Game Dynamics: Discrete versus continuous time

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$\dot{x} = f(x)$ differential equation in $\mathbb{R}^k$

$x \mapsto T_h(x) = x + hf(x)$ discretization map with step size h

if h is small, the dynamics should be similar

general results:

1) **linearized dynamics near an equilibrium/fixed point**

$\dot{x} = Jx \qquad T_h(x) = (I + hJ)x$

a) if J is a stable matrix: $\text{Re}\lambda < 0 \quad \forall \lambda$
then $I + hJ$ is contracting: $|1 + h\lambda| < 1$ for small $h > 0$

b) if J has an eigenvalue λ with $\text{Re}\lambda > 0$
then $|1 + h\lambda| > 1$ for all $h > 0$

For hyperbolic equilibria, small h: same local behaviour

b) applies to $\lambda \neq 0, \text{Re}\lambda = 0$

2) **Attractors are USC under discretization**

Let A be an **attractor** (= asymptotically stable invariant set) of the differential equation. Then for small h, orbits of T_h, i.e., iteration sequences $x, T_h(x), T_h^2(x), \ldots$, converge to a neighborhood of A, for x close to A

3) **The chain recurrent set is USC under discretization**

For small h, **all** orbits of T_h converge to a neighborhood of the set of chain recurrent points of the differential equation works more generally for differential inclusions

$$\dot{x} \in F(x)$$

$F : \mathbb{R}^k \rightrightarrows \mathbb{R}^k$ u.s.c., with compact convex values

$x^\varepsilon_{n+1} - x^\varepsilon_n \in \varepsilon F^{\delta(\varepsilon)}(x^\varepsilon_n)$, $\quad \varepsilon > 0$ small step size
Graph(F^δ) $\subset N^\delta$(Graph(F))
$\delta : (0, +\infty) \to [0, +\infty)$: $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

M. Benaim, JH, S. Sorin, Dynamic Games and Applications, to appear

applications to game dynamics

replicator dynamics

Nash map

BR dynamics

Evolutionary Games

a large population of players

pure strategies: $S = \{1, \ldots, n\}$

mixed strategies: $x \in \Delta(S)$: $x_i \geq 0, \sum_{i \in S} x_i = 1$

payoff to i: $a_i(x)$, $a_i : \Delta \to \mathbb{R}$ continuous (population game)

(Symmetric) 2 Person Game: a_{ij}, $a_i(x) = \sum_j a_{ij} x_j = (Ax)_i$

payoff to mixed strategy $y \in \Delta$: $y \cdot Ax$

$\hat{x} \in \Delta(S)$ is a (symmetric) NE iff $\hat{x} \cdot A\hat{x} \geq x \cdot A\hat{x} \quad \forall x \in \Delta(S)$

Replicator dynamics

$$x_i' = x_i \frac{C + (Ax)_i}{C + x \cdot Ax}, \quad i = 1, \ldots, n \qquad \text{(RM)}$$

as a difference equation: $x_i' - x_i = \frac{(Ax)_i - xAx}{C + xAx}$
$x = x(t), x' = x(t+h), h = 1/C$, $C \to \infty$: differential equation

$$\dot{x}_i = x_i((Ax)_i - x \cdot Ax) \qquad \text{(REP)}$$

(RM) is (for large C) essentially an Euler discretization of (REP)

players replicate, offspring inherits strategy
payoff $\doteq$ fitness $\doteq$ number of offspring

Special case $a_{ij} = a_{ji}$ (potential game)

population genetics

$$x_i' = x_i \frac{(Ax)_i}{x \cdot Ax} \quad (i = 1, \ldots, n) \quad x' = F(x), \quad F : \Delta \to \Delta$$

selection map on simplex $\Delta = \Delta_n = \{x \in \mathbb{R}^n_+ : \sum x_i = 1\}$

x_i frequency of gene (allele) A_i (in gene pool)

$x_i x_j$ frequency of genotype $A_i A_j$ (random mating)

$a_{ij} = a_{ji} \geq 0$ fitness (survival probability) of genotype $A_i A_j$

$a_{ij} x_i x_j$ adults with genotype $A_i A_j$

$x_i' \sim \sum_j a_{ij} x_i x_j$ frequency of gene A_i in next generation

$n = 2$ **Fisher, Haldane, Wright** 1930s

Fundamental Theorem of Natural Selection

Mulholland–Smith 1959, Atkinson–Watterson–Moran 1960, Kingman 1961

Mean fitness $x \cdot Ax = \sum_{ij} a_{ij} x_i x_j$ increases along orbits: $x' \cdot Ax' \geq x \cdot Ax$ with equality only if $x = x'$ (at fixed points)

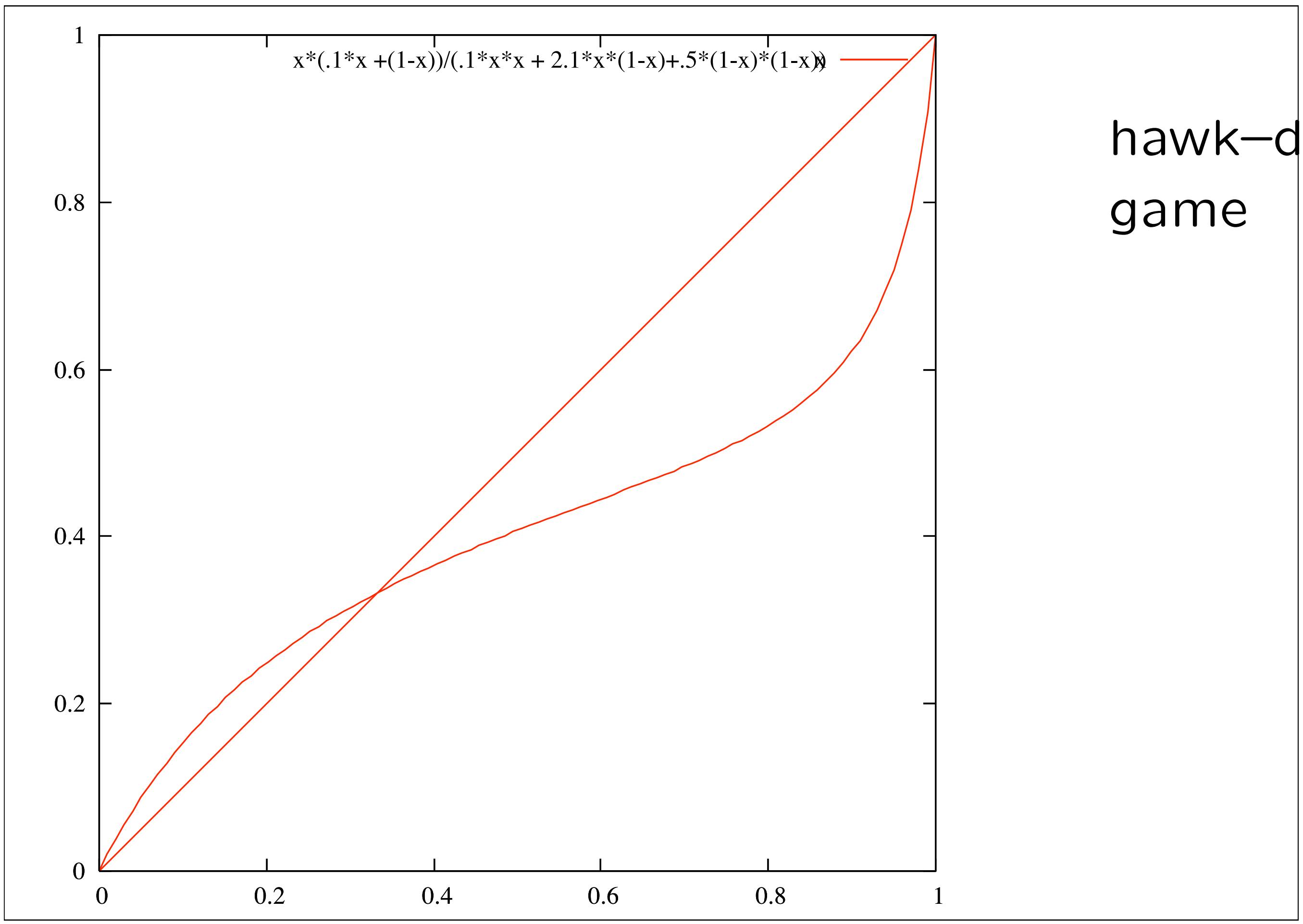
Hence: ω–limits are connected sets of fixed points, of constant mean fitness.

Convergence Theorem (Lyubich et al, Aulbach, Losert & Akin 1983): Each orbit of the selection map converges to a fixed point.

Qu: Does this follow from Lojasiewicz technique?

