_{\mathcal{H}} = \mathcal{Q}_{\mathcal{G}} \circ L$ and therefore the inequalities in the proof of (iii) can be replaced with equalities.

(v): Here $\mathcal{Q}_{\mathcal{G}} = \mathcal{Q}_{\mathcal{H}} \circ L^*$ and thus $\Phi = 0$. Therefore, the result follows from Proposition 3.2(i)–(ii).

(vi): A consequence of (iv) and (v). □

Remark 3.21. Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ is an isometry, let $g\colon \mathcal{G} \to \left]-\infty,+\infty\right]$ be a proper function such that $\operatorname{cam} g \neq \varnothing$, and let $\gamma \in \left]0,+\infty\right[$. Then we recover from [2, Proposition 13.24(v)] as well as items (i), (iv), and (ii) in Proposition 3.20 the inequalities

$$(g^* \circ L)^* \leqslant L^* \blacktriangleright g^{**} \leqslant L \overset{\gamma}{\diamond} g = L \overset{\gamma}{\blacklozenge} g \leqslant g^{**} \circ L, \tag{3.36}$$

which appear in [9, Proposition 5.4] in the special case in which $\gamma = 1$.

Proposition 3.22. *Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$, let $g\colon \mathcal{G} \to]-\infty,+\infty]$ be a proper function such that $\operatorname{cam} g \neq \varnothing$, let $\gamma \in]0,+\infty[$, let $x \in \mathcal{H}$, and set $\Phi = \mathcal{Q}_{\mathcal{G}} - \mathcal{Q}_{\mathcal{H}} \circ L^*$. Then the following hold:*

(i) *Suppose that $y^* \in \partial g(Lx)$. Then $0 \leqslant g(Lx) - (L \overset{\gamma}{\blacklozenge} g)(x) \leqslant \gamma\Phi(y^*)$.*

(ii) *Suppose that $0 \in (\mathrm{Id}_{\mathcal{G}} - L \circ L^*)(\partial g(Lx))$. Then $(L \overset{\gamma}{\blacklozenge} g)(x) = g(Lx)$.*

Proof. (i): By [2, Proposition 16.10], $g(Lx) + g^*(y^*) = \langle Lx \mid y^*\rangle_{\mathcal{G}}$. Further, [2, Proposition 16.5] yields $g^{**}(Lx) = g(Lx) \in \mathbb{R}$. Therefore, we deduce from Propositions 3.20(ii) and 3.2(ii) that $(L \overset{\gamma}{\blacklozenge} g)(x) \in \mathbb{R}$ and that

$$\begin{aligned} 0 &\leqslant g(Lx) - \big(L \overset{\gamma}{\blacklozenge} g\big)(x) \\ &= g(Lx) - \big(g^* + \gamma\Phi\big)^*(Lx) \\ &= g(Lx) - \sup_{y\in\mathcal{G}}\big(\langle Lx \mid y\rangle_{\mathcal{G}} - g^*(y) - \gamma\Phi(y)\big) \\ &\leqslant g(Lx) - \big(\langle Lx \mid y^*\rangle_{\mathcal{G}} - g^*(y^*) - \gamma\Phi(y^*)\big) \\ &= \gamma\Phi(y^*). \end{aligned} \tag{3.37}$$

(ii): There exists $y^* \in \partial g(Lx)$ such that $L(L^*y^*) = y^*$. Therefore, $\Phi(y^*) = 0$ and the conclusion follows from (i). ☐

Proposition 3.23. *Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$, let $\beta \in]0,+\infty[$, let $\gamma \in]0,+\infty[$, and let $g\colon \mathcal{G} \to \mathbb{R}$ be convex and β-Lipschitzian. Then the following hold:*

(i) $0 \leqslant g \circ L - L \overset{\gamma}{\blacklozenge} g \leqslant \gamma\beta^2/2$.

(ii) $L^* \triangleright g^* \leqslant L \overset{1/\gamma}{\diamond} g^* \leqslant (L^* \triangleright g^*) + \gamma\beta^2/2$.

Proof. We recall that a lower semicontinuous convex function $f\colon \mathcal{H} \to \mathbb{R}$ is β-Lipschitzian if and only if $\operatorname{ran}\partial f = \operatorname{dom}\partial f^* \subset B(0;\beta)$ [2, Corollary 17.19]. Moreover, since $\operatorname{dom} g = \mathcal{G}$, we have $\operatorname{dom}\partial g = \mathcal{G}$ by [2, Proposition 16.27].

(i): Let $x \in \mathcal{H}$ and set $\Phi = \mathcal{Q}_{\mathcal{G}} - \mathcal{Q}_{\mathcal{H}} \circ L^*$. Since $\operatorname{dom}\partial g = \mathcal{G}$, there exists $y^* \in \partial g(Lx) \subset \operatorname{ran}\partial g \subset B(0;\beta)$. Thus, $\Phi(y^*) \leqslant \mathcal{Q}_{\mathcal{G}}(y^*) \leqslant \beta^2/2$ and the result follows from Proposition 3.22(i).

(ii): The leftmost inequality follows from Proposition 3.20(i) and Lemma 2.1(iii). On the other hand, Proposition 3.7(i) implies that $L \overset{1/\gamma}{\diamond} g^* \in \Gamma_0(\mathcal{H})$. Additionally, in view of Lemma 2.1(ii) and (i), $(L \overset{\gamma}{\blacklozenge} g)^* \leqslant (g \circ L - \gamma\beta^2/2)^*$. Finally, we deduce from Proposition 3.7(ii) and [2, Proposition 13.24(v)] that

$$L \overset{1/\gamma}{\diamond} g^* = \big(L \overset{\gamma}{\blacklozenge} g\big)^* \leqslant \left(g \circ L - \frac{\gamma\beta^2}{2}\right)^* = \big(g \circ L\big)^* + \frac{\gamma\beta^2}{2} \leqslant \big(L^* \triangleright g^*\big) + \frac{\gamma\beta^2}{2}, \tag{3.38}$$

as required. ☐

Example 3.24. Let $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$, let $g \in \Gamma_0(\mathcal{G})$, let $\gamma \in]0,+\infty[$, and let $\rho \in]0,+\infty[$. Suppose that $L \circ L^* = \rho\mathrm{Id}_{\mathcal{G}}$. Then the following hold:

(i) Set $h = g(\sqrt{\rho}\,\cdot)$ and $S = L/\sqrt{\rho}$. Then $g \circ L = S \overset{\gamma}{\blacklozenge} h$.

(ii) $\operatorname{prox}_{\gamma g \circ L} = \mathrm{Id}_{\mathcal{H}} + \rho^{-1}L^* \circ (\operatorname{prox}_{\gamma\rho g} - \mathrm{Id}_{\mathcal{G}}) \circ L$.

Proof. (i): Since $L \circ L^* = \rho \mathrm{Id}_{\mathcal{G}}$, S is a coisometry, and we deduce from Proposition 3.20(v) and Lemma 2.4(ii) that $S \overset{\gamma}{\blacklozenge} h = h \circ S = g \circ L$.

(ii): This follows from (i) and Proposition 3.10(ii) (see also [2, Proposition 24.14]). ☐

Example 3.25. Let V be a closed vector subspace of $\mathcal{H}$ and $\gamma \in \left]0,+\infty\right[$. Then the following hold:

(i) $\mathrm{proj}_V \overset{\gamma}{\diamond} \|\cdot\| = \iota_V + \|\cdot\|$.

(ii) $\mathrm{proj}_V \overset{\gamma}{\blacklozenge} \|\cdot\| = \|\cdot\| \circ \mathrm{proj}_V$.

Proof. Set $\Phi = \mathcal{Q}_{\mathcal{H}} - \mathcal{Q}_{\mathcal{H}} \circ \mathrm{proj}_V$ and let $x \in \mathcal{H}$.

(i): It follows from Proposition 3.2(i), Lemma 2.4(ii), and the identity $\Phi = \mathcal{Q}_{\mathcal{H}} \circ \mathrm{proj}_{V^\perp}$ that

$$\Big(\mathrm{proj}_V \overset{\gamma}{\diamond} \|\cdot\|\Big)(x) = \inf_{\substack{y\in\mathcal{H}\\ \mathrm{proj}_V y = x}} \left(\|y\| + \frac{1}{2\gamma}\|x-y\|^2\right) = \begin{cases} \|x\|, & \text{if } x \in V \\ +\infty, & \text{if } x \notin V \end{cases} = \iota_V(x) + \|x\|. \tag{3.39}$$

(ii): We recall that $\partial\|\cdot\|(x) = \{x/\|x\|\}$ if $x \neq 0$ and that $\partial\|\cdot\|(0) = B(0;1)$ [2, Example 16.32]. Hence,

$$\begin{aligned} \mathrm{proj}_{V^\perp}\big(\partial\|\cdot\|(\mathrm{proj}_V x)\big) &= \begin{cases} \big\{\mathrm{proj}_{V^\perp}\big(\mathrm{proj}_V x/\|\mathrm{proj}_V x\|\big)\big\}, & \text{if } \mathrm{proj}_V x \neq 0; \\ \mathrm{proj}_{V^\perp}\big(B(0;1)\big), & \text{if } \mathrm{proj}_V x = 0 \end{cases} \\ &= \begin{cases} \{0\}, & \text{if } x \notin V^\perp; \\ \mathrm{proj}_{V^\perp}\big(B(0;1)\big), & \text{if } x \in V^\perp \end{cases} \\ &\ni 0. \end{aligned} \tag{3.40}$$

However, $\mathrm{Id} - \mathrm{proj}_V \circ \mathrm{proj}_V^* = \mathrm{proj}_{V^\perp}$. Therefore, in view of Proposition 3.22(ii), this confirms that $\mathrm{proj}_V \overset{\gamma}{\blacklozenge} \|\cdot\| = \|\cdot\| \circ \mathrm{proj}_V$. ☐

Remark 3.26. In contrast with Proposition 3.20(v), Example 3.25(ii) shows an instance in which the proximal cocomposition coincides with the standard composition for a linear operator which is not a coisometry.

Example 3.27. Let V be a closed vector subspace of $\mathcal{G}$, $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$, $g \in \Gamma_0(\mathcal{G})$, and $\gamma \in \left]0,+\infty\right[$. Suppose that L is surjective and that $L^* \circ L = \mathrm{proj}_V$. Then $L \overset{\gamma}{\diamond} g = L^* \triangleright g$ and $L \overset{\gamma}{\blacklozenge} g = g \circ L$.

Proof. Let $y \in \mathcal{G}$. Since L is surjective, there exists $x \in \mathcal{H}$ such that $Lx = y$. Moreover, since $\ker L = \ker(L^* \circ L) = \ker \mathrm{proj}_V = V^\perp$, we obtain

$$L(L^*y) = L\big(L^*(Lx)\big) = L(\mathrm{proj}_V x) = Lx - L(\mathrm{proj}_{V^\perp} x) = y - 0 = y. \tag{3.41}$$

Hence, L is a coisometry and the assertion follows from Proposition 3.20(v) and Lemma 2.4(ii). ☐

Remark 3.28. In the context of Example 3.27, the identity $L \overset{\gamma}{\blacklozenge} g = g \circ L$ combined with Proposition 3.13(ii) recovers the fact that ${}^{\gamma}(g \circ L) = {}^{\gamma}g \circ L$ (see [21, Lemma 3]).

3.4. Asymptotic properties

We investigate asymptotic properties of the families $(L \overset{\gamma}{\diamond} g)_{\gamma\in]0,+\infty[}$ and $(L \overset{\gamma}{\blacklozenge} g)_{\gamma\in]0,+\infty[}$ as γ varies. These results provide further connections between the compositions (1.3), (1.4), and the proximal compositions of Definition 1.1.

