Monotone Operator Theory in Convex Optimization

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Notation

- $\mathcal{H}$, $\mathcal{G}$, etc are real Hilbert spaces
- $\mathscr{B}(\mathcal{H},\mathcal{G})$ is the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{G}$; $\mathscr{B}(\mathcal{H}) = \mathscr{B}(\mathcal{H},\mathcal{H})$
- Synthetic problem: given $f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$,

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- *Convex optimization* refers to the case when f in (1) is proper, lower semicontinuous, and convex, which we denote by $f \in \Gamma_0(\mathcal{H})$
- We interpret (1) in the strict sense of producing a point in $\operatorname{Argmin} f$, not in the looser sense of making f small
  - Minimizing sequences may have little to do with actually approaching a point in $\operatorname{Argmin} f$ as we can have for $p > 2$ (even in $\mathcal{H} = \mathbb{R}^2$): $f(x_n) - \min f(\mathcal{H}) = 1/(n+1)^p \downarrow 0$ and $d_{\operatorname{Argmin} f}(x_n) \uparrow +\infty$

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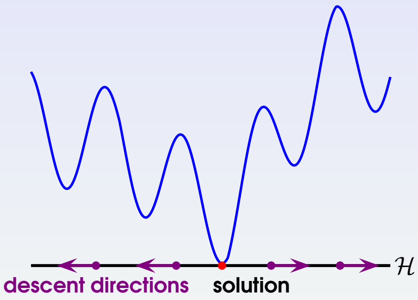

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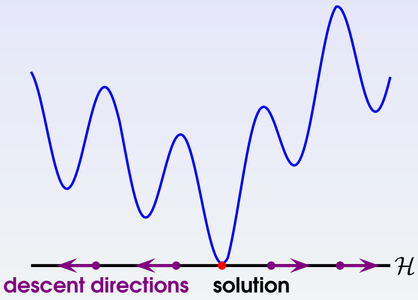

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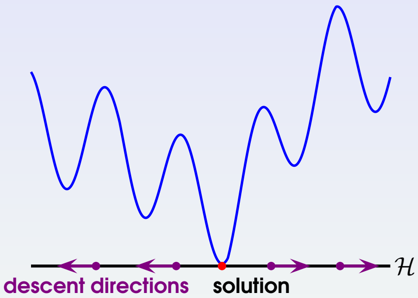

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4. Algorithms may yield trivial solutions:

Let $f: \mathcal{H} \to \{0, \ldots, p\}$ be l.s.c. (e.g., rank etc.), let $C \neq \varnothing$. Then ***any*** point in C is a local minimizer of:

$$\underset{x \in C}{\text{minimize}} \;\; f(x)$$

J.-B. Hiriart-Urruty, When only global optimization matters, *J. Global Optim.*, vol. 56, pp. 761–763, 2013

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$$\underset{x_i \in \mathcal{H}_i,\, i \in I}{\text{minimize}} \;\; \sum_{i \in I} \big(f_i(x_i) - \langle x_i \mid z_i^* \rangle \big) + \sum_{k \in K} g_k\Big(\sum_{i \in I} L_{ki} x_i - r_k \Big)$$

where $f_i \in \Gamma_0(\mathcal{H}_i)$, $g_k \in \Gamma_0(\mathcal{G}_k)$, $L_{ki} \in \mathscr{B}(\mathcal{H}_i, \mathcal{G}_k)$

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- To analyze and solve such complex minimization problem, one must borrow tools from functional and numerical analysis

- Our main message is that *monotone operator theory* plays an increasingly central role in convex optimization and that both fields maintain a tight and productive interplay

Functional analysis: Historical overview

1950's

Linear functional analysis

- *Topological vector spaces*
- *Linear operators*
- *Duality*
- *Theory of distributions*
- *etc.*

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Nonlinear functional analysis: "anything not linear"

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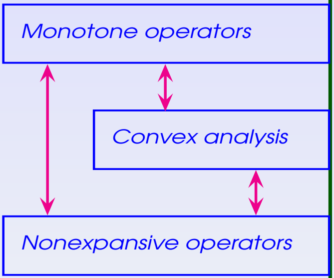

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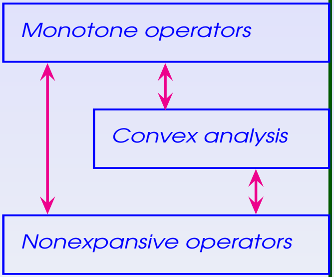

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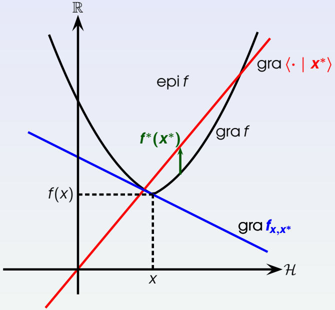
These new structured theories, which often revolve around turning equalities in classical linear analysis into inequalities, benefit from tight connections between each other.

