Tropical considerations in dynamic programming

Stephane.Gaubert@inria.fr

INRIA and CMAP, École Polytechnique

Optimization, Games, and Dynamics
IHP, Paris, November 28-29, 2011

Based on works with Akian,Guterman arXiv:0912.2462 to appear in IJAC, Allamigeon, Katz JCTA11,LAA11, Vigeral Math. Proc. Phil. Soc.11, McEneaney, Qu CDC11

Work partially supported by the Arpege programme of ANR and by Digiteo Labs

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

So

- $\text{“}2 + 3\text{”} =$

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

So

- $\text{“}2 + 3\text{”} = 3$

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a+b\text{”} = \max(a,b) \qquad \text{“}a\times b\text{”} = a+b$$

So

- “$2+3$” $=3$
- “2×3” $=$

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a+b\text{”} = \max(a,b) \qquad \text{“}a\times b\text{”} = a+b$$

So

- $\text{“}2+3\text{”} = 3$
- $\text{“}2\times 3\text{”} = 5$

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

So

- “$2 + 3$” $= 3$
- “2×3” $= 5$
- “$5/2$” $=$

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

So

- "$2 + 3$" $= 3$
- "2×3" $=5$
- "$5/2$" $=3$

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$“a+b” = \max(a,b) \qquad “a\times b” = a+b$$

So

- $“2+3” = 3$
- $“2\times 3” = 5$
- $“5/2” = 3$
- $“2^3” =$

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a+b\text{”} = \max(a,b) \qquad \text{“}a\times b\text{”} = a+b$$

So

- “$2+3$” $=3$
- “2×3” $=5$
- “$5/2$” $=3$
- “2^3” $=$ “$2\times 2\times 2$” $=6$

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

So

- $\text{“}2 + 3\text{”} = 3$
- $\text{“}2 \times 3\text{”} = 5$
- $\text{“}5/2\text{”} = 3$
- $\text{“}2^3\text{”} = \text{“}2 \times 2 \times 2\text{”} = 6$
- $\text{“}\sqrt{-1}\text{”} =$

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

So

- “$2 + 3$” $= 3$
- “2×3” $= 5$
- “$5/2$” $= 3$
- “2^3” $=$ “$2 \times 2 \times 2$” $= 6$
- “$\sqrt{-1}$” $= -0.5$

$$\text{“}\begin{pmatrix} 7 & 0 \\ -\infty & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}\text{”} =$$

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

So

- “$2 + 3$” $= 3$
- “2×3” $= 5$
- “$5/2$” $= 3$
- “2^3” $=$ “$2 \times 2 \times 2$” $= 6$
- “$\sqrt{-1}$” $= -0.5$

$$\text{“}\begin{pmatrix} 7 & 0 \\ -\infty & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}\text{”} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$$

The notation $a \oplus b := \max(a, b)$, $a \otimes b := a + b$, $\mathbb{0} := -\infty$, $\mathbb{1} := 0$ is also used in the tropical/max-plus litterature

The sister algebra: min-plus

$$\text{“}a + b\text{”} = \min(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

- $\text{“}2 + 3\text{”} = 2$
- $\text{“}2 \times 3\text{”} = 5$

The term "tropical" is in the honor of Imre Simon, 1943 - 2009

who lived in Sao Paulo (south tropic).

These algebras were invented by various schools in the world

- Cuninghame-Green 1960- OR (scheduling, optimization)
- Vorobyev ∼65 ... Zimmerman, Butkovic; Optimization
- Maslov ∼ 80'- ... Kolokoltsov, Litvinov, Samborskii, Shpiz... Quasi-classic analysis, variations calculus
- Simon ∼ 78- ... Hashiguchi, Leung, Pin, Krob, ... Automata theory
- Gondran, Minoux ∼ 77 Operations research
- Cohen, Quadrat, Viot ∼ 83- ... Olsder, Baccelli, S.G., Akian initially discrete event systems, then optimal control, idempotent probabilities, combinatorial linear algebra
- Nussbaum 86- Nonlinear analysis, dynamical systems, also related work in linear algebra, Friedland 88, Bapat ˜94
- Kim, Roush 84 Incline algebras
- Fleming, McEneaney ∼00- max-plus approximation of HJB
- Puhalskii ∼99- idempotent probabilities (large deviations)

and now in **tropical geometry, after Viro, Mikhalkin, Passare, Sturmfels and many**.

Menu: connections between. . .

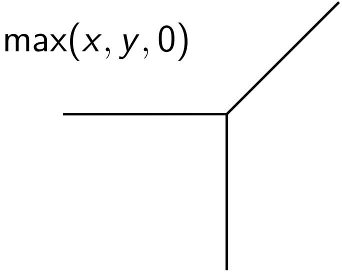
- tropical convexity
- dynamic programming / zero-sum games
- Perron-Frobenius theory
- metric geometry

Some elementary tropical geometry

A tropical line in the plane is the set of (x, y) such that the max in

$$\text{“}ax + by + c\text{”}$$

is attained at least twice.



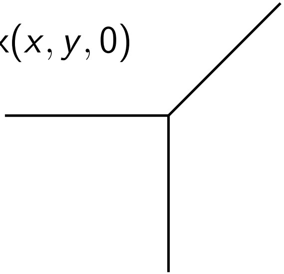
Some elementary tropical geometry

A tropical line in the plane is the set of (x, y) such that the max in

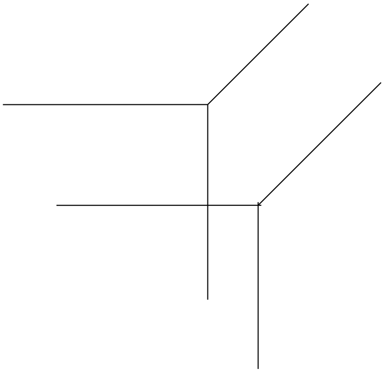
$$\max(a + x, b + y, c)$$

is attained at least twice.

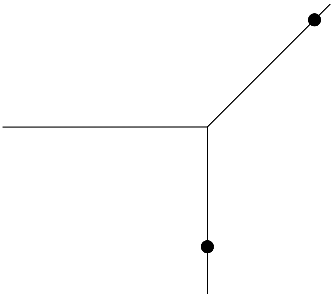
$$\max(x, y, 0)$$



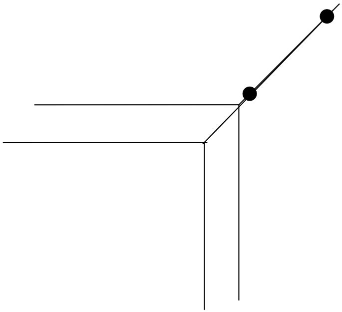
Two generic tropical lines meet at a unique point



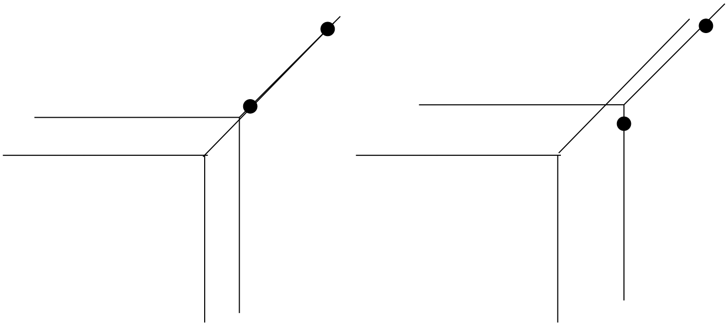
By two generic points passes a unique tropical line



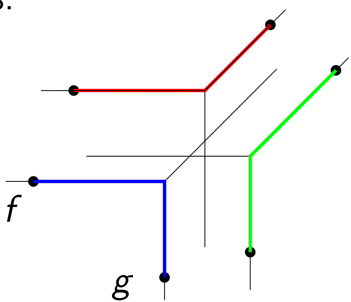
non generic case



non generic case resolved by perturbation



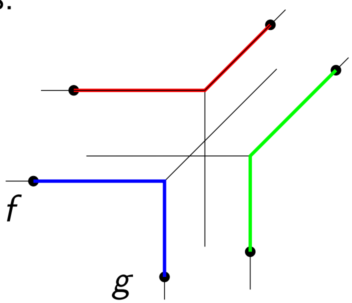
Tropical segments:



$$[f,g] := \{\text{“}\lambda f + \mu g\text{”} \mid \lambda,\mu \in \mathbb{R}\cup\{-\infty\},\ \text{“}\lambda+\mu = 1\text{”}\}.$$

(The condition “$\lambda, \mu \geq 0$” is automatic.)