Proposition 3.29. *Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$ and let $g\colon \mathcal{G} \to]-\infty,+\infty]$ be a proper function such that $\operatorname{cam} g \neq \varnothing$. Suppose that $x \in L^{-1}(\operatorname{dom} g^{**})$ and set, for every $\gamma \in]0,+\infty[$, $x_\gamma = \operatorname{prox}_{\gamma(L \overset{\gamma}{\blacklozenge} g)} x$. Then*

$$\lim_{0<\gamma\to 0} \big(L \overset{\gamma}{\blacklozenge} g\big)(x_\gamma) = g^{**}(Lx). \tag{3.42}$$

Proof. We first observe that, by virtue of Proposition 3.7(i), $(x_\gamma)_{\gamma\in]0,+\infty[}$ is well defined. Appealing to Proposition 3.13(ii), we get

$$\big(L \overset{\gamma}{\blacklozenge} g\big)(x_\gamma) + \frac{1}{\gamma}\mathcal{Q}_{\mathcal{H}}(x - x_\gamma) = {}^{\gamma}\big(L \overset{\gamma}{\blacklozenge} g\big)(x) = {}^{\gamma}\big(g^{**}\big)(Lx). \tag{3.43}$$

On the other hand, by Proposition 3.10(ii),

$$\frac{1}{\gamma}\mathcal{Q}_{\mathcal{H}}(x - x_\gamma) = \frac{1}{\gamma}\mathcal{Q}_{\mathcal{H}}\Big(L^*\big(Lx - \operatorname{prox}_{\gamma g^{**}}(Lx)\big)\Big) \leqslant \frac{1}{\gamma}\|L\|^2\mathcal{Q}_{\mathcal{G}}\big(Lx - \operatorname{prox}_{\gamma g^{**}}(Lx)\big). \tag{3.44}$$

Therefore, since $Lx \in \operatorname{dom} g^{**}$, [2, Proposition 12.33(iii)] implies that $(1/\gamma)\mathcal{Q}_{\mathcal{H}}(x - x_\gamma) \to 0$ as $\gamma \to 0$. Finally, by (3.43) and [2, Proposition 12.33(ii)],

$$\lim_{0<\gamma\to 0} \big(L \overset{\gamma}{\blacklozenge} g\big)(x_\gamma) = \lim_{0<\gamma\to 0} {}^{\gamma}\big(g^{**}\big)(Lx) = \big(g^{**}\big)(Lx), \tag{3.45}$$

as claimed. ☐

Theorem 3.30. *Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$, let $g\colon \mathcal{G} \to]-\infty,+\infty]$ be a proper function such that $\operatorname{cam} g \neq \varnothing$, and let $x \in \mathcal{H}$. Then the following hold:*

(i) *The function $]0,+\infty[\to]-\infty,+\infty]\colon \gamma \mapsto (L \overset{\gamma}{\diamond} g)(x)$ is decreasing.*

(ii) *The function $]0,+\infty[\to]-\infty,+\infty]\colon \gamma \mapsto (L \overset{\gamma}{\blacklozenge} g)(x)$ is decreasing.*

(iii) $\lim_{\gamma\to+\infty} (L \overset{\gamma}{\diamond} g)(x) = (L^* \triangleright g^{**})(x)$.

(iv) $\lim_{0<\gamma\to 0} (L \overset{\gamma}{\blacklozenge} g)(x) = g^{**}(Lx)$.

(v) *Suppose that $\|L\| < 1$. Then* $\lim_{\gamma\to+\infty} (L \overset{\gamma}{\blacklozenge} g)(x) = \inf_{y\in\mathcal{G}} g^{**}(y)$.

(vi) *Suppose that $\|L\| = 1$ and that $V = \operatorname{ran}(\mathrm{Id}_{\mathcal{G}} - L\circ L^*)$ is closed. Then* $\lim_{\gamma\to+\infty} (L \overset{\gamma}{\blacklozenge} g)(x) = \inf_{y\in Lx-V} g^{**}(y)$.

Proof. Set $\Phi = \mathcal{Q}_{\mathcal{G}} - \mathcal{Q}_{\mathcal{H}} \circ L^*$.

(i): Fix $\gamma_1 \in]0,+\infty[$ and $\gamma_2 \in]0,+\infty[$ such that $\gamma_1 \leqslant \gamma_2$. Then we deduce from Proposition 3.2(i) that

$$L \overset{\gamma_2}{\diamond} g = L^* \triangleright \big(g^{**} + \Phi/\gamma_2\big) \leqslant L^* \triangleright \big(g^{**} + \Phi/\gamma_1\big) = L \overset{\gamma_1}{\diamond} g. \tag{3.46}$$

(ii): Fix $\gamma_1 \in \,]0,+\infty[$ and $\gamma_2 \in \,]0,+\infty[$ such that $\gamma_1 \leqslant \gamma_2$. By (i), $L \overset{1/\gamma_1}{\diamond} g^* \leqslant L \overset{1/\gamma_2}{\diamond} g^*$. Therefore, appealing to Definition 1.1 and Lemma 2.1(ii), we get

$$L \overset{\gamma_2}{\blacklozenge} g = \big(L \overset{1/\gamma_2}{\diamond} g^*\big)^* \leqslant \big(L \overset{1/\gamma_1}{\diamond} g^*\big)^* = L \overset{\gamma_1}{\blacklozenge} g. \tag{3.47}$$

(iii): Since $\Phi \geqslant 0$, it follows from (i) and Proposition 3.2(i) that

$$\begin{aligned}
\lim_{\gamma\to+\infty} \big(L \overset{\gamma}{\diamond} g\big)(x) &= \inf_{\gamma\in]0,+\infty[} \left(L^* \triangleright \left(g^{**} + \frac{1}{\gamma}\Phi\right)\right)(x) \\
&= \inf_{\gamma\in]0,+\infty[} \left(\inf_{\substack{y\in\mathcal{G}\\ L^*y=x}} \left(g^{**}(y) + \frac{1}{\gamma}\Phi(y)\right)\right) \\
&= \inf_{\substack{y\in\mathcal{G}\\ L^*y=x}} \left(\inf_{\gamma\in]0,+\infty[} \left(g^{**}(y) + \frac{1}{\gamma}\Phi(y)\right)\right) \\
&= \inf_{\substack{y\in\mathcal{G}\\ L^*y=x}} g^{**}(y) \\
&= \big(L^* \triangleright g^{**}\big)(x).
\end{aligned} \tag{3.48}$$

(iv): By [2, Proposition 12.33(ii)], ${}^{\gamma}(g^{**}) \to g^{**}$ as $0 < \gamma \to 0$. The claim therefore follows from Proposition 3.20(ii).

(v)–(vi): As in the proof of Proposition 3.2(iv), $(g^* + \gamma\Phi)^* = g^{**} \,\square\, (\Phi^*/\gamma)$. Thus, it follows from Proposition 3.2(ii) that

$$L \overset{\gamma}{\blacklozenge} g = \Big(g^{**} \,\square\, \big(\Phi^*/\gamma\big)\Big) \circ L. \tag{3.49}$$

Moreover, since $\Phi \leqslant \mathscr{Q}_{\mathcal{G}}$, Lemma 2.1(ii) yields $\mathscr{Q}_{\mathcal{G}} \leqslant \Phi^*$. Altogether, using (ii) and (3.49), we obtain

$$\begin{aligned}
\lim_{\gamma\to+\infty} \big(L \overset{\gamma}{\blacklozenge} g\big)(x) &= \inf_{\gamma\in]0,+\infty[} \left(g^{**} \,\square\, \frac{\Phi^*}{\gamma}\right)(Lx) \\
&= \inf_{\gamma\in]0,+\infty[} \left(\inf_{y\in\mathcal{G}} \left(g^{**}(y) + \frac{1}{\gamma}\Phi^*(Lx - y)\right)\right) \\
&= \inf_{y\in Lx-\operatorname{dom}\Phi^*} \left(\inf_{\gamma\in]0,+\infty[} \left(g^{**}(y) + \frac{1}{\gamma}\Phi^*(Lx - y)\right)\right) \\
&= \inf_{y\in Lx-\operatorname{dom}\Phi^*} g^{**}(y).
\end{aligned} \tag{3.50}$$

We set $A = \mathrm{Id}_{\mathcal{G}} - L \circ L^*$ and observe that $\Phi\colon y \mapsto \langle y \mid Ay\rangle_{\mathcal{G}}/2$. In case (v), since $\|L\| < 1$, A is invertible and Lemma 2.8 asserts that $\operatorname{dom}\Phi^* = \operatorname{ran} A = \mathcal{G}$ in (3.50). Finally, case (vi) follows from Lemma 2.8 and (3.50). ☐

Corollary 3.31. *Suppose that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ is an isometry, let $g \in \Gamma_0(\mathcal{G})$, and let $x \in \mathcal{H}$. Then the following hold:*

(i) $\displaystyle\lim_{\gamma\to+\infty} \big(L \overset{\gamma}{\diamond} g\big)(x) = \big(L^* \triangleright g\big)(x)$.

(ii) $\lim_{0<\gamma\to 0} (L \overset{\gamma}{\diamond} g)(x) = g(Lx)$.

Proof. By Proposition 3.20(iv), $L \overset{\gamma}{\blacklozenge} g = L \overset{\gamma}{\diamond} g$, whereas Lemma 2.4(ii) yields $g^{**} = g$.

(i): A consequence of Theorem 3.30(iii).

(ii): A consequence of Theorem 3.30(iv). ☐

Example 3.32. Let $V \neq \{0\}$ be a closed vector subspace of $\mathcal{G}$, let $g \in \Gamma_0(\mathcal{G})$, and let $x \in \mathcal{G}$. Then

$$\lim_{\gamma\to+\infty} (\mathrm{proj}_V \overset{\gamma}{\blacklozenge} g)(x) = \inf_{v\in V^\perp} g(x+v). \tag{3.51}$$

Proof. Since $\|\mathrm{proj}_V\| = 1$ and $\mathrm{ran}(\mathrm{Id}_\mathcal{G} - \mathrm{proj}_V \circ \mathrm{proj}_V^*) = V^\perp$, it follows from Theorem 3.30(vi) and Lemma 2.4(ii) that

$$\lim_{\gamma\to+\infty} (\mathrm{proj}_V \overset{\gamma}{\blacklozenge} g)(x) = \inf_{y\in \mathrm{proj}_V x - V^\perp} g(y) = \inf_{y\in x+V^\perp} g(y) = \inf_{v\in V^\perp} g(x+v), \tag{3.52}$$

as announced. ☐

We now turn our attention to epi-convergence. As discussed in [1], this notion plays a central role in the approximation of variational problems. It will allow us to connect asymptotically the proximal composition to the infimal postcomposition, and the proximal cocomposition to the standard composition as γ evolves.

Definition 3.33 ([1, Chapter 1], [17, Chapter 7]). Suppose that $\mathcal{H}$ is finite-dimensional, and let $(f_n)_{n\in\mathbb{N}}$ and f be functions from $\mathcal{H}$ to $[-\infty,+\infty]$. We say that $(f_n)_{n\in\mathbb{N}}$ *epi-converges* to f, in symbols $f_n \overset{e}{\to} f$, if the following hold for every $x \in \mathcal{H}$:

(i) For every sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that $x_n \to x$, $f(x) \leqslant \underline{\lim}\, f_n(x_n)$.

(ii) There exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that $x_n \to x$ and $\overline{\lim}\, f_n(x_n) \leqslant f(x)$.

The *epi-topology* is the topology induced by epi-convergence.

Lemma 3.34. *Suppose that $\mathcal{H}$ and $\mathcal{G}$ are finite-dimensional, let $(L_n)_{n\in\mathbb{N}}$ and L be operators in $\mathcal{B}(\mathcal{H},\mathcal{G})$, let $(g_n)_{n\in\mathbb{N}}$ and g be functions in $\Gamma_0(\mathcal{G})$, and let $(\gamma_n)_{n\in\mathbb{N}}$ and γ be reals in $]0,+\infty[$. Suppose that $L_n \to L$, $g_n \overset{e}{\to} g$, and $\gamma_n \to \gamma$. Then the following hold:*

(i) $\gamma_n g_n \overset{e}{\to} \gamma g$.