(REP) is gradient system w.r.t. a certain Riemannian metric on int Δ

$$x' = F(x), \quad x_i' = x_i \frac{(Ax)_i}{xAx}, \quad i = 1, \ldots, n \qquad \text{(RM)}$$



hawk–dove game

$n = 2$: The replicator map $F : [0, 1] \to [0, 1]$ is strictly increasing.
$\Longrightarrow$ convergence to fixed points $(0, 1, \widehat{x})$

$$x_i' = x_i \frac{(Ax)_i}{x \cdot Ax}, \quad i = 1, \dots, n \qquad \text{(RM)}$$

general n: If $a_{ij} > 0 \quad \forall i, j$ then $F : \Delta \to \Delta$ is a diffeomorphism (Losert & Akin, JMB 1983)

however, for $n \geq 3$, (RM) is more complicated than (REP)

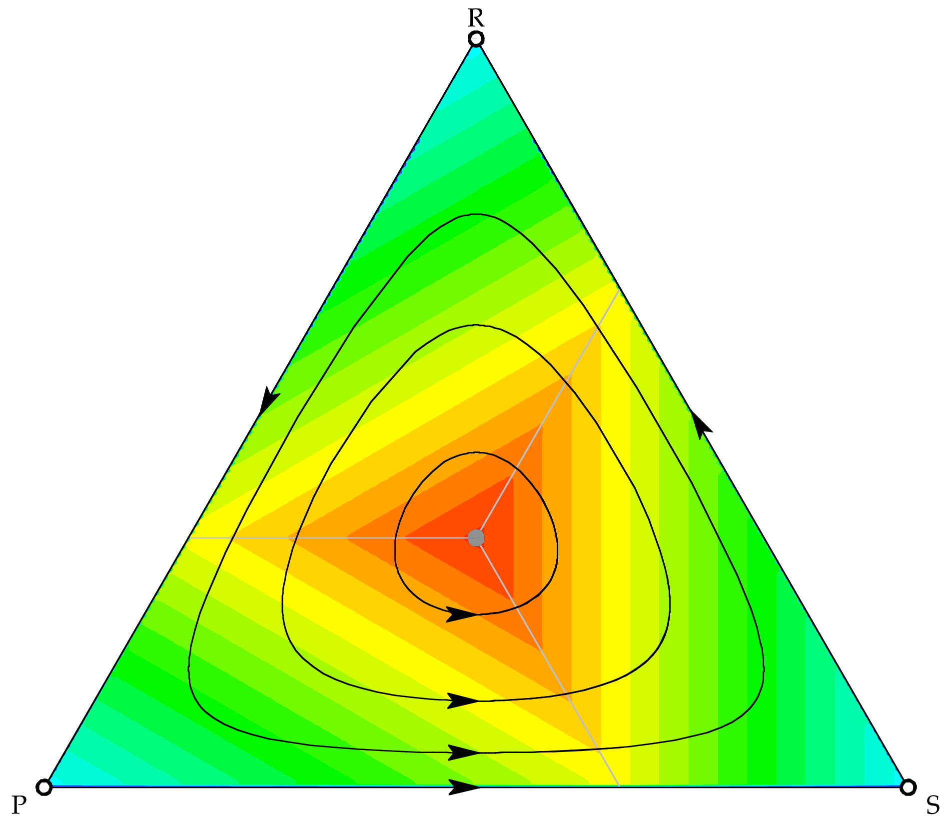
Example: The Rock–Scissors–Paper game

$$A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} \qquad (c > a > b \geq 0\ )$$

unique NE: $E = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

$V(x) = x_1 x_2 x_3$
$V(x) \geq 0$, V is maximal at $E = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

I) If $2a = b + c$ then $\dot{V}(x) = 0 \quad \forall x \in \Delta$: closed orbits
II) If $2a < b + c$ then $\dot{V}(x) \geq 0 \quad \forall x \in \Delta$, E is global attractor.
III) If $2a > b + c$ then $\dot{V}(x) \leq 0 \quad \forall x \in \Delta$. E is repeller,
$\omega(p) = \partial\Delta$ for all $p \neq E$.



$x_1 x_2 x_3 = \text{const.}$

The RSP game: discrete time

$V(x) = \frac{x_1 x_2 x_3}{xAx}$ (JH 1984)
$V(x) \geq 0$, V is maximal at $E = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

I) If $a^2 = bc$ then $V(x') = V(x) \quad \forall x \in \Delta$.
II) If $a^2 < bc$ then $V(x') \geq V(x) \quad \forall x \in \Delta$, E is global attractor.
III) If $a^2 > bc$ then $V(x') \leq V(x) \quad \forall x \in \Delta$. E is repeller.
$\omega(p) \subseteq \partial\Delta$ for all $p \neq E$.

In case (I): invariant closed curves,
dynamics is conjugate to rotation
Case (III): Qu: $\omega(p) = \partial\Delta$?

Stein-Ulam spiral map (1955/60/64): $a = 1, b = 2, c = 0$

$x \cdot Ax = (x_1 + x_2 + x_3)^2 = 1$

Menzel–Stein–Ulam (1955): quadratic maps $\Delta \to \Delta$

BRUCE KITCHENS AND MICHAŁ MISIUREWICZ

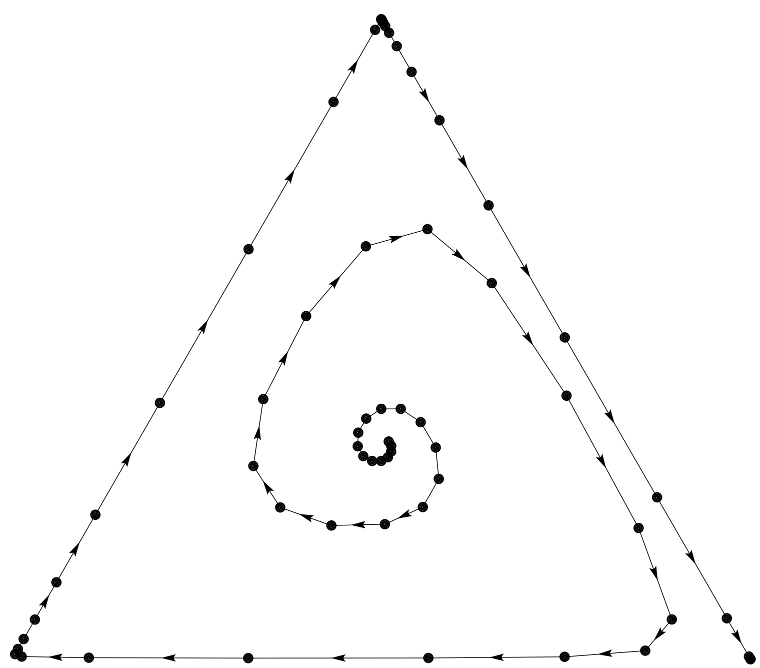


FIGURE 1. A piece of a trajectory of the Stein-Ulam Spiral map.

$$x_1' = x_1(x_1 + 2x_2)$$
$$x_2' = x_2(x_2 + 2x_3)$$
$$x_3' = x_3(x_3 + 2x_1)$$

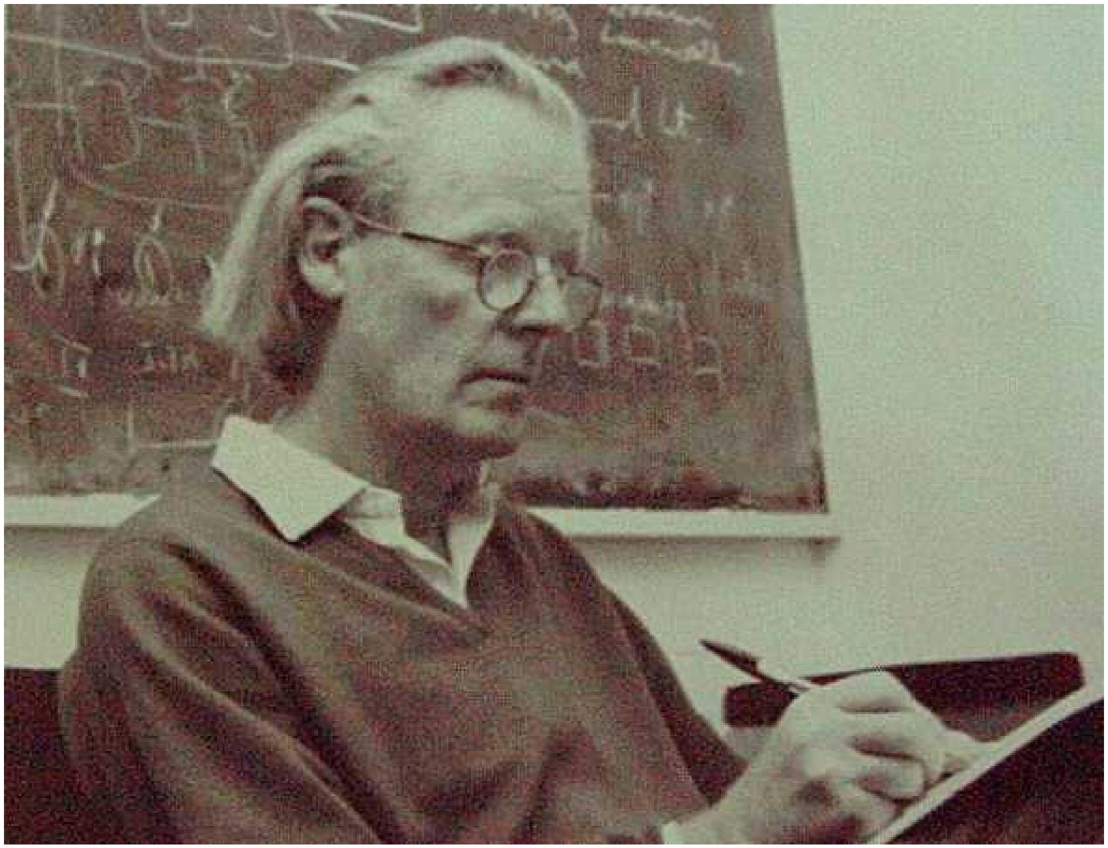
all orbits go to $\partial\Delta$

Vallander (1972): what is the limit set?