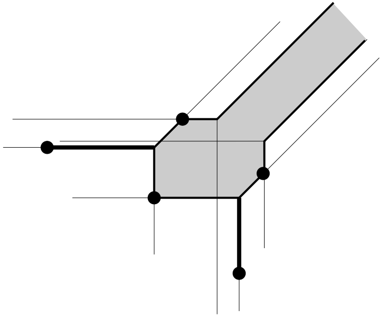
Tropical segments:



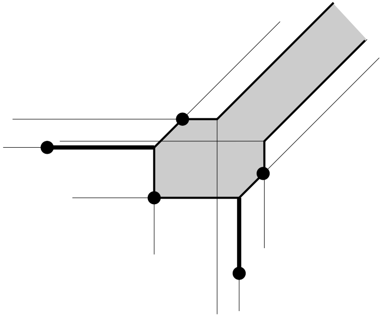
$[f,g] := \{\sup(\lambda + f, \mu + g) \mid \lambda, \mu \in \mathbb{R} \cup \{-\infty\}, \ \max(\lambda, \mu) = 0\}.$

(The condition $\lambda, \mu \geq -\infty$ is automatic.)

Tropical convex set: $f, g \in C \implies [f, g] \in C$

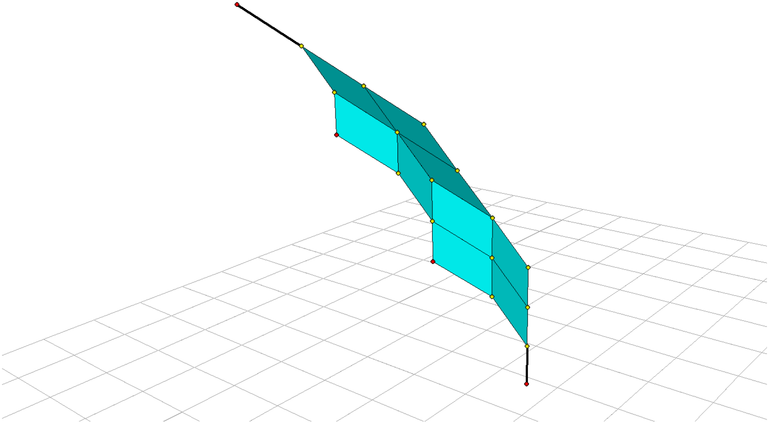


Tropical convex set: $f, g \in C \implies [f, g] \in C$



Tropical convex cone: ommit "$\lambda + \mu = 1$", i.e., replace $[f, g]$ by $\{\sup(\lambda + f, \mu + g) \mid \lambda, \mu \in \mathbb{R} \cup \{-\infty\}\}$

A max-plus "tetrahedron"?

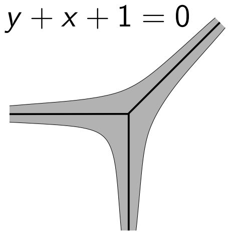


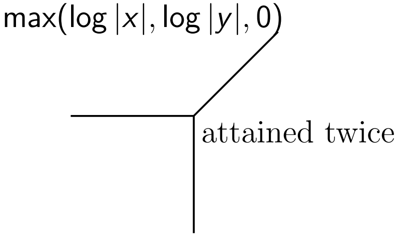
The previous drawing was generated by POLYMAKE of Gawrilow and Joswig, in which an extension allows one to handle easily tropical polyhedra. They were drawn with JAVAVIEW. See Joswig arXiv:0809.4694 for more information.

Why?

Gelfand, Kapranov, and Zelevinsky defined the amoeba of an algebraic variety $V \subset (\mathbb{C}^*)^n$ to be the "log-log plot"

$$A(V) := \{(\log|z_1|, \ldots, \log|z_n|) \mid (z_1, \ldots, z_n) \in V\} \ .$$

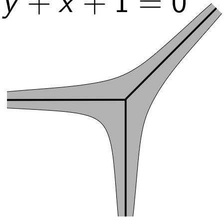




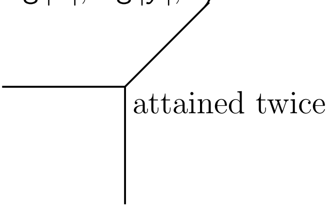
Gelfand, Kapranov, and Zelevinsky defined the amoeba of an algebraic variety $V \subset (\mathbb{C}^*)^n$ to be the "log-log plot"

$$A(V) := \{(\log|z_1|, \ldots, \log|z_n|) \mid (z_1, \ldots, z_n) \in V\} \ .$$

$y + x + 1 = 0$



$\max(\log|x|, \log|y|, 0)$

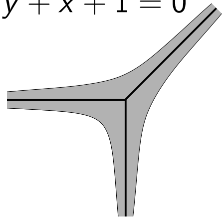


$|y| \leq |x| + 1,\ |x| \leq |y| + 1,\ 1 \leq |x| + |y|$

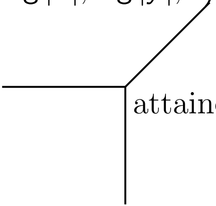
Gelfand, Kapranov, and Zelevinsky defined the amoeba of an algebraic variety $V \subset (\mathbb{C}^*)^n$ to be the "log-log plot"

$$A(V) := \{(\log|z_1|, \dots, \log|z_n|) \mid (z_1, \dots, z_n) \in V\} \ .$$

$y + x + 1 = 0$



$\max(\log|x|, \log|y|, 0)$

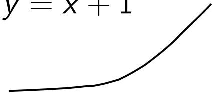


$X := \log|x|, \ Y := \log|y|$

$Y \leq \log(e^X + 1), \ X \leq \log(e^Y + 1), \ 1 \leq e^X + e^Y$

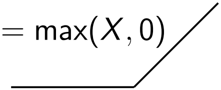
real tropical lines

$y = x + 1$



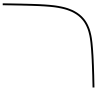
$X = \log(e^X + 1)$

$Y = \max(X, 0)$



real tropical lines

$x + y = 1$

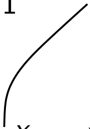


$\log(e^X + e^Y) = 1$

$\max(X, Y) = 0$

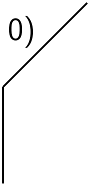
real tropical lines

$x = y + 1$



$X = \log(e^X + 1)$

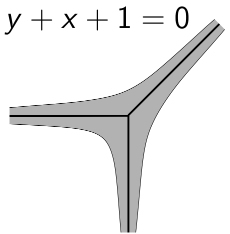
$X = \max(Y, 0)$

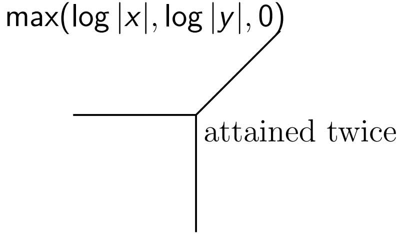


Viro's log-glasses, related to Maslov's dequantization

$$a +_h b := h \log(e^{a/h} + e^{b/h}), \qquad h \to 0^+$$

With h-log glasses, the amoeba of the line retracts to the tropical line as $h \to 0^+$





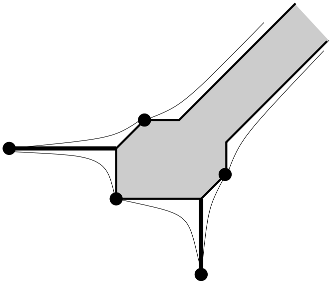
$$\max(a, b) \le a +_h b \le h \log 2 + \max(a, b)$$

Similar to convergence of p-norm to sup-norm

$$[a, b] := \{\lambda a +_p \mu b,\ \lambda, \mu \geq 0, \lambda +_p \mu = 1$$

$$a +_p b = (a^p + b^p)^{1/p}$$

The convex hull in the $+_h$ / $+_p$ sense converges to the tropical convex hull as $h \to 0$ / $p \to \infty$ (Briec and Horvath).



All the results of classical convexity have tropical analogues, sometimes more degenerate...

- generation by extreme points Helbig; SG, Katz 07; Butkovič, Sergeev, Schneider 07
- projection / best-approximation : Cohen, SG, Quadrat 01,04; Singer
- Hahn-Banach analytic Litvinov, Maslov, Shpiz 00; Cohen, SG, Quadrat 04; geometric Zimmermann 77, Cohen, SG, Quadrat 01,05; Develin, Sturmfels 04, Joswig 05
- cyclic projections Butkovic, Cuninghame-Green TCS03; SG, Sergeev 06
- Radon, Helly, Carathéodory, Colorful Carathéodory, Tverberg: SG, Meunier DCG09

See Passare & Rullgard, Duke Math. 04 for more information on amoebas

This talk

Tropical convexity is equivalent to dynamic programming (zero-sum games).

This talk

Tropical convexity is equivalent to dynamic programming (zero-sum games).

- finite dimensional convex sets (cones) $\sim$ stochastic games with finite state spaces

This talk

Tropical convexity is equivalent to dynamic programming (zero-sum games).

- finite dimensional convex sets (cones) $\sim$ stochastic games with finite state spaces
- infinite dimensional convex cones, spaces of functions $\sim$ stationnary solutions of Hamilton-Jacobi(-Bellman) equations (1-player: Fathi's weak KAM solutions)

This talk

Tropical convexity is equivalent to dynamic programming (zero-sum games).

- finite dimensional convex sets (cones) $\sim$ stochastic games with finite state spaces
- infinite dimensional convex cones, spaces of functions $\sim$ stationnary solutions of Hamilton-Jacobi(-Bellman) equations (1-player: Fathi's weak KAM solutions)
- leads to: equivalence (computational complexity) results, algorithms, approximation methods, ...