(ii) $g_n^* \overset{e}{\to} g^*$.

(iii) *Suppose that $h\colon \mathcal{G} \to \mathbb{R}$ is continuous. Then $g_n + \gamma_n h \overset{e}{\to} g + \gamma h$.*

(iv) *Suppose that $0 \in \mathrm{int}(\mathrm{dom}\, g - \mathrm{ran}\, L)$. Then $g_n \circ L_n \overset{e}{\to} g \circ L$.*

Proof. (i): [17, Exercise 7.8(d)].

(ii): [17, Theorem 11.34].

(iii): It follows from (i) and [17, Exercise 7.8(a)] that $g_n/\gamma_n + h \overset{e}{\to} g/\gamma + h$. Invoking (i) once more, we obtain $g_n + \gamma_n h = \gamma_n(g_n/\gamma_n + h) \overset{e}{\to} \gamma(g/\gamma + h) = g + \gamma h$.

(iv): [17, Exercise 7.47(a)]. ☐

Theorem 3.35. *Suppose that $\mathcal{H}$ and $\mathcal{G}$ are finite-dimensional, let $(L_n)_{n\in\mathbb{N}}$ and L be operators in $\mathcal{B}(\mathcal{H},\mathcal{G})$, let $(g_n)_{n\in\mathbb{N}}$ and g be functions in $\Gamma_0(\mathcal{G})$, and let $(\gamma_n)_{n\in\mathbb{N}}$ and γ be reals in $]0,+\infty[$. Then the following hold:*

(i) *Suppose that* $L_n \to L$, $g_n \xrightarrow{e} g$, *and* $\gamma_n \to \gamma$. *Then the following are satisfied:*

   (a) $L_n \overset{\gamma_n}{\diamond} g_n \xrightarrow{e} L \overset{\gamma}{\diamond} g$.

   (b) $L_n \overset{\gamma_n}{\blacklozenge} g_n \xrightarrow{e} L \overset{\gamma}{\blacklozenge} g$.

(ii) *Suppose that* $0 < \|L\| \leqslant 1$. *Then the following are satisfied:*

   (a) *Suppose that* $\gamma_n \uparrow +\infty$. *Then* $L \overset{\gamma_n}{\diamond} g \xrightarrow{e} (L^* \triangleright g)\breve{}$.

   (b) *Suppose that* $\gamma_n \downarrow 0$. *Then* $L \overset{\gamma_n}{\blacklozenge} g \xrightarrow{e} g \circ L$.

Proof. (i)(a): It follows from Lemmas 2.4(viii) and 3.34(ii)–(iii) that

$$\overset{\frac{1}{\gamma_n}}{}(g_n^*) = \left(g_n + \frac{1}{\gamma_n}\mathcal{Q}_{\mathcal{G}}\right)^* \xrightarrow{e} \left(g + \frac{1}{\gamma}\mathcal{Q}_{\mathcal{G}}\right)^* = \overset{\frac{1}{\gamma}}{}(g^*). \tag{3.53}$$

Since Lemmas 2.1(v) and 2.4(vi) yield $\operatorname{dom} \overset{\frac{1}{\gamma}}{}(g^*) = \mathcal{G}$, Lemma 3.34(iv) and (3.53) imply that $\overset{\frac{1}{\gamma_n}}{}(g_n^*) \circ L_n \xrightarrow{e} \overset{\frac{1}{\gamma}}{}(g^*) \circ L$. Finally, appealing to Definition 1.1 and Lemma 3.34(ii)–(iii), we conclude that

$$L_n \overset{\gamma_n}{\diamond} g_n = \left(\overset{\frac{1}{\gamma_n}}{}(g_n^*) \circ L_n\right)^* - \frac{1}{\gamma_n}\mathcal{Q}_{\mathcal{H}} \xrightarrow{e} \left(\overset{\frac{1}{\gamma}}{}(g^*) \circ L\right)^* - \frac{1}{\gamma}\mathcal{Q}_{\mathcal{H}} = L \overset{\gamma}{\diamond} g. \tag{3.54}$$

(i)(b): By Lemma 3.34(ii), $g_n^* \xrightarrow{e} g^*$. Therefore, upon combining (i)(a) and Lemma 3.34(ii), we obtain

$$L_n \overset{\gamma_n}{\blacklozenge} g_n = \left(L_n \overset{1/\gamma_n}{\diamond} g_n^*\right)^* \xrightarrow{e} \left(L \overset{1/\gamma}{\diamond} g^*\right)^* = L \overset{\gamma}{\blacklozenge} g. \tag{3.55}$$

(ii)(a): Set $f = L^* \triangleright g$ and $(\forall n \in \mathbb{N})$ $f_n = L \overset{\gamma_n}{\diamond} g$. It follows from items (i) and (iii) in Theorem 3.30, as well as Lemma 2.4(ii), that $(f_n)_{n\in\mathbb{N}}$ is decreasing and pointwise convergent to f as $n \to +\infty$. Further, since f is convex by [2, Proposition 12.36(ii)], we deduce from [17, Proposition 7.4(c)] and [2, Corollary 9.10] that

$$f_n \xrightarrow{e} \overline{\inf_{n\in\mathbb{N}} f_n} = \overline{f} = \breve{f}. \tag{3.56}$$

(ii)(b): Set $f = g \circ L$ and $(\forall n \in \mathbb{N})$ $f_n = L \overset{\gamma_n}{\blacklozenge} g$. Since $(\gamma_n)_{n\in\mathbb{N}}$ is decreasing, $(f_n)_{n\in\mathbb{N}}$ is increasing by Theorem 3.30(ii). Further, Theorem 3.30(iv) and Lemma 2.4(ii) imply that $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f as $n \to +\infty$. On the other hand, Proposition 3.7(i) implies that $(\forall n \in \mathbb{N})$ $\overline{f_n} = f_n$. Therefore, by virtue of [17, Proposition 7.4(d)],

$$f_n \xrightarrow{e} \sup_{n\in\mathbb{N}} \overline{f_n} = \sup_{n\in\mathbb{N}} f_n = f, \tag{3.57}$$

which concludes the proof. ☐

Corollary 3.36. *Suppose that* $\mathcal{H}$ *and* $\mathcal{G}$ *are finite-dimensional, let* $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$, *let* $g \in \Gamma_0(\mathcal{G})$, *and let* $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence in* $]0,+\infty[$. *Suppose that* L *is an isometry and that* $(\operatorname{ri}\operatorname{dom} g^*) \cap (\operatorname{ran} L) \neq \varnothing$. *Then the following hold:*

(i) *Suppose that* $\gamma_n \uparrow +\infty$. *Then* $L \overset{\gamma_n}{\diamond} g \xrightarrow{e} L^* \triangleright g$.

(ii) *Suppose that* $\gamma_n \downarrow 0$. *Then* $L \overset{\gamma_n}{\diamond} g \xrightarrow{e} g \circ L$.

(iii) *For every* $t \in [0,1]$, *set* $\gamma_t = \tan(\pi t/2)$. *Then the operator*

$$T\colon [0,1] \to \Gamma_0(\mathcal{H})\colon t \to \begin{cases} g \circ L, & \text{if } t = 0;\\ L \overset{\gamma_t}{\diamond} g, & \text{if } 0 < t < 1;\\ L^* \triangleright g, & \text{if } t = 1 \end{cases} \tag{3.58}$$

is continuous with respect to the epi-topology.

Proof. Proposition 3.20(iv) yields $(\forall \gamma \in \,]0,+\infty[)\; L \overset{\gamma}{\blacklozenge} g = L \overset{\gamma}{\diamond} g$. Further, [2, Proposition 6.19(x)] implies that $0 \in \operatorname{sri}(\operatorname{dom} g^* - \operatorname{ran} L)$. Therefore, by virtue of Lemmas 2.5(iii) and 2.4(ii), we get $L^* \triangleright g \in \Gamma_0(\mathcal{H})$.

(i): A consequence of Theorem 3.35(ii)(a).

(ii): See Theorem 3.35(ii)(b).

(iii): Theorem 3.35(i)(a) guarantees the epi-continuity of T on $]0,1[$. Finally, (i) and (ii) imply that $\lim_{0<t\to 0} T(t) = T(0)$ and $\lim_{1>t\to 1} T(t) = T(1)$, respectively. □

Remark 3.37. Suppose that $\mathcal{H}$ and $\mathcal{G}$ are finite-dimensional and that $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ satisfies $0 < \|L\| \leqslant 1$, let $g \in \Gamma_0(\mathcal{G})$, and let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $]0,+\infty[$. Under a qualification condition (see Lemma 2.5(iii)), $L^* \triangleright g \in \Gamma_0(\mathcal{H})$ and, consequently, $L^* \triangleright g = (L^* \triangleright g)\breve{}$. In this case, Theorem 3.30(iii) and Theorem 3.35(ii)(a) show that the proximal composition converges pointwise and epi-converges to the infimal postcomposition as $\gamma_n \uparrow +\infty$. On the other hand, Theorem 3.30(iv) and Theorem 3.35(ii)(b) show that the proximal cocomposition converges pointwise and epi-converges to the standard composition. Further, in the particular case in which $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ is an isometry, Corollary 3.36(iii) asserts that $g \circ L$ and $L^* \triangleright g$ are homotopic via the proximal composition with respect to the epi-topology.

Proposition 3.38. *Suppose that* $\mathcal{H}$ *and* $\mathcal{G}$ *are finite-dimensional and that* $L \in \mathcal{B}(\mathcal{H},\mathcal{G})$ *satisfies* $0 < \|L\| \leqslant 1$, *let* $g \in \Gamma_0(\mathcal{G})$, *and let* $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence in* $]0,+\infty[$ *such that* $\gamma_n \downarrow 0$. *Suppose that* $\operatorname{dom} g \cap \operatorname{ran} L \neq \varnothing$ *and that* $g \circ L$ *is coercive. Then the following hold:*

(i) $\inf_{x\in\mathcal{H}} (L \overset{\gamma_n}{\blacklozenge} g)(x) \to \min_{x\in\mathcal{H}} g(Lx)$.

(ii) *There exists* $N \subset \mathbb{N}$ *such that* $\mathbb{N} \smallsetminus N$ *is finite and* $(\forall n \in N)\; \operatorname{Argmin}(L \overset{\gamma_n}{\blacklozenge} g) \neq \varnothing$. *Further,*

$$\overline{\lim} \operatorname{Argmin}\big(L \overset{\gamma_n}{\blacklozenge} g\big) \subset \operatorname{Argmin}\big(g \circ L\big). \tag{3.59}$$

Proof. Set $f = g \circ L$ and $(\forall n \in \mathbb{N})\; f_n = L \overset{\gamma_n}{\blacklozenge} g$. Since $\operatorname{dom} g \cap \operatorname{ran} L \neq \varnothing$, $f \in \Gamma_0(\mathcal{H})$. Thus, by [2, Proposition 11.15(i)], f has a minimizer over $\mathcal{H}$. Further, by Proposition 3.7(i), for every $n \in \mathbb{N}$, $f_n \in \Gamma_0(\mathcal{H})$ and, by Theorem 3.35(ii)(b), $f_n \overset{e}{\to} f$. On the other hand, [2, Proposition 11.12] asserts that the lower level sets $(\operatorname{lev}_{\leqslant \xi} f)_{\xi\in\mathbb{R}}$ are bounded. Altogether, by virtue of [17, Exercise 7.32(c)], for every $\xi \in \mathbb{R}$, there exists $N_\xi \in \mathbb{N}$ such that $\bigcup_{n\geqslant N_\xi} \operatorname{lev}_{\leqslant \xi} f_n$ is bounded.