Barański & Misiurewicz (2009):

1) For generic initial conditions $p \in \Delta$ (residual set): $\omega(p) = \partial\Delta$

2) For each closed invariant subset $L \subseteq \partial\Delta$ which intersects all three sides of Δ there is a dense set of points $p \in \Delta$ with $\omega(p) = L$



Evolutionary stability (John Maynard Smith)

$\hat{x}$ is an ESS $\Leftrightarrow$

$$(i) \quad x \cdot A\hat{x} \quad \leq \quad \hat{x} \cdot A\hat{x} \quad \forall x \in \Delta,$$

and if there is equality in (i) then

$$(ii) \quad x \cdot Ax \quad < \quad \hat{x} \cdot Ax \quad \text{for} \quad x \neq \hat{x}$$

$\Leftrightarrow \qquad \hat{x} \cdot Ax > x \cdot Ax \quad \forall x \neq \hat{x}$ close to $\hat{x}$.

For a NE $\hat{x} \in \operatorname{int} \Delta$: ESS $\Leftrightarrow$

$$z \cdot Az < 0 \quad \forall z \neq 0, \sum_i z_i = 0$$

Example: The RSP game

$$A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} \qquad (c > a > b \geq 0\ )\ z \in \mathbb{R}^n_0 : \qquad z_1 + z_2 + z_3 = 0$$

$$z \cdot Az = a(z_1^2 + z_2^2 + z_3^2) + (b + c)(z_1 z_2 + z_2 z_3 + z_1 z_3)$$

$$= (a - \frac{b+c}{2})[z_1^2 + z_2^2 + z_3^2]$$

$2a < b + c$: negative definite, E is ESS

$2a > b + c$: positive definite, E is anti-ESS

Theorem. 1) An ESS is asymptotically stable under (REP), and asymp. stable under (RM) for small h (= large C).

2) In a negative definite game:

$$z \cdot Az < 0 \quad \forall z \neq 0, \sum_i z_i = 0$$

The unique NE is an ESS and is globally asymptotically stable under (REP), and under (RM) for small h (= large C).

Liapunov function: $V(x) = \sum_i \widehat{x}_i \log x_i$

$zAz \leq 0 \quad a(x) = Ax \quad (x-y)(a(x)-a(y)) \leq 0 \quad \forall x, y \in Sx$
payoff function '*monotone*'

Replicator dynamics for bimatrix games

two disjoint player populations, playing a two person game
payoff matrices: $A = (a_{ij})\ n \times m$, $B = (b_{ji})\ m \times n$

$$x_i' = x_i \frac{(Ax)_i}{y \cdot Ax}, \quad y_j' = y_j \frac{(By)_j}{x \cdot By} \qquad \text{(RM)}$$

$$i = 1, \ldots, n \qquad j = 1, \ldots, m$$

$$x_i' = x_i \frac{1 + h(Ax)_i}{1 + hy \cdot Ax}, \quad y_j' = y_j \frac{1 + h(By)_j}{1 + hx \cdot By} \qquad \text{(RM)}_h$$

with rescaled payoffs $h > 0$, $h \to 0$

$$\dot{x}_i = x_i \Big((Ax)_i - y \cdot Ax \Big), \qquad \dot{y}_j = y_j \Big((By)_j - x \cdot By \Big) \qquad \text{(REP)}$$

alternative discrete time version

$$x_i' = x_i + h x_i \Big((Ax)_i - y \cdot Ax \Big), \qquad y_j' = y_j + h y_j \Big((By)_j - x \cdot By \Big) \qquad \text{(RM)}_h'$$

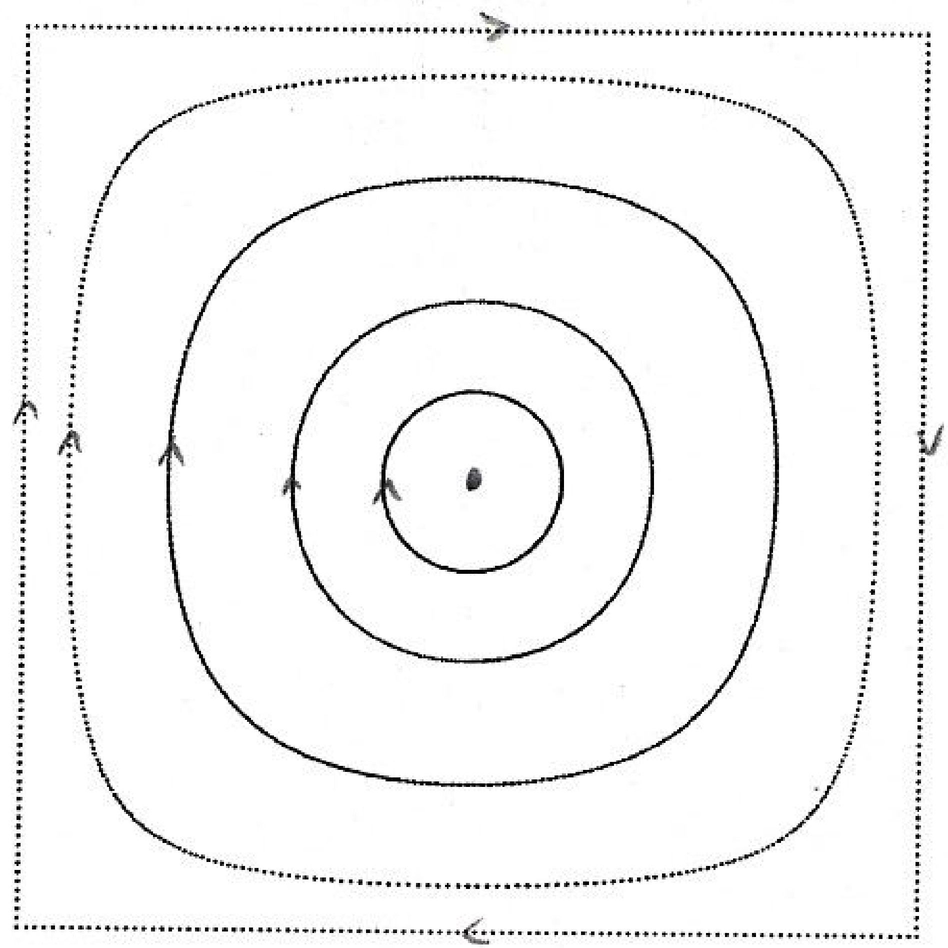
arises from reinforcement learning model (Borgers and Sarin) and imitation model (Schlag, 1998)

$1 - h$ level of inertia
opportunity for switching with probability h between rounds

Constant sum games: $a_{ij} + b_{ji} = 1$

Example: 2×2 cyclic games

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix},\ B = \begin{pmatrix} c & d \\ d & c \end{pmatrix} \qquad (a > b > 0,\ d > c > 0)$$



closed orbits for (REP)

For (RM) (both discrete time versions):
interior equilibrium is repelling:
eigenvalues λ imaginary, hence $|1 + h\lambda| > 1$,
all orbits converge to boundary of $[0, 1]^2$

sophisticated imitation model (Schlag, 1999, Hofbauer & Schlag, 2000):

observe 2 or more agents

(sequential) proportional observation rule

adopt a strategy with payoff p (normalized s.t. $\in (0,1)$) with probability p

$$x_i' = x_i + hx_i\Big((Ax)_i - y{\cdot}Ax\Big)\phi_1(y{\cdot}Ax)$$

$$y_j' = y_j + hy_j\Big((By)_j - x{\cdot}By\Big)\phi_2(x{\cdot}By)$$

ϕ_i decreasing

continuous time limit:

$$\dot{x}_i = x_i\Big((Ax)_i - y\cdot Ax\Big)\phi_1(y\cdot Ax)$$

$$\dot{y}_j = y_j\Big((By)_j - x\cdot By\Big)\phi_2(x\cdot By)$$

E is asymptotically stable for differential equation

eigenvalues at E: $\pm i\omega$

E is repelling for difference equation

Hopf bifurcation through discretization:

$h \to 0$ invariant curve, radius $\sim \sqrt{h}$

The Nash map

Nash's proof of existence of Nash equilibria (Ann. Math. 1951)

Continuous map $f : \Delta \to \Delta$

$$f(x)_i = \frac{x_i + h\hat{a}_i(x)}{1 + h\sum_{j=1}^n \hat{a}_j(x)} \qquad\qquad h > 0$$

with $\hat{a}_i(x) = [(Ax)_i - x \cdot Ax]_+$ excess payoffs
$(u_+ = \max(u, 0))$