Shapley operators

$X = \mathscr{C}(K)$, even $X = \mathbb{R}^n$; Shapley operator T,

$$T_i(x) = \max_{a \in A_i} \min_{b \in B_{i,a}} \Big(r_i^{ab} + \sum_{1 \leq j \leq n} P_{ij}^{ab} x_j \Big) , \qquad i \in [n]$$

- $[n] := \{1, \ldots, n\}$ set of states
- a action of Player I, b action of Player II
- r_i^{ab} payment of Player II to Player I
- P_{ij}^{ab} transition probability $i \to j$

Shapley operators

$X = \mathscr{C}(K)$, even $X = \mathbb{R}^n$; Shapley operator T,

$$T_i(x) = \max_{a \in A_i} \min_{b \in B_{i,a}} \Big(r_i^{ab} + \sum_{1 \le j \le n} P_{ij}^{ab} x_j \Big), \qquad i \in [n]$$

T is order preserving and additively homogeneous:

$$x \le y \implies T(x) \le T(y)$$
$$T(\alpha + x) = \alpha + T(x), \qquad \forall \alpha \in \mathbb{R}$$

Shapley operators

$X = \mathscr{C}(K)$, even $X = \mathbb{R}^n$; Shapley operator T,

$$T_i(x) = \max_{a \in A_i} \min_{b \in B_{i,a}} \Big(r_i^{ab} + \sum_{1 \leq j \leq n} P_{ij}^{ab} x_j \Big), \qquad i \in [n]$$

Conversely, any order preserving additively homogeneous operator is a Shapley operator (Kolokoltsov), even with degenerate transition probabilities (deterministic) Gunawardena, Sparrow; Singer, Rubinov,

$$T_i(x) = \sup_{y \in \mathbb{R}} \Big(T_i(y) + \min_{1 \leq i \leq n} (x_i - y_i) \Big)$$

Variant. T is additively subhomogeneous if

$$T(\alpha + x) \leq \alpha + T(x), \qquad \forall \alpha \in \mathbb{R}_+$$

This corresponds to $1 - \sum_j P_{ij}^{ab} =$ death probability > 0.

Order-preserving + additively (sub)homogeneous $\implies$ sup-norm nonexpansive

$$\|T(x) - T(y)\|_\infty \leq \|x - y\|_\infty \ .$$

Variant. T is additively subhomogeneous if

$$T(\alpha + x) \leq \alpha + T(x), \qquad \forall \alpha \in \mathbb{R}_+$$

This corresponds to $1 - \sum_j P_{ij}^{ab} =$ death probability > 0.

Order-preserving + additively homogeneous $\iff$ top nonexpansive

$$\mathrm{t}(T(x) - T(y)) \leq \mathrm{t}(x - y), \qquad \mathrm{t}(z) := \max_i z_i \quad .$$

Variant. T is additively subhomogeneous if

$$T(\alpha + x) \leq \alpha + T(x), \qquad \forall \alpha \in \mathbb{R}_+$$

This corresponds to $1 - \sum_j P_{ij}^{ab} =$ death probability > 0.

Order-preserving + additively subhomogeneous $\iff$ top$^+$ nonexpansive

$$\mathrm{t}^+(T(x) - T(y)) \leq \mathrm{t}^+(x - y), \quad \mathrm{t}^+(z) := \max(\max_i z_i, 0) \ .$$

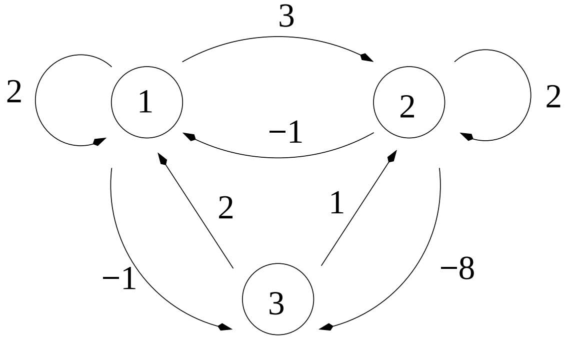
Repeated games

The value of the game in horizon k starting from state i is $(T^k(0))_i$.

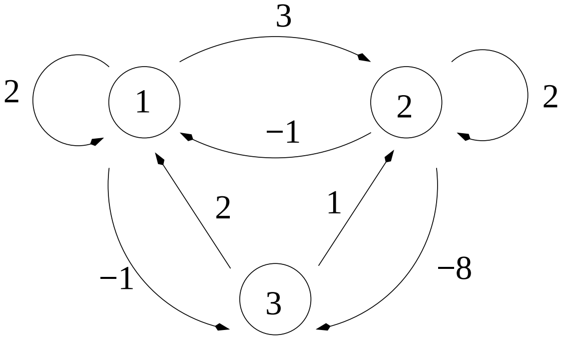
We are interested in the long term payment per time unit

$$\chi(T) := \lim_{k\to\infty} T^k(0)/k$$

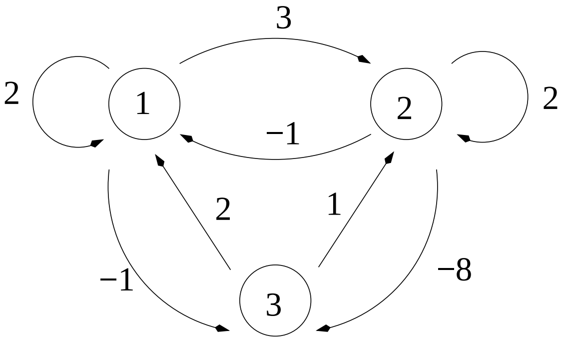
Max and Min flip a coin to decide who makes the move. Min always pay.



Max and Min flip a coin to decide who makes the move. Min always pay.



$$v_i^{k+1} = \frac{1}{2}\Big(\max_{j:\, i\to j}(c_{ij} + v_j^k) + \min_{j:\, i\to j}(c_{ij} + v_j^k)\Big) \enspace .$$



$$\begin{bmatrix}5\\0\\4\end{bmatrix}\quad \begin{aligned} v_1 &= \tfrac{1}{2}(\max(\overline{2+v_1}, 3+v_2, -1+v_3) + \min(2+v_1, \overline{3+v_2}, \overline{-1+v_3}) \\ v_2 &= \tfrac{1}{2}(\max(\overline{-1+v_1}, 2+v_2, -8+v_3) + \min(-1+v_1, 2+v_2, \overline{-8+v_3}) \\ v_3 &= \tfrac{1}{2}(\max(\overline{2+v_1}, 1+v_2) + \min(2+v_1, \overline{1+v_2}) \end{aligned}$$

this game is fair

$$v = \frac{1}{2}(\max_{j:\, i\to j} v_j^k + \min_{j:\, i\to j} v_j^k) \ ,$$

v_i, $i \in$ boundary prescribed:

discrete variant of Laplacian infinity (Oberman), or Richman games (Tug of war).

Optimality certificates

More generally, for $u \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$,

$$T(u) \geq u \implies \chi(T) \geq 0$$

$$T(u) \leq u \implies \chi(T) \leq 0$$

$$T(u) = \lambda + u \implies \chi(T) = (\lambda, \dots, \lambda) \ .$$

Optimality certificates

More generally, for $u \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$,

$$T(u) \geq u \implies \chi(T) \geq 0$$

$$T(u) \leq u \implies \chi(T) \leq 0$$

$$T(u) = \lambda + u \implies \chi(T) = (\lambda, \ldots, \lambda) \ .$$

Sufficient condition SG+Gunawardena, TAMS 2004: if $G(T)$ is strongly connected, then the additive eigenproblem $T(u) = \lambda + u$ with $\lambda \in \mathbb{R}$ is solvable
$G(T)$: arc $i \to j$ if $\lim_{s\to\infty} T_i(se_j) = +\infty$.

$T(u) = \lambda + u$, $u \in \mathbb{R}^n$ may not have a solution.

$T(u) = \lambda + u$, $u \in \mathbb{R}^n$ may not have a solution.

Indeed, it may happen that $\chi_j(T) \neq \chi_k(T)$ for two different initial states j, k.

$T(u) = \lambda + u$, $u \in \mathbb{R}^n$ may not have a solution.

Indeed, it may happen that $\chi_j(T) \neq \chi_k(T)$ for two different initial states j, k.

$\chi(T) = \lim_k T^k(0)/k$ may even not exist if the action spaces are infinite (Kohlberg, Neyman)

$T(u) = \lambda + u$, $u \in \mathbb{R}^n$ may not have a solution.

Indeed, it may happen that $\chi_j(T) \neq \chi_k(T)$ for two different initial states j, k.

$\chi(T) = \lim_k T^k(0)/k$ may even not exist if the action spaces are infinite (Kohlberg, Neyman)

However...

$T(u) = \lambda + u$, $u \in \mathbb{R}^n$ may not have a solution.