(i)–(ii): A consequence of [17, Theorem 7.33]. □

§4. Integral proximal mixtures

4.1. Definition and mathematical setting

Integral proximal mixtures were introduced in [7] as a tool to combine arbitrary families of convex functions and linear operators in such a way that the proximity operator of the mixture can be expressed explicitly in terms of the individual proximity operators. They extend the proximal mixtures of [9], which were designed for finite families. In this section, we use the results of Section 3 to study their variational properties. This investigation is carried out in the same framework as in [7], which hinges on the following assumptions. Henceforth, we adopt the customary convention that the integral of an $\mathcal{F}$-measurable function $\vartheta\colon \Omega \to [-\infty,+\infty]$ is the usual Lebesgue integral $\int_\Omega \vartheta d\mu$, except when the Lebesgue integral $\int_\Omega \max\{\vartheta,0\}d\mu$ is $+\infty$, in which case $\int_\Omega \vartheta d\mu = +\infty$.

Assumption 4.1. Let $(\Omega,\mathcal{F},\mu)$ be a complete σ-finite measure space, let $(\mathsf{G}_\omega)_{\omega\in\Omega}$ be a family of real Hilbert spaces, and let $\prod_{\omega\in\Omega}\mathsf{G}_\omega$ be the usual real vector space of mappings x defined on Ω such that $(\forall\omega\in\Omega)\ x(\omega)\in\mathsf{G}_\omega$. Let $((\mathsf{G}_\omega)_{\omega\in\Omega},\mathfrak{G})$ be an $\mathcal{F}$-measurable vector field of real Hilbert spaces, that is, $\mathfrak{G}$ is a vector subspace of $\prod_{\omega\in\Omega}\mathsf{G}_\omega$ which satisfies the following:

[A] For every $x\in\mathfrak{G}$, the function $\Omega\to\mathbb{R}\colon \omega\mapsto \|x(\omega)\|_{\mathsf{G}_\omega}$ is $\mathcal{F}$-measurable.

[B] For every $x\in\prod_{\omega\in\Omega}\mathsf{G}_\omega$,

$$\Big[\ (\forall y\in\mathfrak{G})\ \ \Omega\to\mathbb{R}\colon \omega\mapsto \langle x(\omega)\mid y(\omega)\rangle_{\mathsf{G}_\omega}\ \text{is } \mathcal{F}\text{-measurable}\ \Big]\quad\Rightarrow\quad x\in\mathfrak{G}. \tag{4.1}$$

[C] There exists a sequence $(e_n)_{n\in\mathbb{N}}$ in $\mathfrak{G}$ such that $(\forall\omega\in\Omega)\ \overline{\operatorname{span}}\{e_n(\omega)\}_{n\in\mathbb{N}}=\mathsf{G}_\omega$.

Set $\mathfrak{H}=\big\{x\in\mathfrak{G}\mid \int_\Omega\|x(\omega)\|^2_{\mathsf{G}_\omega}\mu(d\omega)<+\infty\big\}$, and let $\mathcal{G}$ be the real Hilbert space of equivalence classes of μ-a.e. equal mappings in $\mathfrak{H}$ equipped with the scalar product

$$\langle\cdot\mid\cdot\rangle_{\mathcal{G}}\colon \mathcal{G}\times\mathcal{G}\to\mathbb{R}\colon (x,y)\mapsto\int_\Omega\langle x(\omega)\mid y(\omega)\rangle_{\mathsf{G}_\omega}\mu(d\omega), \tag{4.2}$$

where we adopt the common practice of designating by x both an equivalence class in $\mathcal{G}$ and a representative of it in $\mathfrak{H}$. We write

$$\mathcal{G}=\prescript{\mathfrak{G}}{}{\int_\Omega^{\oplus}}\mathsf{G}_\omega\mu(d\omega) \tag{4.3}$$

and call $\mathcal{G}$ the *Hilbert direct integral of* $((\mathsf{G}_\omega)_{\omega\in\Omega},\mathfrak{G})$ [13].

Assumption 4.2. Assumption 4.1 and the following are in force:

[A] H is a separable real Hilbert space.

[B] For every $\omega\in\Omega$, $\mathsf{L}_\omega\in\mathcal{B}(\mathsf{H},\mathsf{G}_\omega)$.

[C] For every $\mathsf{x}\in\mathsf{H}$, the mapping $\mathfrak{e}_{\mathsf{L}}\mathsf{x}\colon \omega\mapsto\mathsf{L}_\omega\mathsf{x}$ lies in $\mathfrak{G}$.

[D] $0<\int_\Omega\|\mathsf{L}_\omega\|^2\mu(d\omega)\leqslant 1$.

Given a complete σ-finite measure space $(\Omega,\mathcal{F},\mu)$, a separable real Hilbert space H with Borel σ-algebra $\mathcal{B}_{\mathsf{H}}$, and $p\in[1,+\infty[$, we set

$$\mathscr{L}^p(\Omega,\mathcal{F},\mu;\mathsf{H})=\left\{x\colon\Omega\to\mathsf{H}\ \middle|\ x \text{ is } (\mathcal{F},\mathcal{B}_{\mathsf{H}})\text{-measurable and } \int_\Omega\|x(\omega)\|^p_{\mathsf{H}}\,\mu(d\omega)<+\infty\right\}. \tag{4.4}$$

The Lebesgue (also called Bochner) integral of $x\in\mathscr{L}^1(\Omega,\mathcal{F},\mu;\mathsf{H})$ is denoted by $\int_\Omega x(\omega)\mu(d\omega)$. The space of equivalence classes of μ-a.e. equal mappings in $\mathscr{L}^p(\Omega,\mathcal{F},\mu;\mathsf{H})$ is denoted by $L^p(\Omega,\mathcal{F},\mu;\mathsf{H})$.

Assumption 4.3. Assumption 4.1 and the following are in force:

[A] For every $\omega\in\Omega$, $\mathsf{g}_\omega\colon\mathsf{G}_\omega\to\,]-\infty,+\infty]$ satisfies $\operatorname{cam}\mathsf{g}_\omega\neq\varnothing$.

[B] For every $x^* \in \mathfrak{H}$, the mapping $\omega \mapsto \operatorname{prox}_{\mathrm{g}_\omega^*} x^*(\omega)$ lies in $\mathfrak{G}$.

[C] There exists $r \in \mathfrak{H}$ such that the function $\omega \mapsto \mathrm{g}_\omega(r(\omega))$ lies in $\mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$.

[D] There exists $r^* \in \mathfrak{H}$ such that the function $\omega \mapsto \mathrm{g}_\omega^*(r^*(\omega))$ lies in $\mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$.

We introduce below parametrized versions of the integral proximal mixtures of [7, Definition 4.2].

Definition 4.4. Suppose that Assumptions 4.2 and 4.3 are in force, and let $\gamma \in \left]0, +\infty\right[$. The *integral proximal mixture* of $(\mathrm{g}_\omega)_{\omega\in\Omega}$ and $(\mathrm{L}_\omega)_{\omega\in\Omega}$ with parameter γ is

$$\overset{\diamond}{\mathrm{M}}_\gamma(\mathrm{L}_\omega, \mathrm{g}_\omega)_{\omega\in\Omega} = \mathrm{h}^* - \frac{1}{\gamma}\mathcal{Q}_{\mathrm{H}}, \quad \text{where} \quad (\forall \mathrm{x} \in \mathrm{H}) \quad \mathrm{h}(\mathrm{x}) = \int_\Omega {}^{\frac{1}{\gamma}}(\mathrm{g}_\omega^*)(\mathrm{L}_\omega \mathrm{x})\mu(d\omega), \tag{4.5}$$

and the *integral proximal comixture* of $(\mathrm{g}_\omega)_{\omega\in\Omega}$ and $(\mathrm{L}_\omega)_{\omega\in\Omega}$ with parameter γ is

$$\overset{\blacklozenge}{\mathrm{M}}_\gamma(\mathrm{L}_\omega, \mathrm{g}_\omega)_{\omega\in\Omega} = \left(\overset{\diamond}{\mathrm{M}}_{1/\gamma}(\mathrm{L}_\omega, \mathrm{g}_\omega^*)_{\omega\in\Omega}\right)^*. \tag{4.6}$$

The following construct will also be required.

Definition 4.5 ([6, Definition 1.4]). Suppose that Assumption 4.1 is in force and, for every $\omega \in \Omega$, let $\mathrm{g}_\omega \colon \mathrm{G}_\omega \to [-\infty, +\infty]$. Suppose that, for every $x \in \mathfrak{H}$, the function $\Omega \to [-\infty, +\infty] \colon \omega \mapsto \mathrm{g}_\omega(x(\omega))$ is $\mathcal{F}$-measurable. The *Hilbert direct integral* of the functions $(\mathrm{g}_\omega)_{\omega\in\Omega}$ relative to $\mathfrak{G}$ is

$$\int_\Omega^{\mathfrak{G}\,\oplus} \mathrm{g}_\omega \mu(d\omega) \colon \mathcal{G} \to [-\infty, +\infty] \colon x \mapsto \int_\Omega \mathrm{g}_\omega\big(x(\omega)\big)\mu(d\omega). \tag{4.7}$$

4.2. Properties

The following proposition adopts the pattern of [7, Theorem 4.3] by connecting integral proximal mixtures to proximal compositions in the more general context of Definitions 1.1 and 4.4.

Proposition 4.6. *Suppose that Assumptions 4.2 and 4.3 are in force, and let $\gamma \in \left]0, +\infty\right[$. Define*

$$L \colon \mathrm{H} \to \mathcal{G} \colon \mathrm{x} \mapsto \mathfrak{e}_{\mathrm{L}}\mathrm{x} \tag{4.8}$$

and

$$g = \int_\Omega^{\mathfrak{G}\,\oplus} \mathrm{g}_\omega^{**} \mu(d\omega). \tag{4.9}$$

Then the following hold:

(i) $L \in \mathcal{B}(\mathrm{H}, \mathcal{G})$ *and* $0 < \|L\| \leqslant 1$.

(ii) $L^* \colon \mathcal{G} \to \mathrm{H} \colon x^* \mapsto \int_\Omega \mathrm{L}_\omega^*(x^*(\omega))\mu(d\omega)$.

(iii) $g \in \Gamma_0(\mathcal{G})$.

(iv) $\overset{\diamond}{\mathrm{M}}_\gamma(\mathrm{L}_\omega, \mathrm{g}_\omega)_{\omega\in\Omega} = L \overset{\gamma}{\diamond} g$.

(v) $\overset{\blacklozenge}{\mathrm{M}}_\gamma(\mathrm{L}_\omega, \mathrm{g}_\omega)_{\omega\in\Omega} = L \overset{\gamma}{\blacklozenge} g$.

Proof. (i): We deduce from [6, Proposition 3.12(ii)] and Assumption 4.2[D] that $L \in \mathcal{B}(\mathrm{H}, \mathcal{G})$ and that $0 < \|L\|^2 \leqslant \int_\Omega \|\mathrm{L}_\omega\|^2 \mu(d\omega) \leqslant 1$.

(ii): See [6, Proposition 3.12(v)].

To establish (iii)–(v), set $\vartheta \colon \Omega \to \mathbb{R} \colon \omega \mapsto -\mathrm{g}_\omega^{**}(r(\omega))$ and $(\forall \omega \in \Omega)$ $\mathrm{f}_\omega = \mathrm{g}_\omega^*$. Let us show that $(\mathrm{f}_\omega)_{\omega\in\Omega}$ satisfies the following:

[A]' For every $\omega \in \Omega$, $\mathsf{f}_\omega \in \Gamma_0(\mathsf{G}_\omega)$.

[B]' For every $x \in \mathfrak{H}$, the mapping $\omega \mapsto \mathrm{prox}_{\mathsf{f}_\omega}(x(\omega))$ lies in $\mathfrak{G}$.

[C]' The function $\omega \mapsto \mathsf{f}_\omega(r^*(\omega))$ lies in $\mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$.

[D]' $\vartheta \in \mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$ and, for every $\omega \in \Omega$, $\mathsf{f}_\omega \geqslant \langle r(\omega) \mid \cdot \rangle_{\mathsf{G}_\omega} + \vartheta(\omega)$.