Convex analysis (Moreau, Rockafellar, 1962+)

- $\Gamma_0(\mathcal{H})$: lower semicontinuous convex functions $f\colon \mathcal{H} \to]-\infty,+\infty]$ such that $\operatorname{dom} f = \{x \in \mathcal{H} \mid f(x) < +\infty\} \neq \varnothing$
- $f^*\colon x^* \mapsto \sup_{x\in\mathcal{H}} \langle x \mid x^*\rangle - f(x)$ is the conjugate of f; if $f \in \Gamma_0(\mathcal{H})$, then $f^* \in \Gamma_0(\mathcal{H})$ and $f^{**} = f$
- The subdifferential of f at $x \in \mathcal{H}$ is

$$\partial f(x) = \Big\{x^* \in \mathcal{H} \mid (\forall y \in \mathcal{H})\ \underbrace{\langle y - x \mid x^*\rangle + f(x)}_{f_{x,x^*}(y)} \leqslant f(y)\Big\}$$



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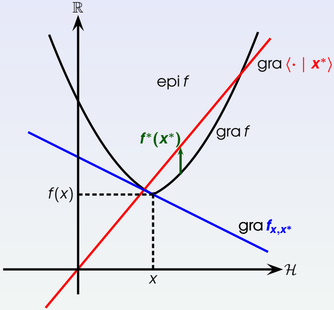


Infimal convolution of $f, g\colon \mathcal{H} \to]-\infty,+\infty]$:
$(f \,\square\, g)\colon x \mapsto \inf_{y\in\mathcal{H}} f(y) + g(x-y)$

Fermat's rule:
x minimizes $f \Leftrightarrow 0 \in \partial f(x)$

Nonexpansive operators (Browder, Minty)

- $T \in \mathscr{B}(\mathcal{H})$ is an *isometry* if $(\forall x \in \mathcal{H})\ \|Tx\| = \|x\|$, i.e.,

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \ \ \|Tx - Ty\| = \|x - y\|.$$

- $T\colon \mathcal{H} \to \mathcal{H}$ is *nonexpansive* if

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \ \ \|Tx - Ty\| \leqslant \|x - y\|,$$

firmly nonexpansive if

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \ \ \|Tx-Ty\|^2+\|(\mathrm{Id}-T)x-(\mathrm{Id}-T)y\|^2 \leqslant \|x-y\|^2.$$

and α-*averaged* ($\alpha \in \,]0,1[$), if

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \ \ \|Tx-Ty\|^2+\frac{1-\alpha}{\alpha}\|(\mathrm{Id}-T)x-(\mathrm{Id}-T)y\|^2 \leqslant \|x-y\|^2$$

Monotone operators (Kačurovskiĭ, Minty, Zarantonello, 1960)

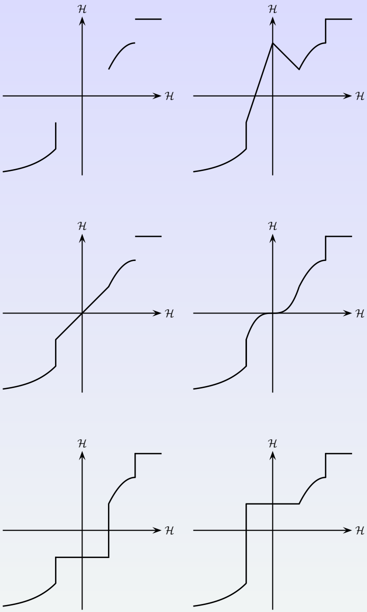
- $A \in \mathscr{B}(\mathcal{H})$ is skew if $(\forall x \in \mathcal{H})\ \langle x \mid Ax \rangle = 0$ and it is positive if $(\forall x \in \mathcal{H})\ \langle x \mid Ax \rangle \geqslant 0$, i.e.,