Indeed, it may happen that $\chi_j(T) \neq \chi_k(T)$ for two different initial states j, k.

$\chi(T) = \lim_k T^k(0)/k$ may even not exist if the action spaces are infinite (Kohlberg, Neyman)

However...

- If the graph of T is semi-algebraic, then $\chi(T)$ does exists. Neyman 04, extending Bewley and Kohlberg 76.

By subadditivity, the following limits (indep of $x \in \mathbb{R}^n$) do exist

$$\lim_{k\to\infty} \frac{\|T^k(x)-x\|_\infty}{k} = \inf_{k\geq 1} \frac{\|T^k(x)-x\|_\infty}{k}$$

$$\overline{\chi}(T) := \lim_{k\to\infty} \frac{\mathrm{t}(T^k(x)-x)}{k} = \inf_{k\geq 1} \frac{\mathrm{t}(T^k(x)-x)}{k}$$

$$\underline{\chi}(T) := \lim_{k\to\infty} \frac{\mathrm{b}(T^k(x)-x)}{k} = \sup_{k\geq 1} \frac{\mathrm{b}(T^k(x)-x)}{k}$$

$$\mathrm{t}(z) := \max_i z_i, \qquad \mathrm{b}(z) := \min_i z_i \quad .$$

In general, think of T as a Perron-Frobenius operator in log-glasses:

$$F = \exp \circ T \circ \log, \qquad \mathbb{R}_+^n \to \mathbb{R}_+^n$$

F extends continuously from $\operatorname{int} \mathbb{R}_+^n$ to $\mathbb{R}_+^n$ Burbanks, Nussbaum, Sparrow.

Theorem (non-linear Collatz-Wielandt, Nussbaum, LAA 86)

$$\begin{aligned}\rho(F) &= \lim_{k\to\infty} \|F^k(x)\|^{1/k}, \qquad x \in \operatorname{Int} \mathbb{R}_+^n \\ &= \max\{\mu \in \mathbb{R}_+ \mid F(v) = \mu v, v \in \mathbb{R}_+^n, v \neq 0\} \\ &= \max\{\mu \in \mathbb{R}_+ \mid F(v) \geq \mu v, v \in \mathbb{R}_+^n, v \neq 0\}\end{aligned}$$

$$\overline{\chi}(T) := \lim_{k\to\infty} \max_{1\leq j\leq n} [T^k(0)]_j/k = \log \rho(F)$$

SG, Gunawardena, TAMS 04: there always exists an initial state which achieves the best payoff

$$\forall x \in \mathbb{R}^n, \qquad \exists j,\ [T^k(x)]_j \geq k\overline{\chi}(T) + x_j, \qquad \forall k$$

Theorem (non-linear Collatz-Wielandt, Nussbaum, LAA 86)

$$\begin{aligned}\rho(F) &= \lim_{k\to\infty} \|F^k(x)\|^{1/k}, \qquad x \in \operatorname{Int} \mathbb{R}^n_+ \\ &= \max\{\mu \in \mathbb{R}_+ \mid F(v) = \mu v, v \in \mathbb{R}^n_+, v \neq 0\} \\ &= \max\{\mu \in \mathbb{R}_+ \mid F(v) \geq \mu v, v \in \mathbb{R}^n_+, v \neq 0\}\end{aligned}$$

Correspondence between tropical convexity and zero-sum games

Theorem (Akian, SG, Guterman, arXiv:0912.2462 → IJAC)

TFAE:

- C *closed tropical convex cone*
- $C = \{u \in (\mathbb{R} \cup \{-\infty\})^n \mid u \leq T(u)\}$ *for some Shapley operator* T

and MAX has at least one winning state ($\overline{\chi}(T) \geq 0$) *if and only if*

$$C \neq \{(-\infty, \dots, -\infty)\} \ .$$

Recall $C \subset (\mathbb{R} \cup \{-\infty\})^n$ is a tropical convex cone if

$$u, v \in C, \lambda \in \mathbb{R} \cup \{-\infty\} \implies \sup(u, v) \in C, \quad \lambda + u \in C \enspace .$$

Easy implication: T order preserving and additively homogeneous $\implies \{u \mid u \leq T(u)\}$ is a closed tropical convex cone

Recall $C \subset (\mathbb{R} \cup \{-\infty\})^n$ is a tropical convex cone if

$$u, v \in C, \lambda \in \mathbb{R}\cup\{-\infty\} \implies \sup(u, v) \in C, \quad \lambda + u \in C \ .$$

Easy implication: T order preserving and additively homogeneous $\implies \{u \mid u \leq T(u)\}$ is a closed tropical convex cone

Remark: $\{u \mid u \geq T(u)\}$ is a dual tropical (min-plus) cone.

Conversely, any closed tropical convex cone can be written as

$$C = \bigcap_{i \in I} H_i$$

where $(H_i)_{i \in I}$ is a family of tropical half-spaces.

$$H_i : \text{“}A_i x \leq B_i x\text{”}$$

Conversely, any closed tropical convex cone can be written as

$$C = \bigcap_{i \in I} H_i$$

where $(H_i)_{i\in I}$ is a family of tropical half-spaces.

$$H_i : \max_{1\leq j\leq n} a_{ij} + x_j \leq \max_{1\leq k\leq n} b_{ik} + x_k , \qquad a_{ij}, b_{ij} \in \mathbb{R} \cup \{-\infty\}$$

$$[T(x)]_j = \inf_{i\in I} -a_{ij} + \max_{1\leq k\leq n} b_{ik} + x_k \quad .$$

Conversely, any closed tropical convex cone can be written as

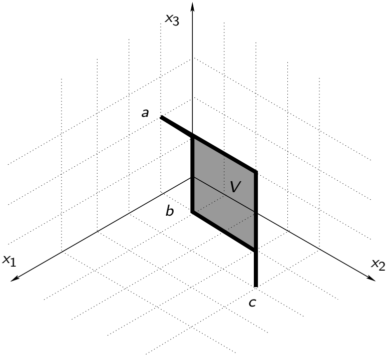
$$C = \bigcap_{i \in I} H_i$$

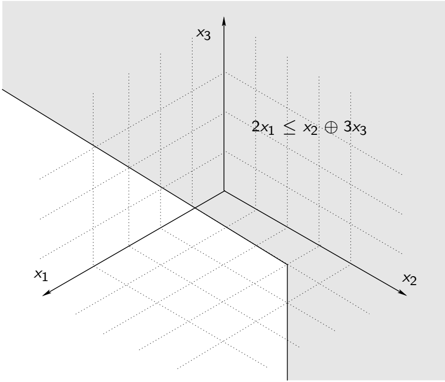
where $(H_i)_{i \in I}$ is a family of tropical half-spaces.

$$H_i : \max_{1 \le j \le n} a_{ij} + x_j \le \max_{1 \le k \le n} b_{ik} + x_k , \qquad a_{ij}, b_{ij} \in \mathbb{R} \cup \{-\infty\}$$

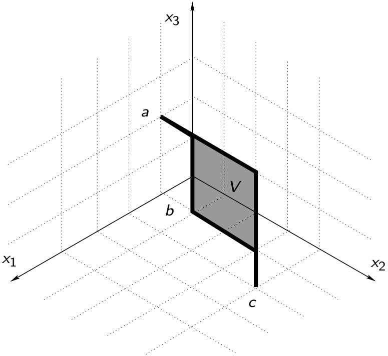
$$[T(x)]_j = \inf_{i \in I} -a_{ij} + \max_{1 \le k \le n} b_{ik} + x_k \quad .$$

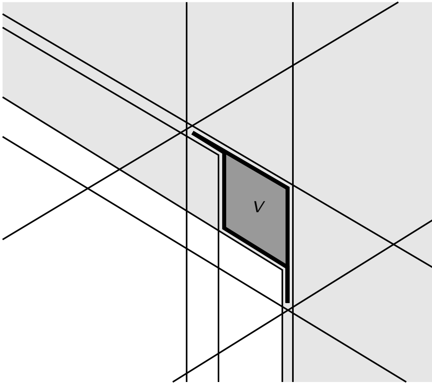
$$x \le T(x) \iff \max_{1 \le j \le n} a_{ij} + x_j \le \max_{1 \le k \le n} b_{ik} + x_k , \; \forall i \in I \quad .$$





$$2 + x_1 \leq \max(x_2, 3 + x_3)$$





$$2 + x_1 \leq \max(x_2, 3 + x_3)$$

$$H_i : \max_{1\le j\le n} a_{ij} + x_j \le \max_{1\le k\le n} b_{ik} + x_k$$

$$[T(x)]_j = \inf_{i\in I} -a_{ij} + \max_{1\le k\le n} b_{ik} + x_k \quad .$$

Interpretation of the game

- State of MIN: variable x_j, $j \in \{1, \dots, n\}$
- State of MAX: half-space H_i, $i \in I$
- In state x_j, Player MIN chooses a tropical half-space H_i with x_j in the LHS
- In state H_i, player MAX chooses a variable x_k at the RHS of H_i
- Payment $-a_{ij} + b_{ik}$.