This will confirm that $(\mathsf{f}_\omega)_{\omega \in \Omega}$ satisfies the properties of [6, Assumption 4.6]. First, it follows from items [A] and [C] in Assumption 4.3 and from Lemma 2.1(v) that [A]' holds. Second, Assumption 4.3[B] implies that [B]' holds, while Assumption 4.3[D] implies that [C]' holds. Let us now show that $\vartheta \in \mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$. As in the proof of [6, Theorem 4.7(ix)], $-\vartheta$ is $\mathcal{F}$-measurable. Further, by (1.1) and Lemma 2.1(i),

$$(\forall \omega \in \Omega) \quad \langle \cdot \mid r^*(\omega) \rangle_{\mathsf{G}_\omega} - \mathsf{g}_\omega^*\big(r^*(\omega)\big) \leqslant \mathsf{g}_\omega^{**} \leqslant \mathsf{g}_\omega. \tag{4.10}$$

Thus, we infer from Assumption 4.3[C]–[D] that g_ω^{**} is bounded by integrable functions, which shows that

$$\vartheta \in \mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathbb{R}). \tag{4.11}$$

On the other hand, it follows from Lemma 2.1(iii) and (1.1) that, for every $\omega \in \Omega$, $\mathsf{f}_\omega = \mathsf{g}_\omega^{***} \geqslant \langle r(\omega) \mid \cdot \rangle_{\mathsf{G}_\omega} - \mathsf{g}_\omega^{**}(r(\omega)) = \langle r(\omega) \mid \cdot \rangle_{\mathsf{G}_\omega} + \vartheta(\omega)$, which provides [D]'. Therefore $(\mathsf{f}_\omega)_{\omega \in \Omega}$ satisfies the conclusions of [6, Theorem 4.7]. In particular, [6, Theorem 4.7(i)–(ii)] entail that

$$f = {}^{\mathfrak{G}}\!\!\int_\Omega^{\oplus} \mathsf{f}_\omega \mu(d\omega) \tag{4.12}$$

is a well-defined function in $\Gamma_0(\mathcal{G})$ and from [6, Theorem 4.7(ix)] and Lemma 2.4(ii) that

$$g = f^* \in \Gamma_0(\mathcal{G}). \tag{4.13}$$

(iii): See (4.13).

(iv): By [6, Theorem 4.7(viii)],

$$\tfrac{1}{\gamma} f = {}^{\mathfrak{G}}\!\!\int_\Omega^{\oplus} \tfrac{1}{\gamma}\mathsf{f}_\omega \mu(d\omega). \tag{4.14}$$

Further, by (iii) and Lemma 2.4(ii), $g^* = f$. In turn, (4.8) and (4.14) imply that

$$\tfrac{1}{\gamma}(g^*) \circ L \colon \mathsf{H} \to \mathbb{R} \colon \mathsf{x} \mapsto \int_\Omega \tfrac{1}{\gamma}\big(\mathsf{g}_\omega^*\big)\big(\mathsf{L}_\omega \mathsf{x}\big)\mu(d\omega). \tag{4.15}$$

In view of Definitions 1.1 and 4.4, the assertion is proved.

(v): Let us show that $(\mathsf{f}_\omega)_{\omega \in \Omega}$ fulfills the properties of Assumption 4.3 by showing that the following hold:

[A]" For every $\omega \in \Omega$, $\mathsf{f}_\omega \colon \mathsf{G}_\omega \to \left]-\infty, +\infty\right]$ satisfies $\operatorname{cam} \mathsf{f}_\omega \neq \varnothing$.

[B]" For every $x^* \in \mathfrak{H}$, the mapping $\omega \mapsto \mathrm{prox}_{\mathsf{f}_\omega^*} x^*(\omega)$ lies in $\mathfrak{G}$.

[C]" The function $\omega \mapsto \mathsf{f}_\omega(r^*(\omega))$ lies in $\mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$.

[D]" The function $\omega \mapsto \mathsf{f}_\omega^*(r(\omega))$ lies in $\mathscr{L}^1(\Omega, \mathcal{F}, \mu; \mathbb{R})$.

We first note that [A]' and Lemma 2.4(i) imply that [A]" holds, and that [C]'⇔[C]". Additionally, it follows from (4.11) that [D]" holds. It remains to establish [B]". Assumption 4.3[B] asserts that, for every $x^* \in \mathfrak{H}$, the mapping $\omega \mapsto \operatorname{prox}_{\mathsf{f}_\omega} x^*(\omega)$ lies in $\mathfrak{G}$. Therefore, the inclusion $\mathfrak{H} \subset \mathfrak{G}$, Lemma 2.4(iv), and the fact the $\mathfrak{G}$ is a vector space imply that, for every $x^* \in \mathfrak{H}$, the mapping $\omega \mapsto \operatorname{prox}_{\mathsf{f}^*_\omega} x^*(\omega) = x^*(\omega) - \operatorname{prox}_{\mathsf{f}_\omega} x^*(\omega)$ lies in $\mathfrak{G}$, which provides [B]". Hence, we combine Definition 4.4, the application of (iv) to $(\mathsf{f}_\omega)_{\omega\in\Omega}$, (4.13), Lemma 2.4(ii), and Definition 1.1, to obtain

$$\overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega,\mathsf{g}_\omega)_{\omega\in\Omega} = \Big(\overset{\diamond}{\mathsf{M}}_{1/\gamma}(\mathsf{L}_\omega,\mathsf{f}_\omega)_{\omega\in\Omega}\Big)^* = \Big(L \overset{1/\gamma}{\diamond} f\Big)^* = \Big(L \overset{1/\gamma}{\diamond} g^*\Big)^* = L \overset{\gamma}{\blacklozenge} g, \tag{4.16}$$

which completes the proof. ☐

Our main results on integral proximal mixtures are the following.

Theorem 4.7. *Suppose that Assumptions 4.2 and 4.3 are in force, and let $\gamma \in \left]0,+\infty\right[$. Then the following hold:*

(i) $\overset{\diamond}{\mathsf{M}}_\gamma(\mathsf{L}_\omega,\mathsf{g}_\omega)_{\omega\in\Omega} \in \Gamma_0(\mathsf{H})$.

(ii) $\overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega,\mathsf{g}_\omega)_{\omega\in\Omega} \in \Gamma_0(\mathsf{H})$.

(iii) $(\overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega,\mathsf{g}_\omega)_{\omega\in\Omega})^* = \overset{\diamond}{\mathsf{M}}_{1/\gamma}(\mathsf{L}_\omega,\mathsf{g}^*_\omega)_{\omega\in\Omega}$.

(iv) $\overset{\diamond}{\mathsf{M}}_\gamma(\mathsf{L}_\omega,\mathsf{g}_\omega)_{\omega\in\Omega} = (\overset{\blacklozenge}{\mathsf{M}}_{1/\gamma}(\mathsf{L}_\omega,\mathsf{g}^*_\omega)_{\omega\in\Omega})^*$.

(v) *Let* $\mathsf{x} \in \mathsf{H}$. *Then* $\operatorname{prox}_{\gamma\overset{\diamond}{\mathsf{M}}_\gamma(\mathsf{L}_\omega,\mathsf{g}_\omega)_{\omega\in\Omega}} \mathsf{x} = \int_\Omega \mathsf{L}^*_\omega(\operatorname{prox}_{\gamma\mathsf{g}^{**}_\omega}(\mathsf{L}_\omega\mathsf{x}))\,\mu(d\omega)$.

(vi) *Let* $\mathsf{x} \in \mathsf{H}$. *Then* $\operatorname{prox}_{\gamma\overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega,\mathsf{g}_\omega)_{\omega\in\Omega}} \mathsf{x} = \mathsf{x} - \int_\Omega \mathsf{L}^*_\omega(\mathsf{L}_\omega\mathsf{x} - \operatorname{prox}_{\gamma\mathsf{g}^{**}_\omega}(\mathsf{L}_\omega\mathsf{x}))\,\mu(d\omega)$.

(vii) *Define g as in (4.9) and L as in (4.8). Then the following are satisfied:*

  (a) $\partial(\overset{\diamond}{\mathsf{M}}_\gamma(\mathsf{L}_\omega,\mathsf{g}_\omega)_{\omega\in\Omega}) = L^* \triangleright (\partial g + (\mathrm{Id}_{\mathcal{G}} - L\circ L^*)/\gamma)$.

  (b) $\partial(\overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega,\mathsf{g}_\omega)_{\omega\in\Omega}) = L^* \circ (\partial g^* + \gamma(\mathrm{Id}_{\mathcal{G}} - L\circ L^*))^{-1} \circ L$.

(viii) *Let* $\mathsf{x} \in \mathsf{H}$. *Then* ${}^{\gamma}(\overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega,\mathsf{g}_\omega)_{\omega\in\Omega})(\mathsf{x}) = \int_\Omega {}^{\gamma}(\mathsf{g}^{**}_\omega)(\mathsf{L}_\omega\mathsf{x})\,\mu(d\omega)$.

(ix) $\operatorname{Argmin}_{\mathsf{x}\in\mathsf{H}}(\overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega,\mathsf{g}_\omega)_{\omega\in\Omega})(\mathsf{x}) = \operatorname{Argmin}_{\mathsf{x}\in\mathsf{H}} \int_\Omega {}^{\gamma}(\mathsf{g}^{**}_\omega)(\mathsf{L}_\omega\mathsf{x})\,\mu(d\omega)$.

(x) *Let* $\mathsf{x} \in \mathsf{H}$. *Then* $(\operatorname{rec} \overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega,\mathsf{g}_\omega)_{\omega\in\Omega})(\mathsf{x}) = \int_\Omega (\operatorname{rec}(\mathsf{g}^{**}_\omega))(\mathsf{L}_\omega\mathsf{x})\,\mu(d\omega)$.

(xi) *Suppose that μ is a probability measure and that there exists $\beta \in \left]0,+\infty\right[$ such that, for every $\omega \in \Omega$, $\mathsf{g}_\omega \colon \mathsf{G}_\omega \to \mathbb{R}$ is convex and β-Lipschitzian. Then $\overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega,\mathsf{g}_\omega)_{\omega\in\Omega}$ is β-Lipschitzian.*

Proof. Define L as in (4.8) and g as in (4.9). Recall from items (i) and (iii) in Proposition 4.6 that $L \in \mathcal{B}(\mathsf{H},\mathcal{G})$, $0 < \|L\| \leqslant 1$, and $g \in \Gamma_0(\mathcal{G})$. Additionally, by Proposition 4.6(iv)–(v),

$$\overset{\diamond}{\mathsf{M}}_\gamma(\mathsf{L}_\omega,\mathsf{g}_\omega)_{\omega\in\Omega} = L \overset{\gamma}{\diamond} g \quad\text{and}\quad \overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega,\mathsf{g}_\omega)_{\omega\in\Omega} = L \overset{\gamma}{\blacklozenge} g. \tag{4.17}$$

Also, proceeding as in the proof of Proposition 4.6, it can be shown that

$$(\mathsf{g}^{**}_\omega)_{\omega\in\Omega} \text{ satisfies the properties of [6, Assumption 4.6].} \tag{4.18}$$

Thus, by [6, Theorem 4.7(iv)],

$$(\forall x \in \mathcal{G}) \quad \big(\mathrm{prox}_{\gamma g}\, x\big)(\omega) = \mathrm{prox}_{\gamma \mathsf{g}_\omega^{**}}\big(x(\omega)\big) \ \text{ for } \mu\text{-almost every } \omega \in \Omega. \tag{4.19}$$

(i)–(iv): These are consequences of (4.17) and Proposition 3.7.
(v): It follows from (4.17), Propositions 3.10(i) and 4.6(ii), and (4.19) that

$$\mathrm{prox}_{\gamma \overset{\diamond}{\mathsf{M}}_\gamma(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega}}\, \mathsf{x} = L^*\big(\mathrm{prox}_{\gamma g}(L\mathsf{x})\big) = \int_\Omega \mathsf{L}_\omega^*\Big(\mathrm{prox}_{\gamma \mathsf{g}_\omega^{**}}(\mathsf{L}_\omega \mathsf{x})\Big)\,\mu(d\omega). \tag{4.20}$$