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \langle x - y \mid Ax - Ay \rangle \geqslant 0. \tag{2}$$

- In 1960, Kačurovskiĭ, Minty, and Zarantonello independently called *monotone* a nonlinear operator $A\colon \mathcal{H} \to \mathcal{H}$ that satisfies (2)
- More generally, a set-valued operator $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ with graph $\operatorname{gra} A = \big\{(x, x^*) \in \mathcal{H} \times \mathcal{H} \mid x^* \in Ax\big\}$ is monotone if

$$(\forall (x, x^*) \in \operatorname{gra} A)(\forall (y, y^*) \in \operatorname{gra} A) \quad \langle x - y \mid x^* - y^* \rangle \geqslant 0,$$

and *maximally monotone* if there is no monotone operator $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ such that $\operatorname{gra} A \subset \operatorname{gra} B \neq \operatorname{gra} A$



monotone, not monotone

monotone, max. monotone

max. monotone, max. monotone

Minty's theorem: A monotone is max. monotone $\Leftrightarrow$ $\operatorname{ran}(\operatorname{Id} + A) = \mathcal{H}$

First examples of maximally monotone operators

- $A\colon \mathbb{R} \to \mathbb{R}$ is a continuous increasing function
- $A \in \mathscr{B}(\mathcal{H})$ is a skew operator
- (Moreau) $f \in \Gamma_0(\mathcal{H})$ and $A = \partial f$
- C is a nonempty closed convex subset of $\mathcal{H}$ and

$$(\forall x \in \mathcal{H}) \;\; Ax = \begin{cases} \big\{u \in \mathcal{H} \mid (\forall y \in C) \;\; \langle y - x \mid u \rangle \leqslant 0\big\} & \text{if } \; x \in C \\ \varnothing & \text{if } \; x \notin C \end{cases}$$

  A is the normal cone operator of C

- V is a closed vector subspace of $\mathcal{H}$ and

$$(\forall x \in \mathcal{H}) \;\; Ax = \begin{cases} V^{\perp} & \text{if } x \in V \\ \varnothing & \text{if } x \notin V \end{cases}$$

What is a maximally monotone operator in general?

- Who knows? ...certainly a complicated object
- The Asplund decomposition

  $$A = \partial f + \text{something (acyclic)}$$

  is not fully understood
- In the Borwein-Wiersma decomposition, "something" is the restriction of a skew operator
- Jon Borwein's conjecture was that in general "something" is locally the restriction (localization) of a skew linear relation

Convexity/Nonexpansiveness/Monotonicity

- If $f \in \Gamma_0(\mathcal{H})$, $A = \partial f$ is maximally monotone
- If $T\colon \mathcal{H} \to \mathcal{H}$ is nonexpansive, $A = \mathrm{Id} - T$ is max. mon. and $\operatorname{Fix} T = \{x \in \mathcal{H} \mid Tx = x\}$ is closed and convex and $\operatorname{Fix} T = \operatorname{zer} A$
- If $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is max. mon., $(\forall x \in \mathcal{H})$ Ax is closed and convex; $\operatorname{zer} A = A^{-1}(0)$ is closed and convex
- If $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone, $\operatorname{int} \operatorname{dom} A$, $\overline{\operatorname{dom}}\, A$, $\operatorname{int} \operatorname{ran} A$, and $\overline{\operatorname{ran}}\, A$ are convex
- (Minty) If $T\colon \mathcal{H} \to \mathcal{H}$ is firmly nonexpansive, then $T = J_A$ for some maximally monotone $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ and $\operatorname{Fix} T = \operatorname{zer} A$
- (Minty) If $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone, the resolvent $J_A = (\mathrm{Id} + A)^{-1}$ is firmly nonexpansive with $\operatorname{dom} J_A = \mathcal{H}$, and the reflected resolvent $R_A = 2J_A - \mathrm{Id}$ is nonexpansive
- If $T\colon \mathcal{H} \to \mathcal{H}$ is an α-averaged ($\alpha \leqslant 1/2$) nonexpansive operator, it is maximally monotone
- If $A = \beta B$ is firmly nonexpansive (hence max. mon.), $0 < \gamma < 2\beta$, and $\alpha = \gamma/(2\beta)$, then $\mathrm{Id} - \gamma B$ is an α-averaged nonexpansive operator