Now, $\overline{\chi}(T) \geq 0 \iff C \neq \{-\infty\}$ follows from Nussbaum's Collatz-Wielandt theorem, $F := \exp \circ T \circ \log$,

$$\begin{aligned}\overline{\chi}(T) &\geq 0\\ \rho(F) &\geq 1\\ \exists v \in \mathbb{R}^n_+, v \not\equiv 0, F(v) &\geq v\\ \exists u \not\equiv -\infty, T(u) &\geq u\end{aligned}$$

Polyhedral case

Theorem (Akian, SG, Guterman arXiv:0912.2462 $\to$ IJAC)

If the game is deterministic with finite action spaces (i.e. C is a tropical polyhedron), then the set of winning states is the support of C:

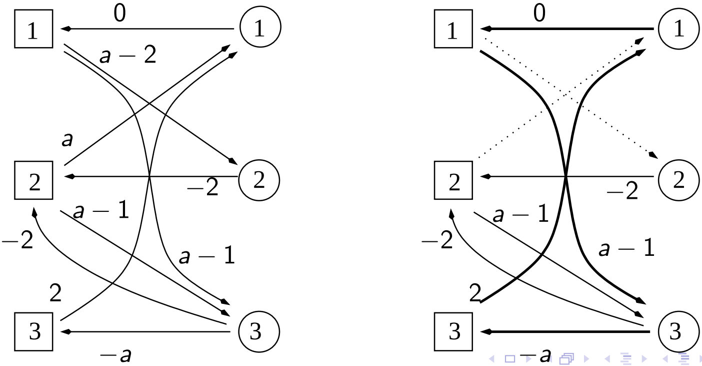
$$\{i \mid \exists u \in C,\ u_i \neq -\infty\} = \{i \mid \chi_i(T) \geq 0\}$$

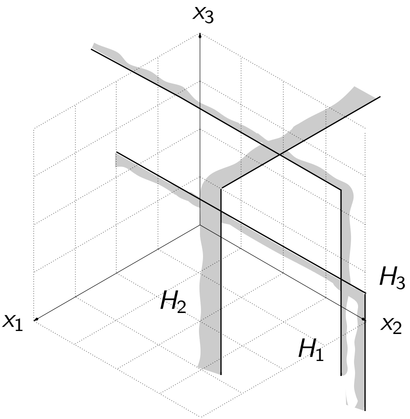
$$x_1 \leq a + \max(x_2 - 2, x_3 - 1) \qquad (H_1)$$

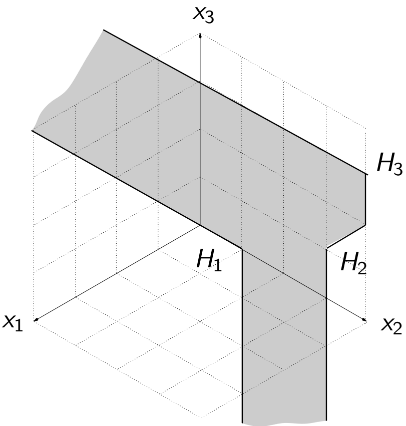
$$-2 + x_2 \leq a + \max(x_1, x_3 - 1) \qquad (H_2)$$

$$\max(x_2 - 2, x_3 - a) \leq x_1 + 2 \qquad (H_3)$$

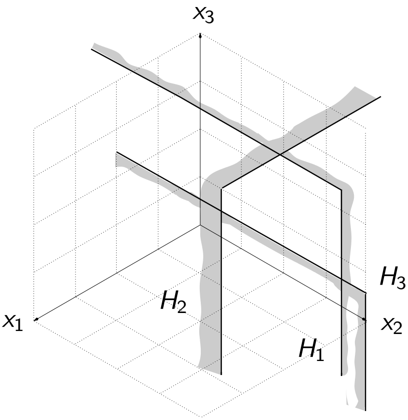
value $\chi(T)_j = (2a+1)/2, \forall j$.

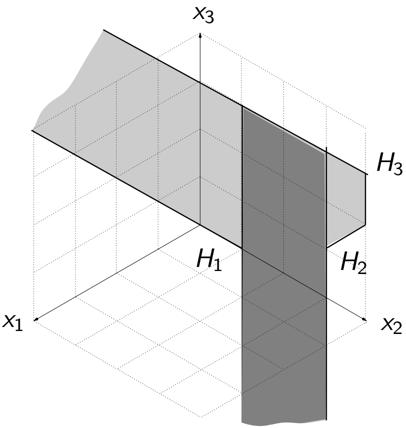






$a = -3/2$, victorious strategy of Min: certificate of emptyness involving $\leq d$ inequalities (Helly)





$a = 1$, victorious strategy of Max: tropical polytrope $\neq \emptyset$ included in the convex set

Corollary

Feasibiliby in tropical linear programming, i.e.,

$$\exists ? u \in (\mathbb{R} \cup \{-\infty\})^n, \ \max_j a_{ij} + u_j \leq \max_j b_{ij} + u_j, \ 1 \leq i \leq p$$

is polynomial-time equivalent to mean payoff games.

Mean payoff games: Gurvich, Karzanov, Khachyan 86; are in NP $\cap$ coNP: Zwick, Paterson 96.

Tropical convex sets are log-limits of classical convex sets: polynomial time solvability of mean payoff games might follow from a strongly polynomial-time algorithm in linear programming (Schewe).

Other problems in tropical programming, like tropical Farkas ($Ax \leq Bx \implies cx \leq dx$?) also equivalent to mean payoff games by Allamigeon, SG, Katz, LAA 11.

See also SG, Katz, Sergeev for linear-fractional tropical programming

An application: perturbation of eigenvalues

Exercise.

$$\mathcal{A}_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix} \quad ,$$

An application: perturbation of eigenvalues

Exercise.

$$\mathcal{A}_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix} ,$$

Show without computation that the eigenvalues have the following asymptotics as $\epsilon \to 0$

$$\mathcal{L}_\varepsilon^1 \sim \varepsilon^{-1/3}, \mathcal{L}_\varepsilon^2 \sim j\varepsilon^{-1/3}, \mathcal{L}_\varepsilon^3 \sim j^2\varepsilon^{-1/3}.$$

Assume that the entries of $\mathcal{A}_\varepsilon$ have Puiseux series expansions in ϵ, or even that $\mathcal{A}_\varepsilon = a + \epsilon b$, $a, b \in \mathbb{C}^{n\times n}$.

$\mathcal{L}_1, \dots, \mathcal{L}_n$ eigenvalues of $\mathcal{A}_\varepsilon$.

$v(s)$: opposite of the smallest exponent of a Puiseux series s.

$\gamma_1 \geq \cdots \geq \gamma_n$: tropical eigenvalues of $v(A_\epsilon)$.

Theorem (Akian, Bapat, SG CRAS04, arXiv:0402090)

$$v(\mathcal{L}_1) + \cdots + v(\mathcal{L}_n) \leq \gamma_1 + \cdots + \gamma_n$$

and equality holds under generic (Lidski-type) conditions.

The maximal tropical eigenvalue γ_1 coincides with the ergodic constant of the one-player game

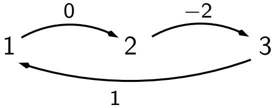
$$\lambda + u_i = \max_{1 \leq j \leq n} \big(\mathrm{val}(A_\epsilon)_{ij} + u_j \big), \forall i$$

λ is the maximal circuit mean.

In general, tropical eigenvalues are non-differentiability points of a parametric optimal assignment problem = Legendre transform a the generic Newton polygon

$$\mathcal{A}_\varepsilon = \begin{bmatrix} \varepsilon & 1 & \varepsilon^4 \\ 0 & \varepsilon & \varepsilon^{-2} \\ \varepsilon & \varepsilon^2 & 0 \end{bmatrix} , \qquad A = \begin{bmatrix} 1 & 0 & 4 \\ \infty & 1 & -2 \\ 1 & 2 & \infty \end{bmatrix} .$$

We have $\gamma_1 = -1/3$, corresponding to the critical circuit:



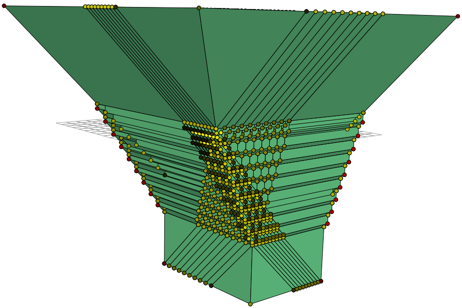
Eigenvalues:

$$\mathcal{L}^1_\varepsilon \sim \varepsilon^{-1/3}, \mathcal{L}^2_\varepsilon \sim j\varepsilon^{-1/3}, \mathcal{L}^3_\varepsilon \sim j^2\varepsilon^{-1/3}.$$

More algorithmic issues . . .