Brouwer: $\hat{x} = f(\hat{x})$

$$\Leftrightarrow \quad \hat{a}_i(\hat{x}) = 0 \quad \forall i \quad \Leftrightarrow \quad \hat{x} \in \text{NE}$$

difference equation

$$f(x)_i - x_i = h \frac{\widehat{a}_i(x) - x_i \sum_{j=1}^{n} \widehat{a}_j(x)}{1 + h \sum_{j=1}^{n} \widehat{a}_j(x)}$$

$h \to 0$

$$\dot{x}_i = \widehat{a}_i(x) - x_i \sum_{j=1}^{n} \widehat{a}_j(x) \qquad \text{(BNN)}$$

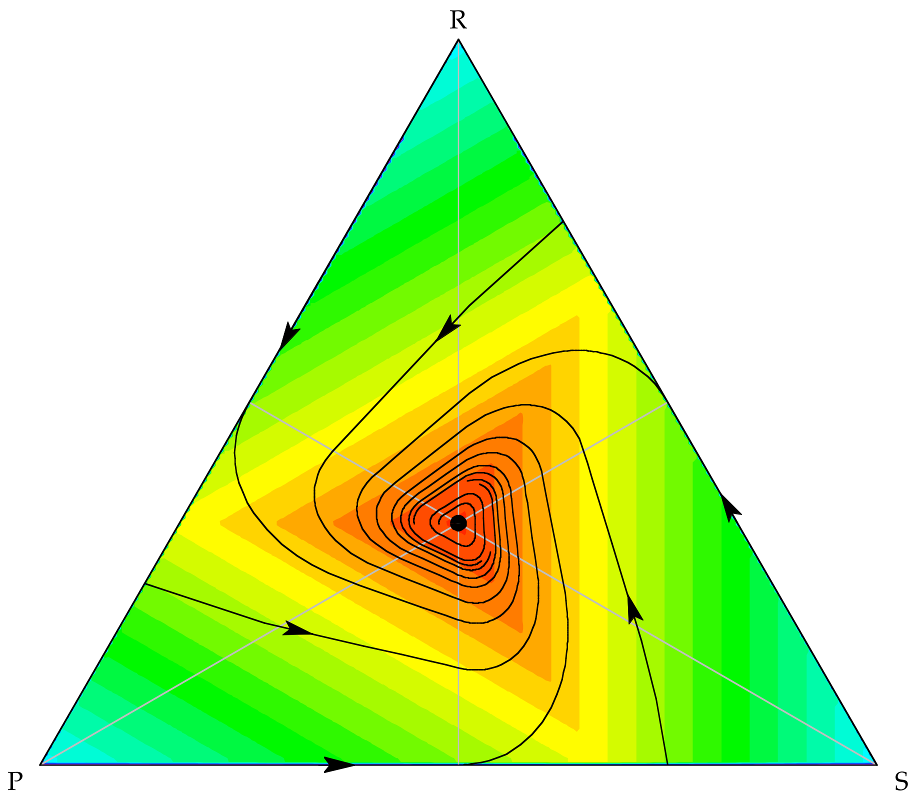
Brown–von Neumann (1950) differential equation:

2 person symmetric zero–sum games

convergence to set of equilibria

players switch to strategies better than average

Nash map, (BNN) are not smooth, but Lipschitz



(BNN)

$2a = b + c$

Stability result:

ESS are asympt. stable, interior ESS are globally asympt. stable for (BNN), and for Nash map for small h.

But not for large h!
hawk–dove game: Nash map can converge to a period 2 orbit for large h.

Cyclic 2×2 games: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Becker et al(JDEA 2007): $h = 2$: convergence to a (semistable) period 8 orbit

Geller, Kitchens, Misiurewicz (DCDS 2010):
for small h: attracting invariant closed curve, radius grows linearly with h, like $3\pi h/16$

supercritical Hopf bifurcation through discretization:
NE is asympt stable for (BNN), with quadratic terms ensuring convergence

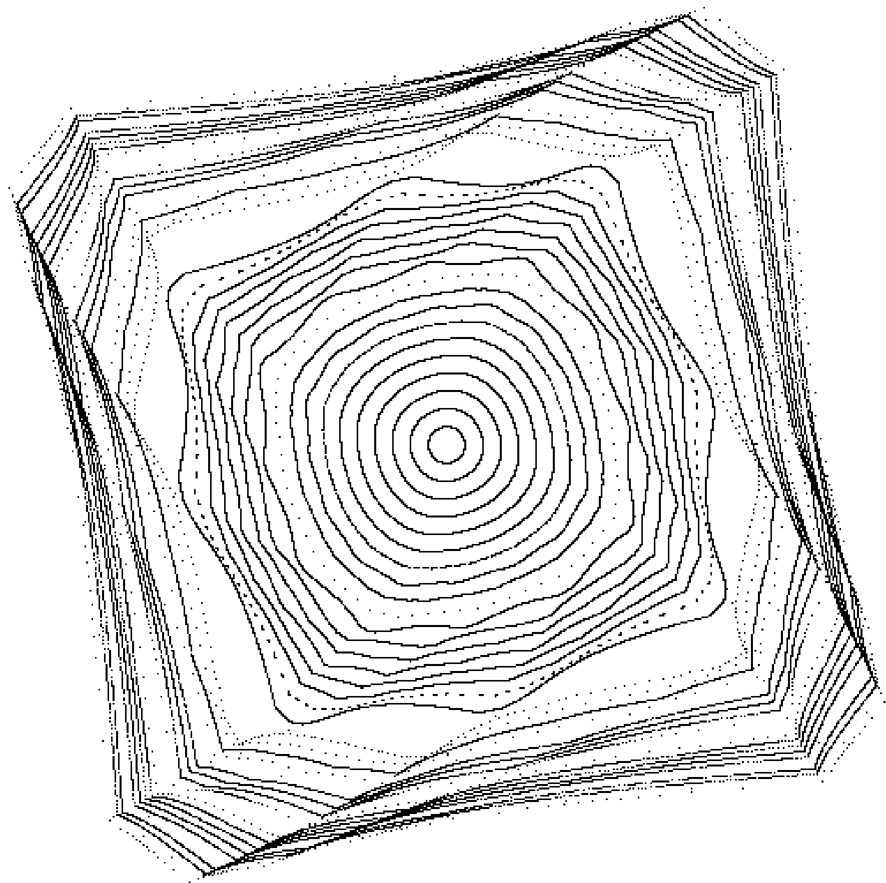


FIGURE 1. Attractors for various values of c; the phase space

Discretization of the BR dynamics

$$\text{BR}(x) = \underset{y \in \Delta}{\text{Argmax}}\; y \cdot a(x) = \{y \in \Delta : y \cdot a(x) \geq z \cdot a(x) \forall z \in \Delta\} \subseteq \Delta$$

A simple discretization of the BR dynamics with constant step size ε is

$$x(t+\varepsilon) \in \varepsilon \text{BR}(x(t)) + (1-\varepsilon)x(t) \tag{1}$$

or

$$x' = T_h(x) \in \frac{1}{1+h}(x + h\text{BR}(x))$$

In each time unit a small proportion of the population switches to a best reponse.

limit $h \to 0$: $\dot{x} \in BR(x) - x$ (BR dynamics)

More general is a discretization with variable step sizes

$$x(t_{n+1}) \in \varepsilon_n \mathsf{BR}(x(t_n)) + (1-\varepsilon_n)x(t_n), \quad t_n + \varepsilon_n = t_{n+1} \tag{2}$$

For $\varepsilon_n = \frac{1}{n}$ this is fictitious play.

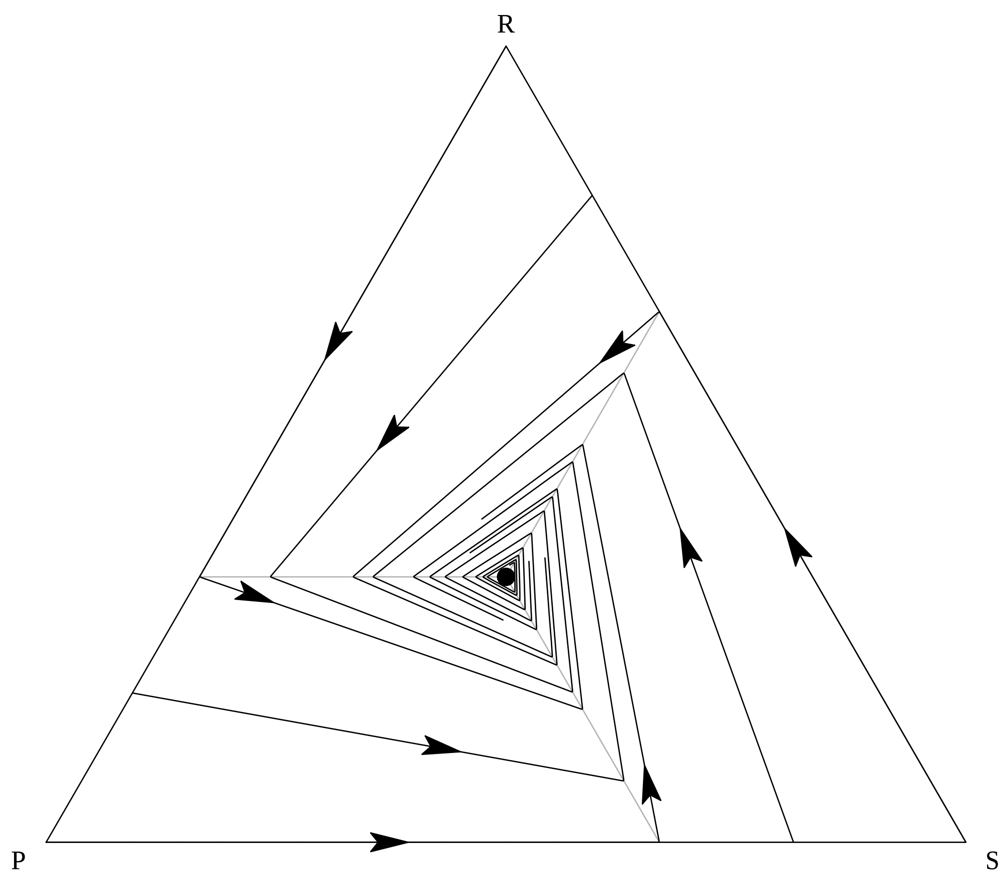
For $\varepsilon_n = \frac{1-\rho}{1-\rho^n}$ (with $0 < \rho < 1$) this is geometric fictitious play with discount rate ρ which tends to (1) with $\varepsilon = 1-\rho$, as $n \to \infty$.