(vi): It follows from (4.17), Propositions 3.10(ii) and 4.6(ii), and (4.19) that

$$\begin{aligned} \mathrm{prox}_{\gamma \overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega}}\, \mathsf{x} &= \mathsf{x} - L^*\big(L\mathsf{x} - \mathrm{prox}_{\gamma g}(L\mathsf{x})\big) \\ &= \mathsf{x} - \int_\Omega \mathsf{L}_\omega^*\Big(\mathsf{L}_\omega \mathsf{x} - \mathrm{prox}_{\gamma \mathsf{g}_\omega^{**}}(\mathsf{L}_\omega \mathsf{x})\Big)\mu(d\omega). \end{aligned} \tag{4.21}$$

(vii): A consequence of (4.17) and Proposition 3.11.
(viii): By (4.18) and [6, Theorem 4.7(viii)],

$$^{\gamma}g = {}^{\mathfrak{G}}\!\!\int_\Omega^{\oplus} {}^{\gamma}\big(\mathsf{g}_\omega^{**}\big)\mu(d\omega). \tag{4.22}$$

However, by Lemma 2.4(ii), $g = g^{**}$. Therefore, (4.17), Proposition 3.13(ii) and (4.22) yield

$$^{\gamma}\Big(\overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega}\Big)(\mathsf{x}) = {}^{\gamma}\big(L \overset{\gamma}{\blacklozenge} g\big)(\mathsf{x}) = {}^{\gamma}g(L\mathsf{x}) = \int_\Omega {}^{\gamma}\big(\mathsf{g}_\omega^{**}\big)(\mathsf{L}_\omega \mathsf{x})\,\mu(d\omega). \tag{4.23}$$

(ix): The assertion is obtained by using successively (4.17), Corollary 3.14, and (viii).
(x): By (4.18) and [6, Theorem 4.7(x)],

$$\mathrm{rec}\, g = {}^{\mathfrak{G}}\!\!\int_\Omega^{\oplus} \mathrm{rec}\big(\mathsf{g}_\omega^{**}\big)\mu(d\omega) \tag{4.24}$$

However, by Lemma 2.4(ii), $g = g^{**}$. Hence, it results from (4.17), Proposition 3.16, and (4.24) that

$$\Big(\mathrm{rec}\, \overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega}\Big)(\mathsf{x}) = \Big(\mathrm{rec}\big(L \overset{\gamma}{\blacklozenge} g\big)\Big)(\mathsf{x}) = (\mathrm{rec}\, g)(L\mathsf{x}) = \int_\Omega \big(\mathrm{rec}(\mathsf{g}_\omega^{**})\big)(\mathsf{L}_\omega \mathsf{x})\,\mu(d\omega). \tag{4.25}$$

(xi): It follows from (4.9), Lemma 2.4(ii), and Jensen's inequality ([2, Proposition 9.24]) that

$$\begin{aligned} (\forall x \in \mathcal{G})(\forall y \in \mathcal{G}) \quad |g(x) - g(y)|^2 &= \left|\int_\Omega \Big(\mathsf{g}_\omega\big(x(\omega)\big) - \mathsf{g}_\omega\big(y(\omega)\big)\Big)\mu(d\omega)\right|^2 \\ &\leqslant \int_\Omega \big|\mathsf{g}_\omega\big(x(\omega)\big) - \mathsf{g}_\omega\big(y(\omega)\big)\big|^2\,\mu(d\omega) \\ &\leqslant \beta^2 \int_\Omega \|x(\omega) - y(\omega)\|_{\mathsf{G}_\omega}^2 \mu(d\omega) \\ &= \beta^2 \|x - y\|_{\mathcal{G}}^2. \end{aligned} \tag{4.26}$$

Therefore, g is β-Lipschitzian, and the conclusion follows from (4.17) and Corollary 3.12. ☐

Our second batch of results focuses on approximation properties.

Theorem 4.8. *Suppose that Assumptions 4.2 and 4.3 are in force. For every* $\mathsf{x} \in \mathsf{H}$*, define*

$$\Big(\overset{\triangleright}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega}\Big)(\mathsf{x}) = \inf\left\{\int_\Omega \mathsf{g}_\omega^{**}(x(\omega))\mu(d\omega) \;\middle|\; x \in \mathcal{G} \text{ and } \int_\Omega \mathsf{L}_\omega^*(x(\omega))\mu(d\omega) = \mathsf{x}\right\} \tag{4.27}$$

and write $(\overset{\triangleright}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega})(\mathsf{x}) = (\overset{\blacktriangleright}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\omega})(\mathsf{x})$ *if the infimum is attained. Then the following hold:*

(i) *Let* $\gamma \in \left]0, +\infty\right[$*. Then* $\overset{\diamond}{\mathsf{M}}_\gamma(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega} \geqslant \overset{\triangleright}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega}$.

(ii) *Let* $\gamma \in \left]0, +\infty\right[$ *and* $\mathsf{x} \in \mathsf{H}$*. Then*

$$\int_\Omega {}^\gamma(\mathsf{g}_\omega^{**})(\mathsf{L}_\omega \mathsf{x})\,\mu(d\omega) \leqslant \Big(\overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega}\Big)(\mathsf{x}) \leqslant \int_\Omega \mathsf{g}_\omega^{**}(\mathsf{L}_\omega \mathsf{x})\,\mu(d\omega). \tag{4.28}$$

(iii) *Let* $\gamma \in \left]0, +\infty\right[$*. Then* $\overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega} \leqslant \overset{\diamond}{\mathsf{M}}_\gamma(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega}$.

(iv) *Let* $\gamma \in \left]0, +\infty\right[$ *and suppose that* μ *is a probability measure and that, for every* $\omega \in \Omega$, L_ω *is an isometry. Then* $\overset{\diamond}{\mathsf{M}}_\gamma(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega} = \overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega}$.

(v) *Let* $\gamma \in \left]0, +\infty\right[$ *and suppose that* L *in* (4.8) *is a coisometry. Then the following are satisfied:*

  (a) $\overset{\diamond}{\mathsf{M}}_\gamma(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega} = \overset{\triangleright}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega}$.

  (b) *Let* $\mathsf{x} \in \mathsf{H}$*. Then* $(\overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega})(\mathsf{x}) = \displaystyle\int_\Omega \mathsf{g}_\omega^{**}(\mathsf{L}_\omega \mathsf{x})\,\mu(d\omega)$.

(vi) *Let* $\mathsf{x} \in \mathsf{H}$*. Then the following are satisfied:*

  (a) $\displaystyle\lim_{\gamma\to+\infty} (\overset{\diamond}{\mathsf{M}}_\gamma(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega})(\mathsf{x}) = (\overset{\triangleright}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega})(\mathsf{x})$.

  (b) $\displaystyle\lim_{0<\gamma\to 0} (\overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega})(\mathsf{x}) = \int_\Omega \mathsf{g}_\omega^{**}(\mathsf{L}_\omega \mathsf{x})\,\mu(d\omega)$.

(vii) *Suppose that* H *and* $\mathcal{G}$ *are finite-dimensional, and let* $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence in* $\left]0, +\infty\right[$*. Then the following are satisfied:*

  (a) *Suppose that* $\gamma_n \uparrow +\infty$*. Then* $\overset{\diamond}{\mathsf{M}}_{\gamma_n}(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega} \overset{e}{\to} \Big(\overset{\triangleright}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega}\Big)\breve{}$.

  (b) *Suppose that* $\gamma_n \downarrow 0$*. Then* $\overset{\blacklozenge}{\mathsf{M}}_{\gamma_n}(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega} \overset{e}{\to} \mathsf{f}$*, where* $(\forall \mathsf{x} \in \mathsf{H})\ \mathsf{f}(\mathsf{x}) = \displaystyle\int_\Omega \mathsf{g}_\omega^{**}(\mathsf{L}_\omega \mathsf{x})\,\mu(d\omega)$.

  (c) *Suppose that* $\gamma_n \downarrow 0$ *and that the function* $\mathsf{x} \mapsto \displaystyle\int_\Omega \mathsf{g}_\omega^{**}(\mathsf{L}_\omega \mathsf{x})\mu(d\omega)$ *is proper and coercive. Then* $\inf_{\mathsf{x}\in\mathsf{H}}(\overset{\blacklozenge}{\mathsf{M}}_{\gamma_n}(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega})(\mathsf{x}) \to \min_{\mathsf{x}\in\mathsf{H}} \displaystyle\int_\Omega \mathsf{g}_\omega^{**}(\mathsf{L}_\omega \mathsf{x})\,\mu(d\omega)$.

Proof. Define L as in (4.8) and g as in (4.9), and recall from items (i) and (iii) of Proposition 4.6 that $L \in \mathcal{B}(\mathsf{H}, \mathcal{G})$, $0 < \|L\| \leqslant 1$, and $g \in \Gamma_0(\mathcal{G})$. Further, by Proposition 4.6(iv)–(v),

$$\overset{\diamond}{\mathsf{M}}_\gamma(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega} = L \overset{\gamma}{\diamond} g, \quad \text{and} \quad \overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega} = L \overset{\gamma}{\blacklozenge} g. \tag{4.29}$$

Additionally, Proposition 4.6(ii) yields

$$(\forall \mathsf{x} \in \mathsf{H}) \quad (L^* \triangleright g)(\mathsf{x}) = \inf_{\substack{x\in\mathcal{G} \\ L^*x=\mathsf{x}}} g(x) = \Big(\overset{\triangleright}{\mathsf{M}}(\mathsf{L}_\omega, \mathsf{g}_\omega)_{\omega\in\Omega}\Big)(\mathsf{x}). \tag{4.30}$$

On the other hand,

$$(\forall \mathsf{x} \in \mathsf{H}) \quad g(L\mathsf{x}) = \int_{\Omega} \mathsf{g}_{\omega}^{**}\big((\mathfrak{e}_{\mathsf{L}}\mathsf{x})(\omega)\big)\mu(d\omega) = \int_{\Omega} \mathsf{g}_{\omega}^{**}(\mathsf{L}_{\omega}\mathsf{x})\mu(d\omega). \tag{4.31}$$

(i): The assertion follows from (4.29), (4.30), and Proposition 3.20(i).
(ii): Combine (4.29), (4.31), and Proposition 3.20(ii).
(iii): This is a consequence of (4.29) and Proposition 3.20(iii).
(iv): We have

$$(\forall \mathsf{x} \in \mathsf{H}) \quad \|L\mathsf{x}\|_{\mathcal{G}}^2 = \int_{\Omega} \|\mathsf{L}_{\omega}\mathsf{x}\|_{\mathsf{G}_{\omega}}^2 \mu(d\omega) = \int_{\Omega} \|\mathsf{x}\|_{\mathsf{H}}^2 \mu(d\omega) = \mu(\Omega)\|\mathsf{x}\|_{\mathsf{H}}^2 = \|\mathsf{x}\|_{\mathsf{H}}^2. \tag{4.32}$$

Therefore, L is an isometry and the assertion follows from (4.29) and Proposition 3.20(iv).
(v)(a): This follows from (4.29), (4.30), and Proposition 3.20(v).
(v)(b): This follows from (4.29), (4.31), and Proposition 3.20(v).
(vi)(a): This follows from (4.29), (4.30), and Theorem 3.30(iii).
(vi)(b): This follows from (4.29), (4.31), and Theorem 3.30(iv).
(vii)(a): This follows from (4.29), (4.30), and Theorem 3.35(ii)(a).
(vii)(b): This follows from (4.29), (4.31), and Theorem 3.35(ii)(b).
(vii)(c): This follows from (4.29), (4.31), and Proposition 3.38(i). ☐