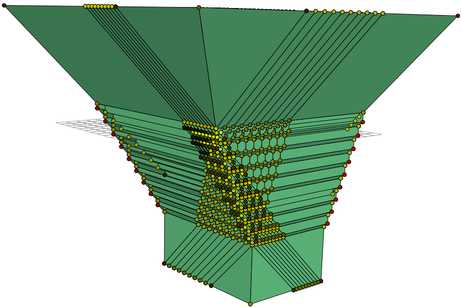
Tropical double description, Allamigeon,SG, Goubault, STACS 10

Can compute efficiently all the extreme generators of $P := \{x \mid Ax \leq Bx\}$, where $A, B \in \mathbb{R}_{\max}^{p\times d}$ (analogue of Fukuda/Motzkin).



Tropical double description, Allamigeon,SG, Goubault, STACS 10

For $d = 4$ and $p = 10$, only 24 vertices, but 1215 pseudo-vertices



(Coarse) worst case bound of double description $O(p^2 d\alpha(d)(p+d)^{d-1})$ where α is the inverse of the Ackermann function.

Better experimental behavior. Implementation in TPLib/caml (Allamigeon).

Tropical polyhedra have fewer extreme points than in the classical case (McMullen bound is not tight, Allamigeon, SG, Katz, JCTA 11, exact bound for the number of extreme points of "$Ax \leq Bx$": open).

Bubble sort

```
Variables: i, j, k, x, y, z
Program:
local t {
i:=x;
j:=y;
k:=z;
if x > y then
  i:=y;
  j:=x;
fi;
if j > z then
  k:=j;
  j:=z;
fi;
if i > j then
  t:=j;
  j:=i;
  i:=t;
fi;
};
```

Can prove automatically that $\mathtt{k} = \mathtt{max}(\mathtt{x}, \mathtt{y}, \mathtt{z})$?

and even that...

$$-y = \max(-k,-y);\ \max(-k,-z) = -z;$$
$$\max(-j,-x,-z) = \max(-x,-z);$$
$$-j = \max(-j,-k);\ \max(-y,-z) = \max(-j,-y,-z);$$
$$\max(j,y,z) = \max(y,z);$$
$$z = \max(i,z);\ -x = \max(-k,-x);$$
$$\max(-x,-y) = \max(-j,-x,-y);$$
$$-i = \max(-i,-x);$$
$$\max(-x,-y,-z) = \max(-i,-k);\ x = \max(i,x);$$
$$\max(j,x,z) = \max(x,z);$$
$$\max(i,y) = y;\ \max(j,x,y) = \max(x,y);$$
$$j = \max(i,j);\ k = \max(x,y,z)$$

Allamigeon, SG, Goubault, SAS'08

Such invariants can be found by abstract interpretation.

Equivalent to solving a game (monotone fixed point problem)

An application of infinite dimensional tropical convexity

Lagrange problem / Lax-Oleinik semigroup

$$v(t,x) = \sup_{\mathbf{x}(0)=x,\ \mathbf{x}(\cdot)} \int_0^t L(\mathbf{x}(s),\dot{\mathbf{x}}(s))ds + \phi(\mathbf{x}(t))$$

Lax-Oleinik semigroup: $(S^t)_{t\geq 0}$, $S^t\phi := v(t,\cdot)$.

Superposition principle: $\forall \lambda \in \mathbb{R}$, $\forall \phi, \psi$,

$$\begin{aligned} S^t(\sup(\phi,\psi)) &= \sup(S^t\phi, S^t\psi) \\ S^t(\lambda+\phi) &= \lambda + S^t\phi \end{aligned}$$

So S^t is max-plus linear.

Lagrange problem / Lax-Oleinik semigroup

$$v(t,x) = \sup_{\mathbf{x}(0)=x,\ \mathbf{x}(\cdot)} \int_0^t L(\mathbf{x}(s), \dot{\mathbf{x}}(s))ds + \phi(\mathbf{x}(t))$$

Lax-Oleinik semigroup: $(S^t)_{t\geq 0}$, $S^t\phi := v(t,\cdot)$.

Superposition principle: $\forall \lambda \in \mathbb{R}$, $\forall \phi, \psi$,

$$\begin{aligned} S^t(\text{“}\phi + \psi) &= \text{“}S^t\phi + S^t\psi \\ S^t(\text{“}\lambda\phi) &= \text{“}\lambda S^t\phi \end{aligned}$$

So S^t is max-plus linear.

The function v is solution of the Hamilton-Jacobi equation

$$\frac{\partial v}{\partial t} = H(x, \frac{\partial v}{\partial x}) \qquad v(0,\cdot) = \phi$$

Max-plus linearity $\Leftrightarrow$ Hamiltonian convex in p

$$H(x,p) = \sup_u (L(x,u) + p \cdot u)$$

Hopf formula, when $L = L(u)$ concave:

$$v(t,x) = \sup_{y \in \mathbb{R}^n} tL(\frac{x-y}{t}) + \phi(y) \ .$$

The function v is solution of the Hamilton-Jacobi equation

$$\frac{\partial v}{\partial t} = H(x, \frac{\partial v}{\partial x}) \qquad v(0,\cdot) = \phi$$

Max-plus linearity $\Leftrightarrow$ Hamiltonian convex in p

$$H(x,p) = \sup_u (L(x,u) + p\cdot u)$$

Hopf formula, when $L = L(u)$ concave:

$$v(t,x) = \text{``}\int G(x-y)\phi(y)dy\ .$$

Max-plus basis / finite-element method

Fleming, McEneaney 00-; Akian, Lakhoua, SG 04-

Approximate the value function by a "linear comb. of "basis" functions with coeffs. $\lambda_i(t) \in \mathbb{R}$:

$$v(t,\cdot) \simeq \text{``}\sum_{i\in[p]} \lambda_i(t) w_i$$

The w_i are given finite elements, to be chosen depending on the regularity of $v(t,\cdot)$

Max-plus basis / finite-element method

Fleming, McEneaney 00-; Akian, Lakhoua, SG 04-

Approximate the value function by a "linear comb. of "basis" functions with coeffs. $\lambda_i(t) \in \mathbb{R}$:

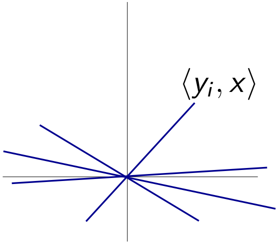
$$v(t,\cdot) \simeq \sup_{i\in[p]} \lambda_i(t) + w_i$$

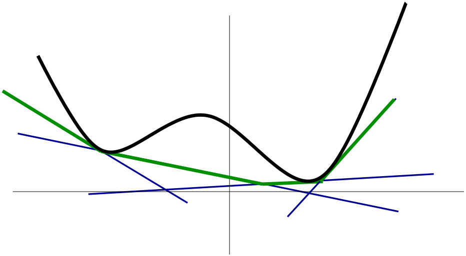
The w_i are given finite elements, to be chosen depending on the regularity of $v(t,\cdot)$

Best max-plus approximation

$$P(f) := \max\{g \leq f \mid g \text{ “linear comb. of } w_i\}$$

linear forms $w_i : x \mapsto \langle y_i, x \rangle$



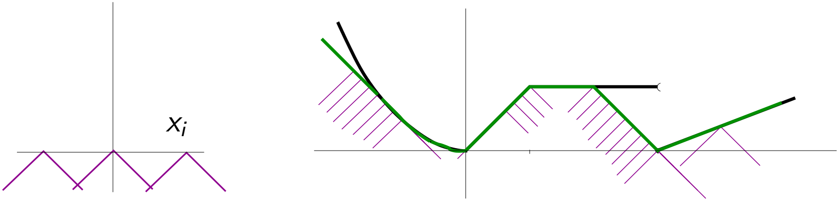


adapted if v is convex

Best max-plus approximation

$$P(f) := \max\{g \le f \mid g \text{ “linear comb. of } w_i\}$$

cone like functions $w_i : x \mapsto -C\|x - x_i\|$



adapted if v is C-Lip

Use max-plus linearity of S^h:

$$v^t = \text{``}\sum_{i \in [p]} \lambda_i(t) w_i$$

and look for new coefficients $\lambda_i(t+h)$ such that

$$v^{t+h} \simeq \text{``}\sum_{i \in [p]} \lambda_i(t+h) w_i$$

Use max-plus linearity of S^h:

$$v^{t+h} = S^h v^t \simeq \text{``}\sum_{i\in[p]} \lambda_i(t) S^h w_i$$

and look for new coefficients $\lambda_i(t+h)$ such that

$$v^{t+h} \simeq \text{``}\sum_{i\in[p]} \lambda_i(t+h) w_i$$

Use max-plus linearity of S^h:

$$v^{t+h} = S^h v^t \simeq \sup_{i \in [p]} \lambda_i(t) + S^h w_i$$

and look for new coefficients $\lambda_i(t+h)$ such that

$$v^{t+h} \simeq \sup_{i \in [p]} \lambda_i(t+h) + w_i$$

Max-plus variational approach

Max-plus scalar product

$$``\langle w, z\rangle := ``\int w(x)z(x)dx$$

For all test functions z_j, $j \in [q]$

$$``\langle v^{t+h}, z_j\rangle = ``\sum_{i\in[p]} \lambda_i(t+h)``\langle w_i, z_j\rangle$$
$$= ``\sum_{k\in[p]} \lambda_k(t)\langle S^h w_k, z_j\rangle$$