$$x' \in \frac{1}{1+h}(x + h\mathsf{BR}(x))$$

general result: global attractor is USC against discretization (H. and Sorin, 2006)

Example: RPS game (zero sum):

global attractor of the BR dynamics $\dot{x} \in BR(x)$ is the unique equilibrium E



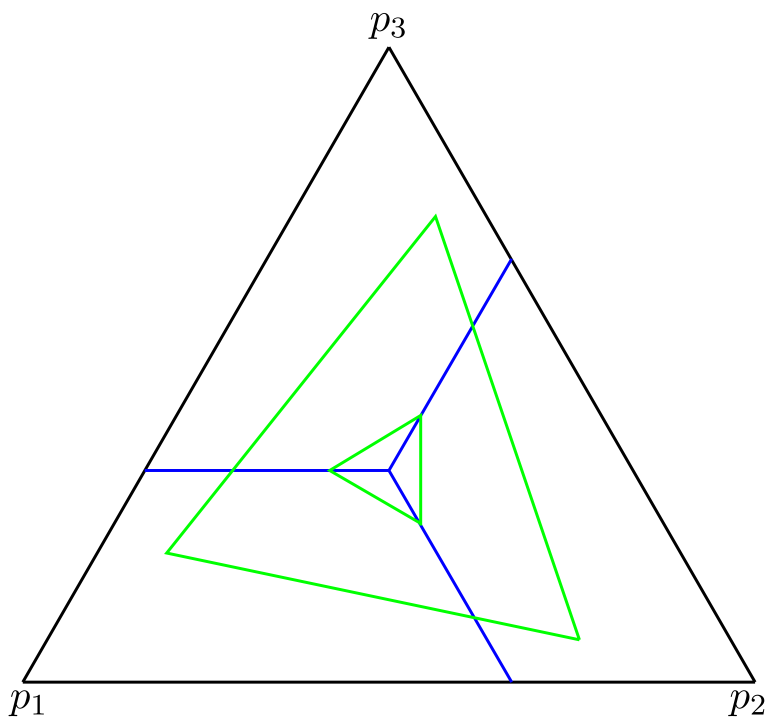
$$V(x) = \max(Ax)_i$$
$$\dot{V}(x) = -V(x)$$

hence, for small h, orbits of

$$x' \in \frac{1}{1+h}\left(x + h\mathsf{BR}(x)\right)$$

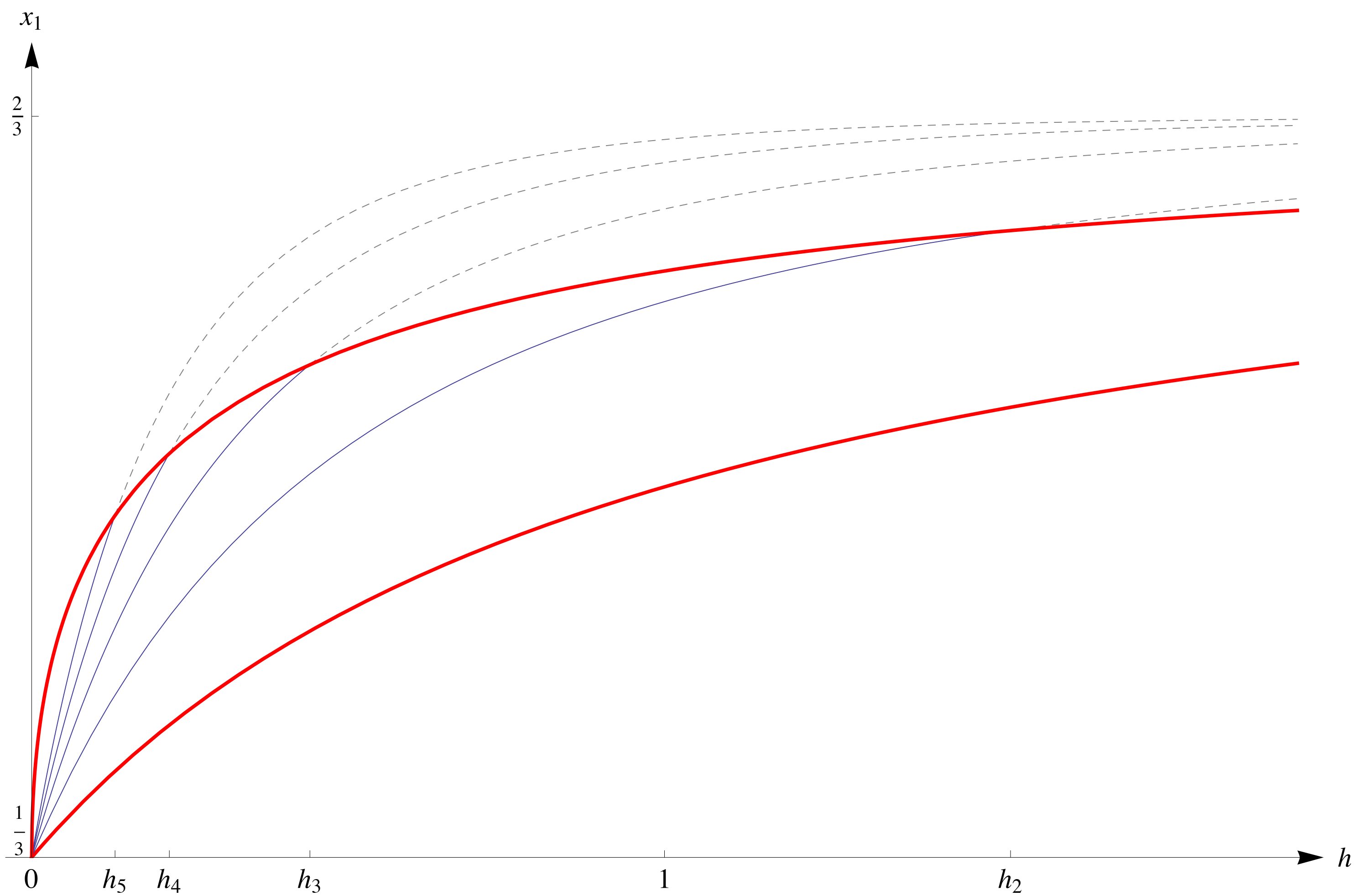
converge to a small neighborhood of the unique equilibrium E.

What is the limit set? (with Peter Bednarik)

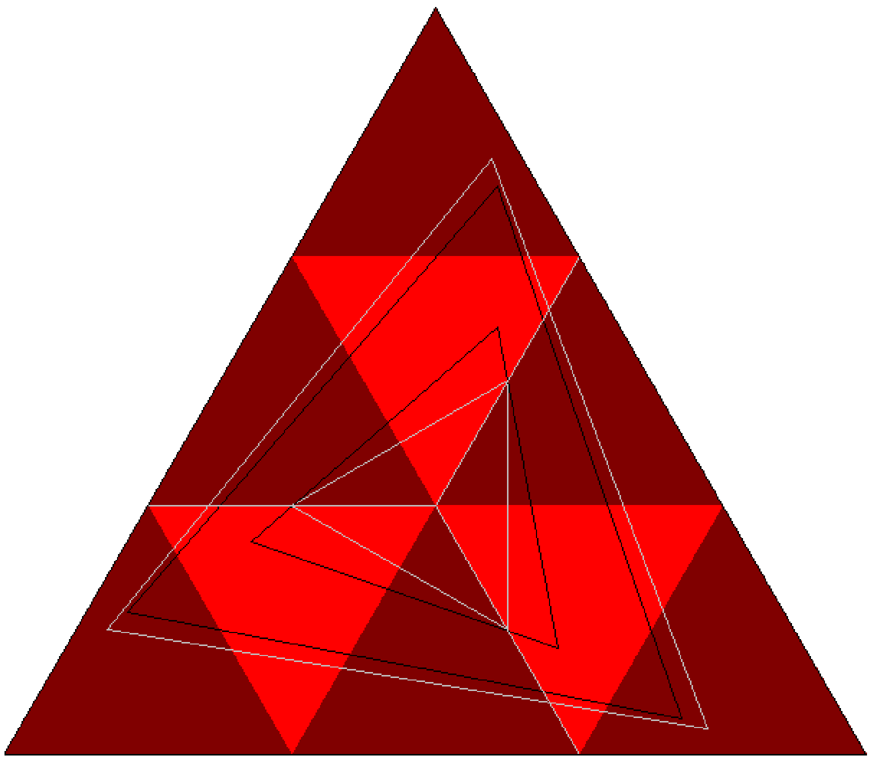


The region bounded by the two green triangles is globally attracting

E is a repellor, attractor lies between the two triangles
shrink to E as $h \to 0$ (like h, resp. $\sqrt{h}$)