Example 4.9. Let $p \in \mathbb{N} \smallsetminus \{0\}$, let $(\alpha_k)_{1 \leqslant k \leqslant p}$ be a family in $]0, +\infty[$, let H and $(\mathsf{G}_k)_{1 \leqslant k \leqslant p}$ be separable real Hilbert spaces, let $\mathfrak{G} = \mathsf{G}_1 \times \cdots \times \mathsf{G}_p$ be the usual Cartesian product vector space, with generic element $x = (\mathsf{x}_k)_{1 \leqslant k \leqslant p}$, and, for every $k \in \{1, \ldots, p\}$, let $\mathsf{L}_k \in \mathcal{B}(\mathsf{H}, \mathsf{G}_k)$ and let $\mathsf{g}_k \in \Gamma_0(\mathsf{G}_k)$. Suppose that $0 < \sum_{k=1}^p \alpha_k \|\mathsf{L}_k\|^2 \leqslant 1$ and set

$$\Omega = \{1, \ldots, p\}, \quad \mathcal{F} = 2^{\{1, \ldots, p\}}, \quad \text{and} \quad (\forall k \in \{1, \ldots, p\}) \;\; \mu(\{k\}) = \alpha_k, \tag{4.33}$$

Then $((\mathsf{G}_k)_{1 \leqslant k \leqslant p}, \mathfrak{G})$ is an $\mathcal{F}$-measurable vector field of real Hilbert spaces and ${}^{\mathfrak{G}}\!\int_{\Omega}^{\oplus} \mathsf{G}_{\omega}\mu(d\omega)$ is the weighted Hilbert direct sum of $(\mathsf{G}_k)_{1 \leqslant k \leqslant p}$, namely the Hilbert space obtained by equipping $\mathfrak{G}$ with the scalar product $(x, y) \mapsto \sum_{k=1}^p \alpha_k \langle \mathsf{x}_k \mid \mathsf{y}_k \rangle_{\mathsf{G}_k}$. Further, $\int_{\Omega} \|\mathsf{L}_{\omega}\|^2 \mu(d\omega) = \sum_{k=1}^p \alpha_k \|\mathsf{L}_k\|^2 \in]0, 1]$. Therefore, Assumptions 4.2 and 4.3 are satisfied, and (4.5) becomes a parametrized version of the *proximal mixture* of [9, Example 5.9], namely,

$$\overset{\diamond}{\mathsf{M}}_{\gamma}(\mathsf{L}_k, \mathsf{g}_k)_{1 \leqslant k \leqslant p} = \left(\sum_{k=1}^p \alpha_k \, {}^{\frac{1}{\gamma}}(\mathsf{g}_k^*) \circ \mathsf{L}_k\right)^* - \frac{1}{\gamma}\mathcal{Q}_{\mathsf{H}}, \tag{4.34}$$

while (4.6) becomes a parametrized version of the *proximal comixture*

$$\overset{\blacklozenge}{\mathsf{M}}_{\gamma}(\mathsf{L}_k, \mathsf{g}_k)_{1 \leqslant k \leqslant p} = \left(\left(\sum_{k=1}^p \alpha_k \, {}^{\gamma}(\mathsf{g}_k^{**}) \circ \mathsf{L}_k\right)^* - \gamma\mathcal{Q}_{\mathsf{H}}\right)^*. \tag{4.35}$$

In particular, for every $\mathsf{x} \in \mathsf{H}$, we derive from Theorem 4.8(vi) the following new facts:

(i) $\displaystyle \lim_{\gamma \to +\infty} \left(\overset{\diamond}{\mathsf{M}}_{\gamma}(\mathsf{L}_k, \mathsf{g}_k)_{1 \leqslant k \leqslant p}\right)(\mathsf{x}) = \left(\overset{\triangleright}{\mathsf{M}}(\mathsf{L}_k, \mathsf{g}_k)_{1 \leqslant k \leqslant p}\right)(\mathsf{x}) = \inf_{\substack{\mathsf{y}_1 \in \mathsf{G}_1, \ldots, \mathsf{y}_p \in \mathsf{G}_p \\ \sum_{k=1}^p \alpha_k \mathsf{L}_k^* \mathsf{y}_k = \mathsf{x}}} \left(\sum_{k=1}^p \alpha_k \mathsf{g}_k^{**}(\mathsf{y}_k)\right).$

(ii) $\displaystyle \lim_{0 < \gamma \to 0} \left(\overset{\blacklozenge}{\mathsf{M}}_{\gamma}(\mathsf{L}_k, \mathsf{g}_k)_{1 \leqslant k \leqslant p}\right)(\mathsf{x}) = \sum_{k=1}^p \alpha_k \mathsf{g}_k^{**}(\mathsf{L}_k\mathsf{x}).$

Example 4.10. In the context of Example 4.9, suppose that H is finite-dimensional and that, for every $k \in \{1, \ldots, p\}$, G_k is finite-dimensional and $\mathsf{g}_k \in \Gamma_0(\mathsf{G}_k)$. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $]0, +\infty[$. Then we obtain the following new results on proximal mixtures and comixtures.

(i) Suppose that $\gamma_n \uparrow +\infty$. Then Theorem 4.8(vii)(a) implies that

$$\overset{\diamond}{\mathsf{M}}_{\gamma_n}(\mathsf{L}_k, \mathsf{g}_k)_{1\leqslant k\leqslant p} \xrightarrow{e} \Big(\overset{\triangleright}{\mathsf{M}}(\mathsf{L}_k, \mathsf{g}_k)_{1\leqslant k\leqslant p}\Big)^{\breve{}}. \tag{4.36}$$

(ii) Suppose that $\gamma_n \downarrow 0$. Then Theorem 4.8(vii)(b) implies that $\overset{\blacklozenge}{\mathsf{M}}_{\gamma_n}(\mathsf{L}_k, \mathsf{g}_k)_{1\leqslant k\leqslant p} \xrightarrow{e} \sum_{k=1}^{p} \alpha_k \mathsf{g}_k \circ \mathsf{L}_k$.

(iii) Suppose that $\gamma_n \downarrow 0$ and that the function $\sum_{k=1}^{p} \alpha_k \mathsf{g}_k \circ \mathsf{L}_k$ is proper and coercive. Then Theorem 4.8(vii)(c) implies that

$$\inf_{\mathsf{x}\in\mathsf{H}} \Big(\overset{\blacklozenge}{\mathsf{M}}_{\gamma_n}(\mathsf{L}_k, \mathsf{g}_k)_{1\leqslant k\leqslant p}\Big)(\mathsf{x}) \to \min_{\mathsf{x}\in\mathsf{H}} \sum_{k=1}^{p} \alpha_k \mathsf{g}_k(\mathsf{L}_k\mathsf{x}). \tag{4.37}$$

Remark 4.11. In connection with Example 4.10, it was empirically argued in [11] (see also [14, 15, 18, 20] for the special cases of proximal averages) that, in variational formulations arising in image recovery and machine learning, combining linear operators $(\mathsf{L}_k)_{1\leqslant k\leqslant p}$ and convex functions $(\mathsf{g}_k)_{1\leqslant k\leqslant p}$ by means of the proximal comixture (4.35) instead of the standard averaging operation $\sum_{k=1}^{p} \alpha_k \mathsf{g}_k \circ \mathsf{L}_k$ had modeling and numerical advantages. For instance, the proximity of the former is intractable in general [12], while that of the latter is explicitly given by Theorem 4.7(vi) to be $\mathrm{Id}_{\mathsf{H}} - \sum_{k=1}^{p} \alpha_k (\mathsf{L}_k^* \circ (\mathrm{Id}_{\mathsf{G}_k} - \mathrm{prox}_{\gamma \mathsf{g}_k}) \circ \mathsf{L}_k)$, which makes the implementation of first-order optimization algorithms [10] straightforward. The results of Example 4.10 provide a theoretical context that sheds more light on such an approximation.

4.3. Proximal expectations

We specialize the results of Section 4.2 to the proximal expectation. This operation, introduced in [7, Definition 4.6] as an extension of the proximal average for finite families, performs a nonlinear averaging of an arbitrary family of functions. We study here the following extension of it which incorporates a parameter.

Definition 4.12. Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a complete probability space, let H be a separable real Hilbert space, let $(\mathsf{f}_\omega)_{\omega\in\Omega}$ be a family of functions in $\Gamma_0(\mathsf{H})$ such that the function

$$\Omega \times \mathsf{H} \to\,]-\infty, +\infty] \colon (\omega, \mathsf{x}) \mapsto \mathsf{f}_\omega(\mathsf{x}) \tag{4.38}$$

is $\mathcal{F} \otimes \mathcal{B}_{\mathsf{H}}$-measurable. Suppose that there exist $r \in \mathscr{L}^2(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{H})$ and $r^* \in \mathscr{L}^2(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{H})$ such that the functions $\omega \mapsto \mathsf{f}_\omega(r(\omega))$ and $\omega \mapsto \mathsf{f}_\omega^*(r^*(\omega))$ lie in $\mathscr{L}^1(\Omega, \mathcal{F}, \mathsf{P}; \mathbb{R})$. The *proximal expectation* of $(\mathsf{f}_\omega)_{\omega\in\Omega}$ with parameter $\gamma \in\,]0, +\infty[$ is

$$\overset{\diamond}{\mathsf{E}}_\gamma(\mathsf{f}_\omega)_{\omega\in\Omega} = \mathsf{h}^* - \frac{1}{\gamma}\mathcal{Q}_{\mathsf{H}}, \quad \text{where} \quad (\forall \mathsf{x} \in \mathsf{H}) \quad \mathsf{h}(\mathsf{x}) = \int_\Omega {}^{\frac{1}{\gamma}}(\mathsf{f}_\omega^*)(\mathsf{x})\mathsf{P}(d\omega). \tag{4.39}$$

An inspection of Definition 4.4 suggests that the proximal expectation can be viewed as the instance of the integral proximal mixture in which $(\forall \omega \in \Omega)$ $\mathsf{G}_\omega = \mathsf{H}$ and $\mathsf{L}_\omega = \mathrm{Id}_{\mathsf{H}}$. This fact opens the possibility of specializing the results of Section 4.2 to obtain properties of the proximal expectation. Let us formalize these ideas.

Proposition 4.13. *Consider the setting of Definition 4.12 and let $\gamma \in]0,+\infty[$. Then the following hold:*

(i) $\overset{\diamond}{\mathsf{E}}_\gamma(\mathsf{f}_\omega)_{\omega\in\Omega} = \overset{\diamond}{\mathsf{M}}_\gamma\big(\mathsf{Id}_\mathsf{H},\mathsf{f}_\omega\big)_{\omega\in\Omega} = \overset{\blacklozenge}{\mathsf{M}}_\gamma\big(\mathsf{Id}_\mathsf{H},\mathsf{f}_\omega\big)_{\omega\in\Omega}$.

(ii) $\overset{\diamond}{\mathsf{E}}_\gamma(\mathsf{f}_\omega)_{\omega\in\Omega} \in \Gamma_0(\mathsf{H})$.

(iii) $(\overset{\diamond}{\mathsf{E}}_\gamma(\mathsf{f}_\omega)_{\omega\in\Omega})^* = \overset{\diamond}{\mathsf{E}}_{1/\gamma}(\mathsf{f}^*_\omega)_{\omega\in\Omega}$.

(iv) *Let* $\mathsf{x} \in \mathsf{H}$*. Then* $\operatorname{prox}_{\gamma\overset{\diamond}{\mathsf{E}}_\gamma(\mathsf{f}_\omega)_{\omega\in\Omega}} \mathsf{x} = \displaystyle\int_\Omega \operatorname{prox}_{\gamma\mathsf{f}_\omega} \mathsf{x}\, \mathsf{P}(d\omega)$.

(v) *Let* $\mathsf{x} \in \mathsf{H}$*. Then* ${}^{\gamma}(\overset{\diamond}{\mathsf{E}}_\gamma(\mathsf{f}_\omega)_{\omega\in\Omega})(\mathsf{x}) = \displaystyle\int_\Omega {}^{\gamma}\mathsf{f}_\omega(\mathsf{x})\, \mathsf{P}(d\omega)$.