Max-plus variational approach

Max-plus scalar product

$$“\langle w, z\rangle := \sup_x w(x) + z(x)$$

For all test functions z_j, $j \in [q]$

$$\sup_{i\in[p]} \lambda_i(t+h) + “\langle w_i, z_j\rangle$$
$$= \sup_{k\in[p]} \lambda_k(t) + “\langle S^h w_k, z_j\rangle$$

Theorem (Akian, SG, Lakhoua, SICON 04)

The approximation error of the max-plus finite element method satisfies

$$\|v_h^t - v^t\|_\infty \leq C(t) \sup_{0\leq s\leq t} \|v^s - P(v^s)\|$$

Results of the same nature (but no so simple) for other versions of the method (Fleming, McEneaney; McEneaney, Kluberg)

McEneaney's curse of dimensionality reduction

Suppose the Hamiltonian is a finite max of Hamiltonians arising from LQ problems

$$H = \sup_{i\in[r]} H_i,\ \ H_i = -(\frac{1}{2}x^*D_ix + x^*A_i^*p + \frac{1}{2}p^*\Sigma_i p)$$

(=LQ with switching). Let S^t and S_i^t denote the corresponding Lax-Oleinik semigroups, S_i^t is exactly known (Riccati!)

Want to solve $v = S^t v, \forall t \geq 0$

Choose a quadratic function ϕ such that $S^t\phi \to v$ as $t \to \infty$. Then, for $t = hk$ large enough,

$$v \simeq (S^h)^k \phi \ .$$

This is a sup of quadratic forms. Inessential terms are trimmed dynamically using Shor relaxation (SDP) $\to$ solution of a typical instance in dim 6 on a single processor

McEneaney, Desphande, SG; ACC 08; SG, McEneaney, Qu CDC 11

Choose a quadratic function ϕ such that $S^t\phi \to v$ as $t \to \infty$. Then, for $t = hk$ large enough,

$$v \simeq \text{“}(\sum_{i\in[r]} S_i^h)^k\phi \ .$$

This is a sup of quadratic forms. Inessential terms are trimmed dynamically using Shor relaxation (SDP) $\to$ solution of a typical instance in dim 6 on a single processor

McEneaney, Desphande, SG; ACC 08; SG, McEneaney, Qu CDC 11

Choose a quadratic function ϕ such that $S^t\phi \to v$ as $t \to \infty$. Then, for $t = hk$ large enough,

$$v \simeq \text{“} \sum_{i_1,\cdots,i_k \in [r]} S^h_{i_1} \cdots S^h_{i_k} \phi \ .$$

This is a sup of quadratic forms. Inessential terms are trimmed dynamically using Shor relaxation (SDP) $\to$ solution of a typical instance in dim 6 on a single processor

McEneaney, Desphande, SG; ACC 08; SG, McEneaney, Qu CDC 11

Choose a quadratic function ϕ such that $S^t\phi \to v$ as $t \to \infty$. Then, for $t = hk$ large enough,

$$v \simeq \sup_{i_1,\cdots,i_k \in [r]} S^h_{i_1} \cdots S^h_{i_k} \phi \ .$$

This is a sup of quadratic forms. Inessential terms are trimmed dynamically using Shor relaxation (SDP) $\to$ solution of a typical instance in dim 6 on a single processor

McEneaney, Desphande, SG; ACC 08; SG, McEneaney, Qu CDC 11

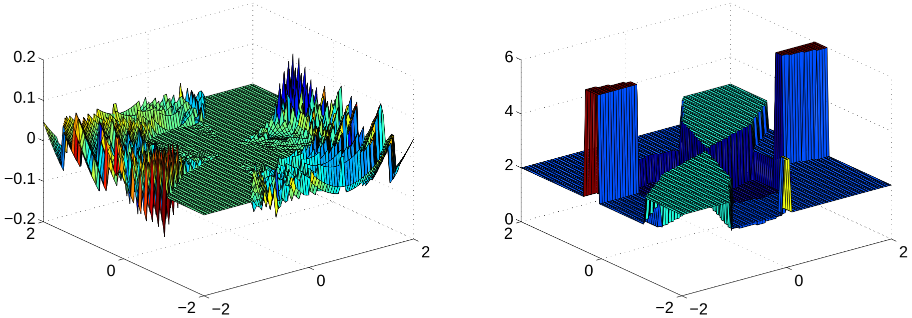


Figure: Backsubstitution error and optimal policy on the x_1,x_2 plane, $h = 0.1$ SG, McEneaney, Qu 11

Error estimates: in terms of projection errors Akian, Lakhoua, SG 04-, curse of dim free estimates (still exp. blowup) Kluberg, McEneaney 09

SG, McEneaney, Qu CDC 11: cant approximate a $\mathcal{C}^2$ strictly convex function by N affine max-plus finite elements in dimension d with an approximation error better than

$$\text{cst} \times \frac{1}{N^{2/d}}$$

Corollary of techniques/results of Grüber on approximation of convex bodies.
Curse of dim is unavoidable, but certified rough approximations is possible.

In program verification, the template method of Manna, Sankaranarayanan, Sipma is a level-set version of max-plus basis methods (in such applications $10^2, 10^3$ typically)

Dessert: from games to metric geometry (generalizations of Denjoy-Wolff)

Beyond games, still with a tropical flavor: nonexpansive mappings

(X, d) metric space, $T : X \to X$,

$$d(T(x), T(y)) \leq d(x, y) \quad .$$

Beyond games, still with a tropical flavor: nonexpansive mappings

(X, d) metric space, $T : X \to X$,

$$d(T(x), T(y)) \leq d(x, y) \ .$$

Define the escape rate

$$\rho(T) := \lim_{k \to \infty} \frac{d(x, T^k(x))}{k}$$

(independent of $x \in X$ by nonexpansiveness, existence by subadditivity).

Theorem (Kohlberg & Neyman, Isr. J. Math., 81)

Assume $\rho(T) > 0$. Then, there exists a linear form $\varphi \in X^*$ of norm one such that for all $x \in X$,

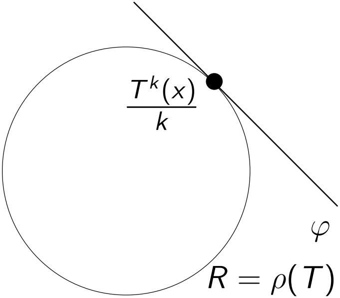
$$\rho(T) = \lim_{k\to\infty} \varphi\big(T^k(x)/k\big) = \inf_{y\in X} \|T(y) - y\|$$

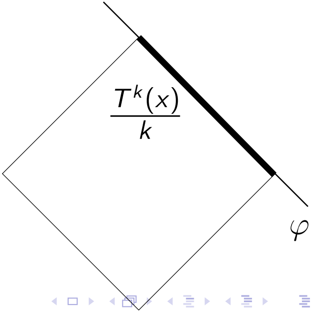
Corollary (Kohlberg & Neyman, Isr. J. Math., 81, extending Reich 73 and Pazy 71)

The limit

$$\lim_{k\to\infty} \frac{T^k(x)}{k}$$

exists in the weak (resp. strong) topology if X is reflexive and strictly convex (resp. if the norm of the dual space $X^\star$ is Frechet differentiable).





Compare Collatz-Wielandt

$$
\begin{aligned}
\rho(F) &= \max\{\mu \in \mathbb{R}_+ \mid F(v) \geq \mu v, v \in \mathbb{R}_+^n, v \neq 0\} \\
&= \inf\{\mu > 0 \mid F(w) \leq \mu w, w \in \operatorname{int}\mathbb{R}_+^n\} \\
&= \lim_{k\to\infty} \|F^k(x)\|^{1/k}, \qquad \forall x \in \operatorname{int}\mathbb{R}_+^n
\end{aligned}
$$

and so

$$
\inf_{w\in \operatorname{int}\mathbb{R}_+^n} \max_{1\leq i\leq n} \frac{(F(w))_i}{w_i} = \rho(F) = \max_{\substack{v\in\mathbb{R}_+^n \\ v\neq 0}} \min_{\substack{1\leq i\leq n \\ v_i\neq 0}} \frac{(F(v))_i}{v_i} \quad .
$$

with Kohlberg and Neyman

$$
\rho(T) := \lim_{k\to\infty} \left\| \frac{T^k(x)}{k} \right\| = \inf_{y\in X} \|T(y) - y\| = \lim_{k\to\infty} \varphi\big(T^k(x)/k\big) \quad .
$$

Is there an explanation of this analogy ?

Collatz-Wielandt and Kohlberg-Neyman are special cases of a general result.

Theorem (SG and Vigeral, Math. Proc. Phil. Soc. 11)

Let T be a nonexpansive self-map of a complete hemi-metric space (X, d) of non-positive curvature in the sense of Busemann. Let

$$\rho(T) := \lim_{k\to\infty} \frac{d(x, T^k(x))}{k}$$

Then, there exists a Martin function h such that

$$h(T(x)) \geq \rho(T) + h(x), \qquad \forall x$$

Moreover,

$$\rho(T) = \inf_{y\in X} d(y, T(y)) \ .$$

If in addition X is a metric space and $\rho(T) > 0$, then h is an horofunction.