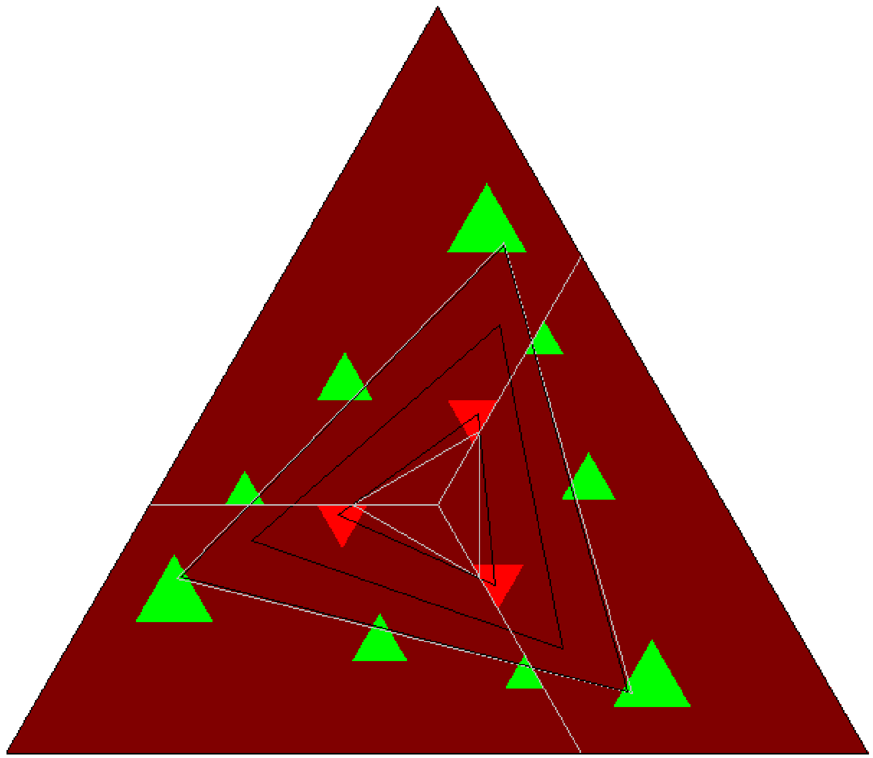


Orbits of period $3n$ exist for $0 < h < h_n$



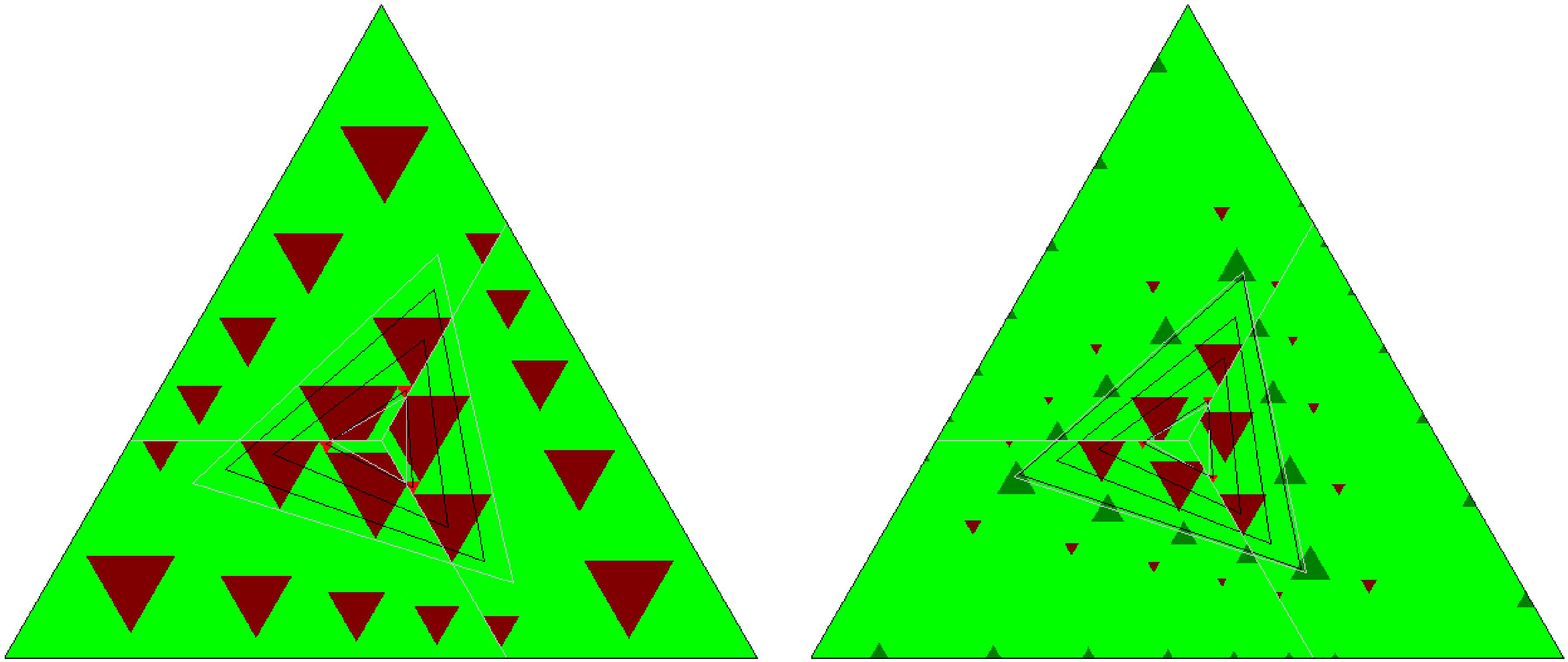
(a) $h = 1.00$

$h = 1, \varepsilon = \frac{1}{2}$: periods 3 (red) and 6 (dark red)



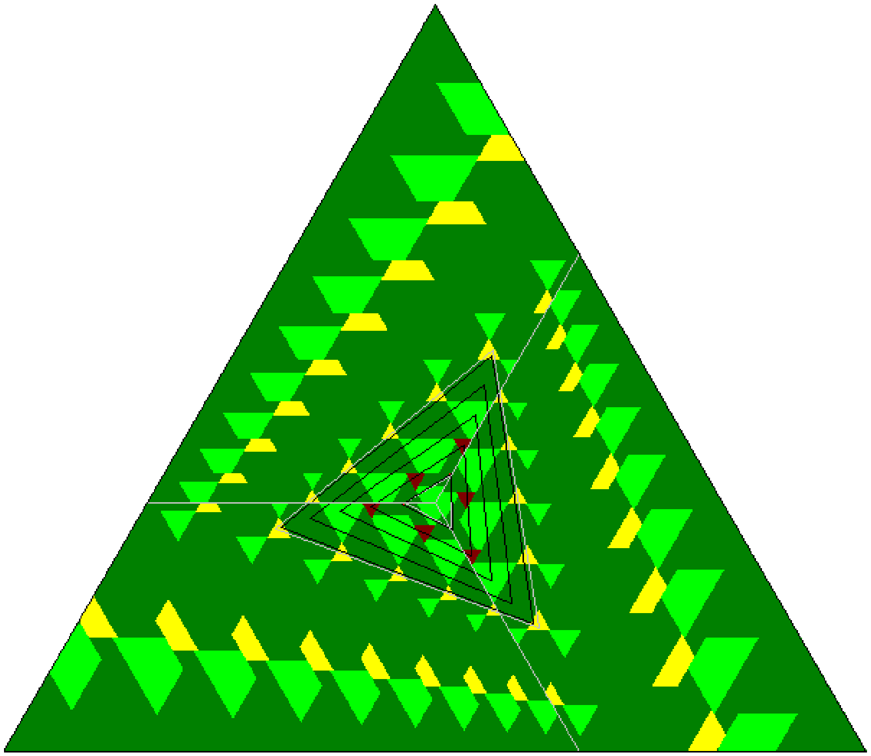
(b) $h = 0.30$

$h = .3$: periods 3 (red), 6 (dark red) and 9 (green)