(vi) $\operatorname{Argmin}_{\mathsf{x}\in\mathsf{H}}(\overset{\diamond}{\mathsf{E}}_\gamma(\mathsf{f}_\omega)_{\omega\in\Omega})(\mathsf{x}) = \operatorname{Argmin}_{\mathsf{x}\in\mathsf{H}} \displaystyle\int_\Omega {}^{\gamma}\mathsf{f}_\omega(\mathsf{x})\, \mathsf{P}(d\omega)$.

(vii) *Let* $\mathsf{x} \in \mathsf{H}$*. Then* $(\operatorname{rec} \overset{\diamond}{\mathsf{E}}_\gamma(\mathsf{f}_\omega)_{\omega\in\Omega})(\mathsf{x}) = \displaystyle\int_\Omega (\operatorname{rec} \mathsf{f}_\omega)(\mathsf{x})\, \mathsf{P}(d\omega)$.

(viii) *Suppose that there exists $\beta \in]0,+\infty[$ such that, for every $\omega \in \Omega$, $\mathsf{f}_\omega\colon \mathsf{H} \to \mathbb{R}$ is β-Lipschitzian. Then $\overset{\diamond}{\mathsf{E}}_\gamma(\mathsf{f}_\omega)_{\omega\in\Omega}$ is β-Lipschitzian.*

Proof. (i): As in the proof of [7, Proposition 4.7], the family $(\mathsf{f}_\omega)_{\omega\in\Omega}$ fulfills the properties of Assumption 4.3. Therefore, the conclusion follows from (4.39), (4.5), and Theorem 4.8(iv).
(ii)–(viii): Combine (i) and Theorem 4.7. ☐

Remark 4.14. Item (iv) in Proposition 4.13 justifies calling $\overset{\diamond}{\mathsf{E}}_\gamma(\mathsf{f}_\omega)_{\omega\in\Omega}$ the proximal expectation of $(\mathsf{f}_\omega)_{\omega\in\Omega}$: its proximity operator is the expected value of the individual ones.

Proposition 4.15. *Consider the setting of Definition 4.12. For every* $\mathsf{x} \in \mathsf{H}$*, define*

$$\Big(\overset{\triangleright}{\mathsf{E}}(\mathsf{f}_\omega)_{\omega\in\Omega}\Big)(\mathsf{x}) = \inf\bigg\{\int_\Omega \mathsf{f}_\omega(x(\omega))\mathsf{P}(d\omega)\ \bigg|\ x \in L^2(\Omega,\mathcal{F},\mathsf{P};\mathsf{H}) \text{ and } \int_\Omega x(\omega)\mathsf{P}(d\omega) = \mathsf{x}\bigg\}. \tag{4.40}$$

Then the following hold:

(i) *Let* $\gamma \in]0,+\infty[$ *and* $\mathsf{x} \in \mathsf{H}$*. Then* $(\overset{\diamond}{\mathsf{E}}_\gamma(\mathsf{f}_\omega)_{\omega\in\Omega})(\mathsf{x}) \geqslant \displaystyle\int_\Omega {}^{\gamma}\mathsf{f}_\omega(\mathsf{x})\, \mathsf{P}(d\omega)$.

(ii) *Let* $\gamma \in]0,+\infty[$ *and* $\mathsf{x} \in \mathsf{H}$*. Then*

$$\Big(\overset{\triangleright}{\mathsf{E}}(\mathsf{f}_\omega)_{\omega\in\Omega}\Big)(\mathsf{x}) \leqslant \Big(\overset{\diamond}{\mathsf{E}}_\gamma(\mathsf{f}_\omega)_{\omega\in\Omega}\Big)(\mathsf{x}) \leqslant \int_\Omega \mathsf{f}_\omega(\mathsf{x})\, \mathsf{P}(d\omega). \tag{4.41}$$

(iii) *Let* $\mathsf{x} \in \mathsf{H}$*. Then the following are satisfied:*

(a) $\displaystyle\lim_{\gamma\to+\infty} (\overset{\diamond}{\mathsf{E}}_\gamma(\mathsf{f}_\omega)_{\omega\in\Omega})(\mathsf{x}) = (\overset{\triangleright}{\mathsf{E}}(\mathsf{f}_\omega)_{\omega\in\Omega})(\mathsf{x})$.

(b) $\displaystyle\lim_{0<\gamma\to0} (\overset{\diamond}{\mathsf{E}}_\gamma(\mathsf{f}_\omega)_{\omega\in\Omega})(\mathsf{x}) = \int_\Omega \mathsf{f}_\omega(\mathsf{x})\, \mathsf{P}(d\omega)$.

(iv) *Suppose that* H *and* $\mathcal{G}$ *are finite-dimensional, and let* $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence in* $]0,+\infty[$*. Then the following are satisfied:*

(a) *Suppose that* $\gamma_n \uparrow +\infty$*. Then* $\overset{\diamond}{\mathsf{E}}_{\gamma_n}(\mathsf{f}_\omega)_{\omega\in\Omega} \overset{e}{\to} \big(\overset{\triangleright}{\mathsf{E}}(\mathsf{f}_\omega)_{\omega\in\Omega}\big)\breve{}$.

(b) *Suppose that* $\gamma_n \downarrow 0$*. Then* $\overset{\diamond}{\mathsf{E}}_{\gamma_n}(\mathsf{f}_\omega)_{\omega\in\Omega} \overset{e}{\to} \mathsf{f}$*, where* $(\forall \mathsf{x} \in \mathsf{H})\ \mathsf{f}(\mathsf{x}) = \displaystyle\int_\Omega \mathsf{f}_\omega(\mathsf{x})\, \mathsf{P}(d\omega)$.

(c) *Suppose that* $\gamma_n \downarrow 0$ *and that the function* $\mathsf{x} \mapsto \int_\Omega \mathsf{f}_\omega(\mathsf{x})\mathsf{P}(d\omega)$ *is proper and coercive. Then* $\inf_{\mathsf{x}\in\mathsf{H}}(\overset{\diamond}{\mathsf{E}}_{\gamma_n}(\mathsf{f}_\omega)_{\omega\in\Omega})(\mathsf{x}) \to \min_{\mathsf{x}\in\mathsf{H}} \int_\Omega \mathsf{f}_\omega(\mathsf{x})\mathsf{P}(d\omega)$.

Proof. Combine Proposition 4.13(i) and Theorem 4.8. ☐

Remark 4.16. Suppose that $(\mathsf{f}_k)_{1\leqslant k\leqslant p}$ is a finite family of functions in $\Gamma_0(\mathsf{H})$ and define P as in (4.33), with the additional assumption that $\sum_{k=1}^p \alpha_k = 1$. Then $\overset{\diamond}{\mathsf{E}}(\mathsf{f}_k)_{1\leqslant k\leqslant p}$ is the *proximal average* of $(\mathsf{f}_k)_{1\leqslant k\leqslant p}$, studied in [3] (see also [9, Example 5.9]), namely,

$$\overset{\diamond}{\mathsf{E}}_\gamma(\mathsf{f}_k)_{1\leqslant k\leqslant p} = \left(\sum_{k=1}^{p} \alpha_k \,{}^{\frac{1}{\gamma}}(\mathsf{f}_k^*)\right)^* - \frac{1}{\gamma}\mathcal{Q}_\mathsf{H} = \mathrm{pav}_\gamma(\mathsf{f}_k)_{1\leqslant k\leqslant p}. \tag{4.42}$$

In this context, Propositions 4.13(i)–(vi) and 4.15 recover properties presented in [3]. On the other hand, Proposition 4.13(vii)–(viii) yields the following new properties of the proximal average:

(i) $\mathrm{rec}(\mathrm{pav}_\gamma(\mathsf{f}_k)_{1\leqslant k\leqslant p}) = \sum_{k=1}^p \alpha_k \,\mathrm{rec}\,\mathsf{f}_k$.

(ii) Suppose that there exists $\beta \in \left]0,+\infty\right[$ such that, for every $k \in \{1,\ldots,p\}$, $\mathsf{f}_k\colon \mathsf{H} \to \mathbb{R}$ is β-Lipschitzian. Then $\mathrm{pav}_\gamma(\mathsf{f}_k)_{1\leqslant k\leqslant p}$ is β-Lipschitzian.

We conclude by making a connection between proximal expectations and integral proximal comixtures that extends Proposition 4.13(i).

Proposition 4.17. *Let* $(\Omega,\mathcal{F},\mathsf{P})$ *be a complete probability space, suppose that Assumptions 4.2 and 4.3 are in force, and let* $\gamma \in \left]0,+\infty\right[$. *Further, for every* $\omega \in \Omega$, *suppose that* $0 < \|\mathsf{L}_\omega\| \leqslant 1$ *and set* $\mathsf{f}_\omega = \mathsf{L}_\omega \overset{\gamma}{\blacklozenge} \mathsf{g}_\omega$. *Suppose that the function* $\Omega\times\mathsf{H} \to \left]-\infty,+\infty\right]\colon (\omega,\mathsf{x}) \mapsto \mathsf{f}_\omega(\mathsf{x})$ *is* $\mathcal{F}\otimes\mathcal{B}_\mathsf{H}$*-measurable and that there exist* $s \in \mathscr{L}^2(\Omega,\mathcal{F},\mathsf{P};\mathsf{H})$ *and* $s^* \in \mathscr{L}^2(\Omega,\mathcal{F},\mathsf{P};\mathsf{H})$ *such that the functions* $\omega \mapsto \mathsf{f}_\omega(s(\omega))$ *and* $\omega \mapsto \mathsf{f}_\omega^*(s^*(\omega))$ *lie in* $\mathscr{L}^1(\Omega,\mathcal{F},\mathsf{P};\mathbb{R})$. *Then*

$$\overset{\diamond}{\mathsf{E}}_\gamma\left(\mathsf{L}_\omega \overset{\gamma}{\blacklozenge} \mathsf{g}_\omega\right)_{\omega\in\Omega} = \overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega,\mathsf{g}_\omega)_{\omega\in\Omega}. \tag{4.43}$$

Proof. As in the proof of [7, Proposition 4.7], the family $(\mathsf{f}_\omega)_{\omega\in\Omega}$ fulfills the properties of Assumption 4.3. On the other hand, Proposition 4.13(ii) and Theorem 4.7(ii) assert that $\overset{\diamond}{\mathsf{E}}_\gamma(\mathsf{f}_\omega)_{\omega\in\Omega}$ and $\overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega,\mathsf{g}_\omega)_{\omega\in\Omega}$ are in $\Gamma_0(\mathsf{H})$. Further, Propositions 4.13(v) and 3.13(ii), together with Theorem 4.7(viii) yield

$$\begin{aligned}(\forall \mathsf{x}\in\mathsf{H})\quad {}^{\gamma}\!\left(\overset{\diamond}{\mathsf{E}}_\gamma(\mathsf{f}_\omega)_{\omega\in\Omega}\right)(\mathsf{x}) &= \int_\Omega {}^{\gamma}\mathsf{f}_\omega(\mathsf{x})\,\mathsf{P}(d\omega)\\ &= \int_\Omega {}^{\gamma}\!\left(\mathsf{L}_\omega \overset{\gamma}{\blacklozenge} \mathsf{g}_\omega\right)(\mathsf{x})\,\mathsf{P}(d\omega)\\ &= \int_\Omega {}^{\gamma}\!\left(\mathsf{g}_\omega^{**}\right)(\mathsf{L}_\omega\mathsf{x})\,\mathsf{P}(d\omega)\\ &= {}^{\gamma}\!\left(\overset{\blacklozenge}{\mathsf{M}}_\gamma(\mathsf{L}_\omega,\mathsf{g}_\omega)_{\omega\in\Omega}\right)(\mathsf{x}),\end{aligned} \tag{4.44}$$

and the assertion therefore follows from Lemma 2.6. ☐

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