Let us explain the different notions appearing in this theorem . . .

Hemi-metric

δ is an hemi-metric on X if

- $\delta(x,z) \leq \delta(x,y) + \delta(y,z)$
- $\delta(x,y) = \delta(y,x) = 0$ if and only if $x = y$.

Variant: weak metric of Papadoupoulos, Troyanov.
(X,δ) is complete if X is complete for the metric
$d(x,y) := \max(\delta(x,y), \delta(y,x))$.

The (reverse) Funk (hemi-)metric on a cone

C closed pointed cone, $X = \operatorname{int} C \neq \varnothing$,

$$\delta(x,y) = \mathrm{RFunk}(x,y) := \log \inf\{\lambda > 0 | \lambda x \geq y\}$$

Lemma

$F : C \to C$ is order preserving and homogeneous of degree 1 iff

$$\mathrm{RFunk}(F(x),F(y)) \leq \mathrm{RFunk}(x,y), \qquad \forall x,y \in \operatorname{int} C \enspace .$$

[simple but useful: Gunawardena, Keane, Sparrow, Lemmens, Scheutzow, Walsh.]

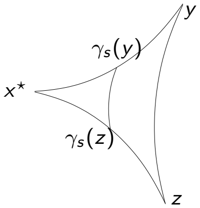
$$\mathrm{RFunk}(x,y) = \log \max_{\varphi \in C^* \setminus \{0\}} \frac{\varphi(y)}{\varphi(x)} = \log \max_{\varphi \in \operatorname{Extr} C^*} \frac{\varphi(y)}{\varphi(x)}$$

$$= \log \max_{1 \leq i \leq n} \frac{y_i}{x_i} \qquad \text{if} \quad C = \mathbb{R}^n_+ \enspace ,$$

Busemann convexity / nonpositive curvature condition

We say that (X, δ) is metrically star-shaped with center $x^\star$ if there exists a family of geodesics $\{\gamma_y\}_{y\in X}$, such that γ_y joins the center $x^\star$ to the point y, and

$$\delta\left(\gamma_y(s), \gamma_z(s)\right) \leq s\delta(y, z), \qquad \forall (y, z) \in X^2, \qquad \forall s \in [0, 1] \ .$$



The horoboundary of a metric space

Defined by Gromov (81), see also Rieffel (Doc. Math. 02).
Fix a basepoint $\bar{x} \in X$.
$i : X \to \mathscr{C}(X)$,

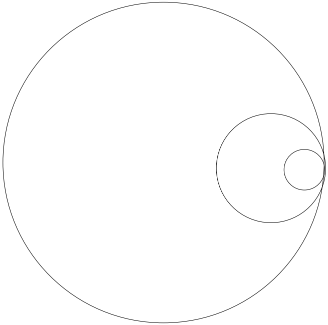
$$i(x) : y \to [i(x)](y) := \delta(\bar{x}, x) - \delta(y, x).$$

so that

$$i(x)(\bar{x}) = 0, \quad \forall x \in X$$

Martin space: $\mathcal{M} := \overline{i(X)}$ (eg: product topology)
Boundary: $\mathcal{H} := \mathcal{M} \setminus i(X)$. An element of $\mathcal{H}$ is an horofunction.
A Busemann point is the limit $\lim_t i(x_t)$, where $(x_t)_{t \geq 0}$ is an infinite (almost) geodesic.
Busemann points $\subseteq$ boundary points, with equality for a polyhedral norm.

In the Poincare disk model, the level lines of horofunctions are horocircles



The Wolff-Denjoy theorem (1926) says that the orbits of a fixed point free analytic function leaving invariant the open disk converge to a boundary point (and that horodisks are invariants).

Theorem (SG and Vigeral)

Let T be a nonexpansive self-map of a complete hemi-metric space (X, d) of non-positive curvature in the sense of Busemann. Then, there exists a Martin function h such that

$$h(T(x)) \geq \rho(T) + h(x), \qquad \forall x$$

If in addition X is a metric space and $\rho(T) > 0$, then h is an horofunction.

Kohlberg-Neyman is a direct corollary. Since $h = \lim_\alpha -\| \cdot - x_\alpha\|$ modulo constants, h is concave. Take any $\varphi \in \partial h(x)$. Then,

$$\varphi(T^k(x) - x) \geq h(T^k(x)) - h(x) \geq k\rho(T) \ .$$

Collatz-Wielandt revisited

Let $F : C \to C$, where C is a symmetric cone (self-dual cone with a group of automorphisms acting transitively on it), say $C = \mathbb{R}_+^n$ or $C = S_n^+$.

Recall F is nonexpansive in RFunk iff it is order preserving and homogeneous of degree one.

Walsh (Adv. Geom. 08): the horoboundary of C in the (reverse) Funk metric is the Euclidean boundary: any Martin function h corresponds to some $u \in C \setminus \{0\}$:

$$h(x) = -\operatorname{RFunk}(x, u) + \operatorname{RFunk}(x^*, u) \ , \forall x \in \operatorname{int} C,$$

h is a horofunction iff $u \in \partial C \setminus \{0\}$.

Corollary (Collatz-Wielandt recovered, and more)

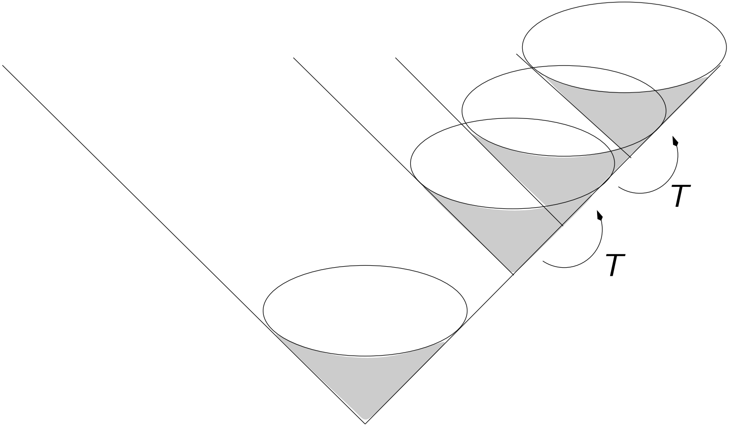
Let $T : \operatorname{int} C \to \operatorname{int} C$*, order-preserving and positively homogeneous,* C *symmetric cone. Then,*

$$\begin{aligned}
\rho(T) &:= \lim_{k\to\infty} \frac{\operatorname{RFunk}(x, T^k(x))}{k}, \qquad \forall x \in \operatorname{int} C \\
&= \inf_{y\in \operatorname{int} C} \operatorname{RFunk}(y, T(y)) \\
&= \log \inf\{\lambda > 0 \mid \exists y \in \operatorname{int} C,\ T(y) \le \lambda y\} \\
&= \max_{u\in C\setminus\{0\}} -\operatorname{RFunk}(T(u), u) \\
&= \log \max\{\mu \ge 0 \mid \exists u \in C \setminus \{0\},\ T(u) \ge \mu u\}
\end{aligned}$$

and there is a generator w *of an extreme ray of* C *such that*

$$\log \left(w, T^k(x)\right) \ge \log \left(w, x\right) + k\rho(T), \qquad \forall k \in \mathbb{N}$$

Refines Gunawardena and Walsh, Kibernetica, 03.


T
T

The special case of games recovered

$$T_i(x) = \max_{a \in A_i} \min_{b \in B_{i,a}} \big(r_i^{ab} + \sum_{1 \leq j \leq n} P_{ij}^{ab} x_j \big), \qquad 1 \leq i \leq n$$

$$\rho(T) = \overline{\chi}(T) = \lim_{k \to \infty} \max_{1 \leq j \leq n} \frac{(T^k(x))_j}{k}$$

Corollary (SG, Gunawardena, TAMS 04 recovered)

For all $x \in \mathbb{R}^n$, there exists $1 \leq i \leq n$ such that

$$\big(T^k(x)\big)_i \geq x_i + k\rho(T), \qquad \forall k \in \mathbb{N} \ .$$

Initial state i guarantees the best reward per time unit.

Conclusion

- Nonexpansive maps/Perron-Frobenius techniques: tools to prove combinatorial results.
- Symmetric cones have a tropical flavor (log glasses, nonpositive curvature)
- Order preserving homogeneous maps should be thought of as nonexpansive maps in $\mathrm{RFunk}(x, y) := \log \inf\{\lambda > 0 | \lambda x \geq y\}$.
- this leads to Denjoy-Wolff type results (nested invariant horoballs)
- Collatz-Wielandt and Kohlberg-Neyman recovered as special cases.
- Generalization of Edmonds's good characterizations (NP $\cap$ coNP membership of mean payoff games is a special case).
- Current work SG+Zheng Qu: application to various Riccati-type equations.