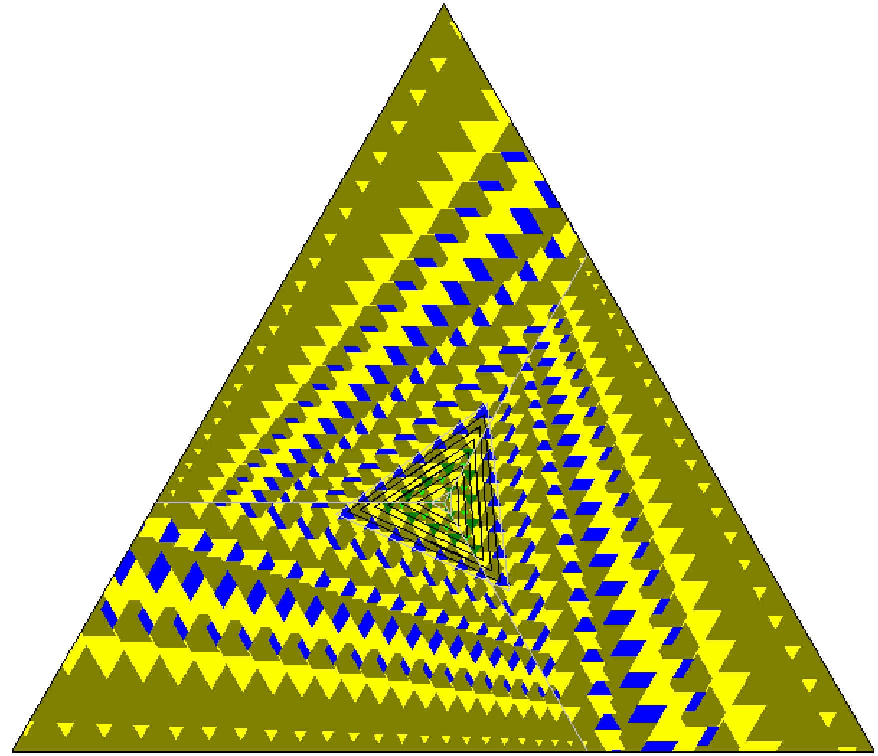


(c) $h = 0.25$ (d) $h = 0.20$

periods 3 (red), 6 (dark red), 9 (green), 12 (dark green)

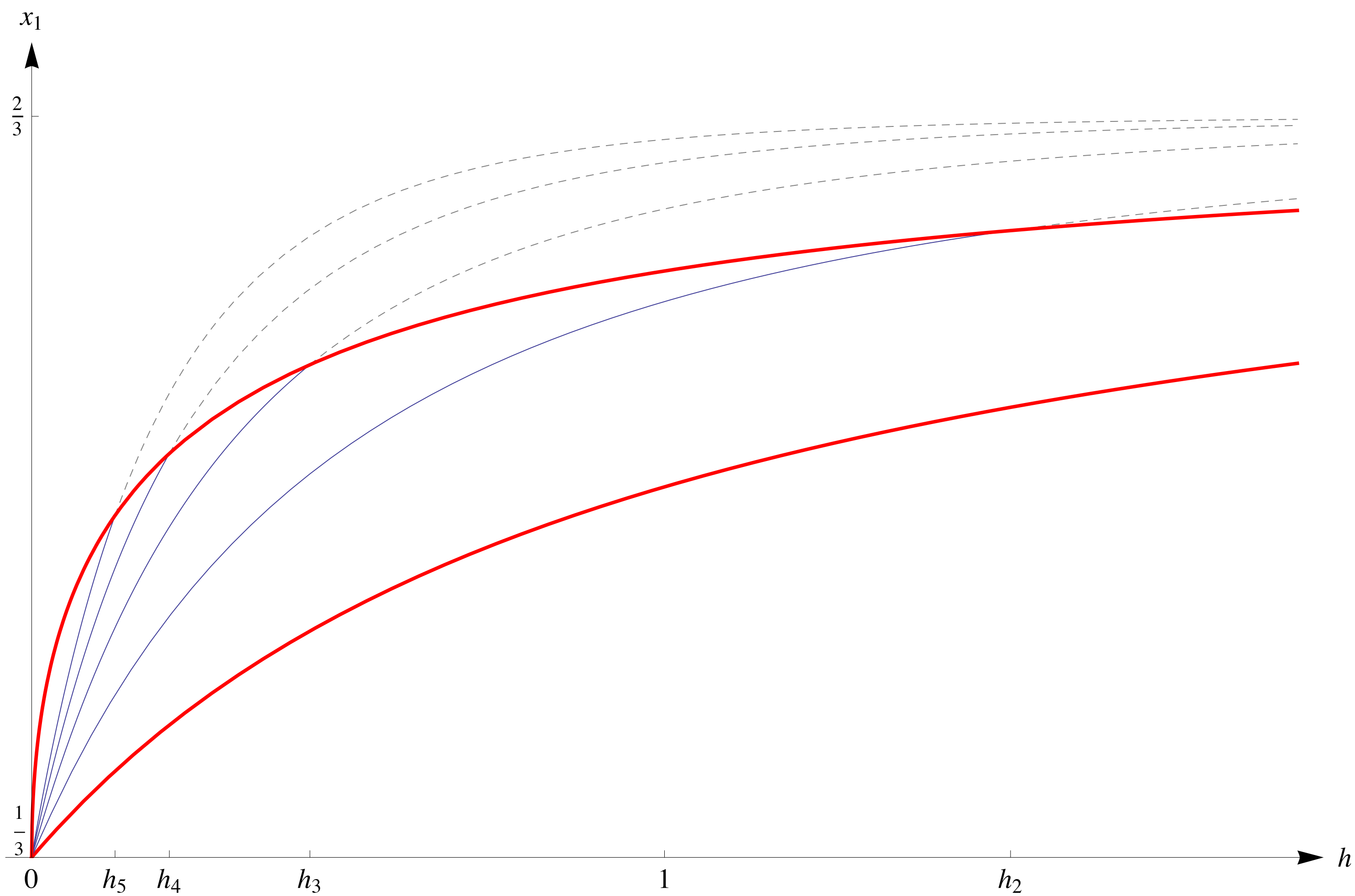


(e) $h = 0.12$



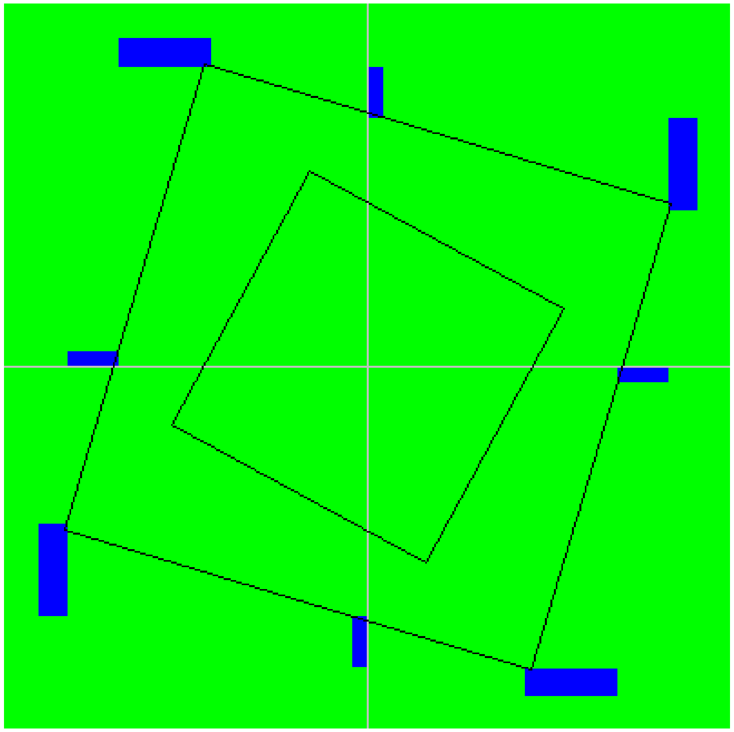
(f) $h = 0.05$

periods 3 (red), 6 (dark red), 9 (green), 12 (dark green), 15 (yellow), 18 (khaki), 21 (blue)

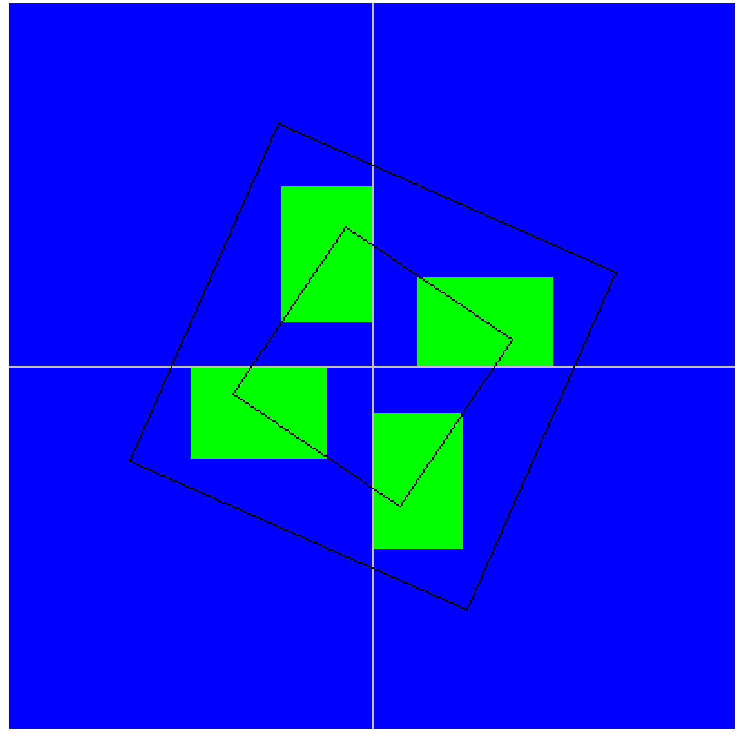


Orbits of period $3n$ exist for $0 < h < h_n$

For cyclic 2×2 games: similar behavior, orbits of period $4n$

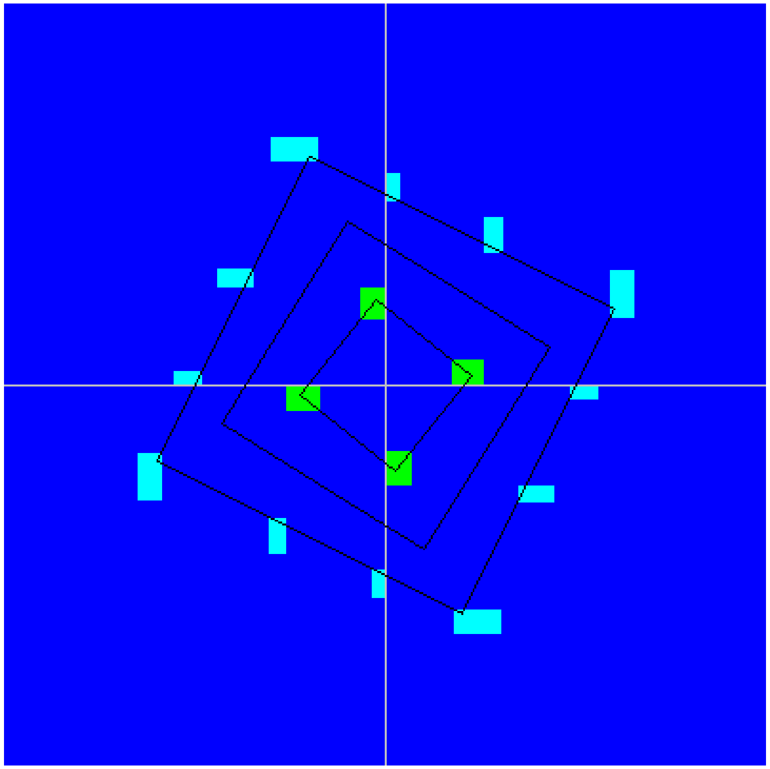


(a) $h = 0.83$

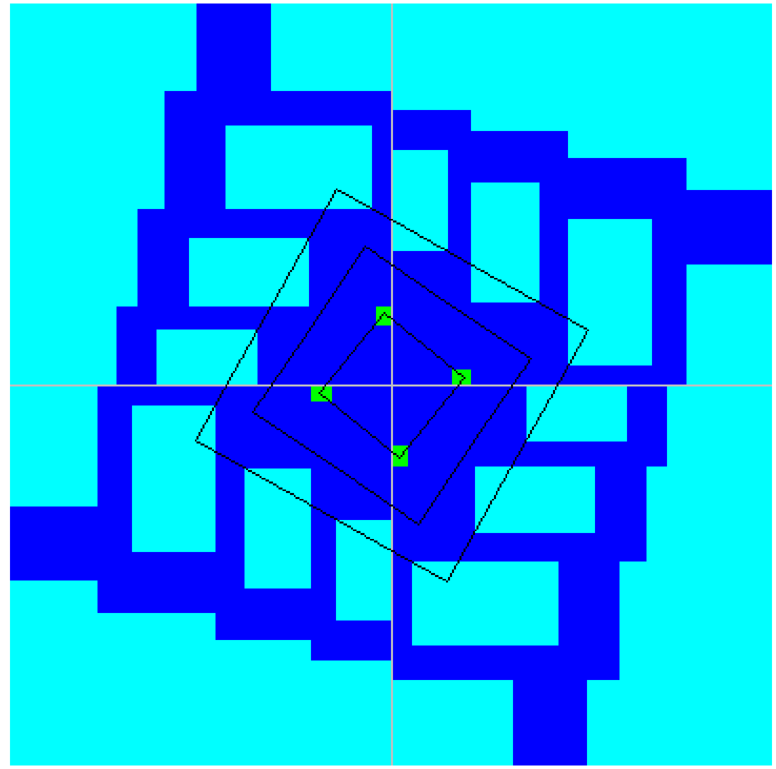


(b) $h = 0.50$

periods 4 (green), 8 (blue)

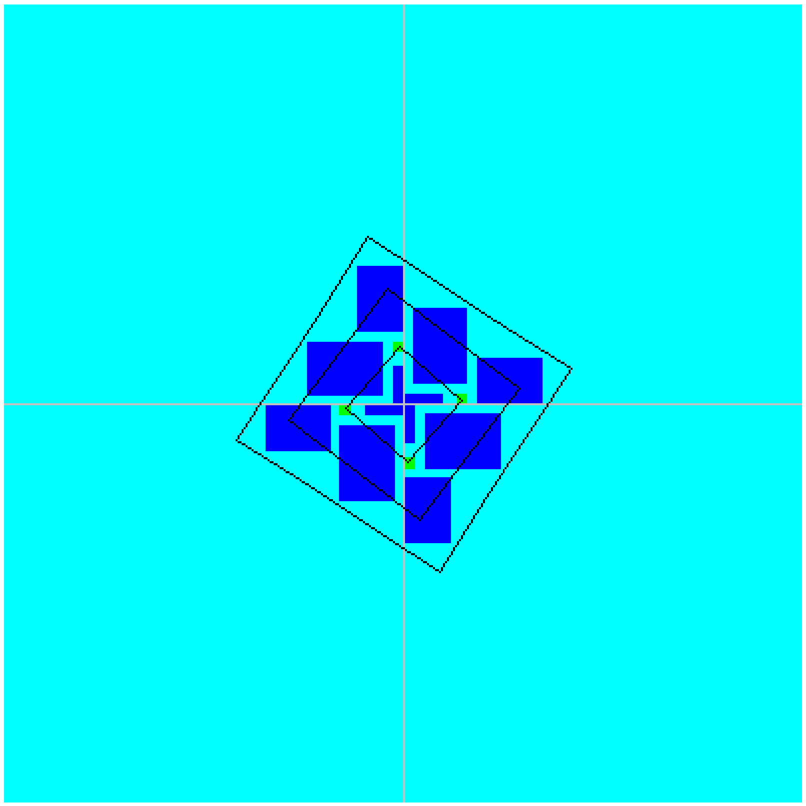


(c) $h = 0.26$

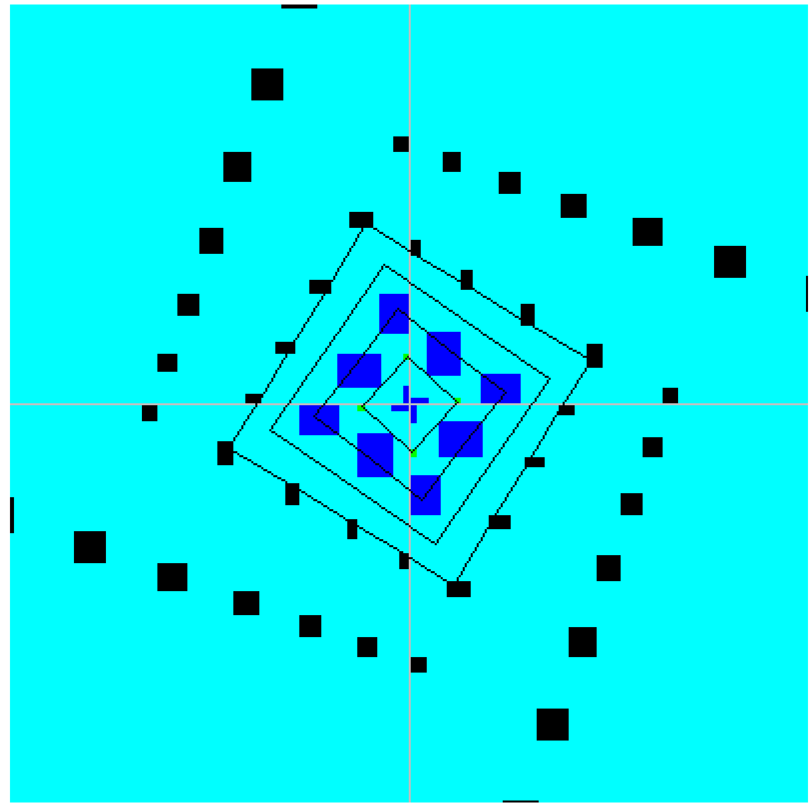


(d) $h = 0.21$

periods 4 (green), 8 (blue), 12 (teal)



(e) $h = 0.16$



(f) $h = 0.13$

periods 4 (green), 8 (blue), 12 (teal), 16 (black) discretization of BR dynamics, stepsize h:

attractor shrinks like $\sqrt{h}$ towards the equilibrium

the smaller h the more complex is the dynamics!