Moreau's proximity operator

- In 1962, Jean Jacques Moreau (1923–2014) introduced the proximity operator of $f \in \Gamma_0(\mathcal{H})$

$$\operatorname{prox}_f \colon x \mapsto \underset{y \in \mathcal{H}}{\operatorname{argmin}}\, f(y) + \frac{1}{2}\|x - y\|^2$$

  and derived all its main properties

- Set $q = \|\cdot\|^2/2$. Then $f \,\Box\, q + f^* \,\Box\, q = q$ and

$$\operatorname{prox}_f = \nabla(f + q)^* = \nabla(f^* \,\Box\, q) = \operatorname{Id} - \operatorname{prox}_{f^*} = (\operatorname{Id} + \partial f)^{-1}$$

- $\operatorname{prox}_f = J_{\partial f}$, hence
  - $\operatorname{Fix} \operatorname{prox}_f = \operatorname{zer} \partial f = \operatorname{Argmin} f$
  - $(\operatorname{prox}_f x, x - \operatorname{prox}_f x) \in \operatorname{gra} \partial f$
  - $\|\operatorname{prox}_f x - \operatorname{prox}_f y\|^2 + \|\operatorname{prox}_{f^*} x - \operatorname{prox}_{f^*} y\|^2 \leqslant \|x - y\|^2$

- This suggests that (Martinet's proximal point algorithm, 1970/72) $x_{n+1} = \operatorname{prox}_f x_n \rightharpoonup x \in \operatorname{Argmin} f$

Subdifferentials as Maximally Monotone Operators

- Rockafellar (1966) has fully characterized subdifferentials as those maximally monotone operators which are cyclically maximally monotone
- Moreau (1965) has fully characterized proximity operators as those (firmly) nonexpansive operators which are gradients of convex functions
- Moreau (1963) showed that a convex average of proximity operator is again a proximity operator. A number of additional "proximity preserving" transformations are identified in the accompanying paper (PLC, 2018), which lead to:
  - A new example of weakly but not strongly convergent proximal iteration
  - New explicit expressions for proximity operators of certain composite functions
  - A study of self-dual classes of firmly nonexpansive operators

The need for monotone operators in optimization

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- ... some key maximal monotone operators arising in the analysis and the numerical solution of convex minimization problems are not subdifferentials, for instance:
  - (Rockafellar, 1970) The saddle operator

    $$A\colon (x_1, x_2) \mapsto \partial\mathcal{L}(\cdot, x_2)(x_1) \times \partial(-\mathcal{L}(x_1, \cdot))(x_2)$$

    associated with a closed convex-concave function $\mathcal{L}$
  - (Spingarn, 1983) The partial inverse of a maximally monotone operator (and even of a subdifferential)
  - Some operators which arise in the perturbation of optimization problems are no longer subdifferentials
  - Skew linear operators arising in composite primal-dual minimization problems (PLC et al., 2011+)

Interplay: The proximal point algorithm

- First derived by Martinet (1970/72) for $f \in \Gamma_0(\mathcal{H})$ with constant proximal parameters, and then by Brézis-Lions (1978)

$$x_{n+1} = \operatorname{prox}_{\gamma_n f} x_n \rightharpoonup x \in \operatorname{Argmin} f \quad \text{if} \quad \sum_{n\in\mathbb{N}} \gamma_n = +\infty \tag{3}$$

- Then extended to a maximally monotone operator A by Rockafellar (1976) and Brézis-Lions (1978)

$$x_{n+1} = J_{\gamma_n A} x_n \rightharpoonup x \in \operatorname{zer} A \quad \text{if} \quad \sum_{n\in\mathbb{N}} \gamma_n^2 = +\infty \tag{4}$$

- Note that (3) has more general parameters. However (4) is considerably more useful **to optimization** than (3)

Interplay: The proximal point algorithm

- (Rockafellar, 1976) Applying the general proximal point algorithm (4) to the saddle operator leads to various minimization algorithms (e.g., the proximal method of multipliers in the case of the ordinary Lagrangian)
- Applying the general proximal point algorithm (4) to the partial inverse of a suitably constructed partial inverse makes it possible to solve (Alghamdi, Alotaibi, PLC, Shahzad, 2014)

$$\underset{(\forall i\in I)\ x_i\in\mathcal{H}_i}{\text{minimize}} \ \sum_{i\in I}\big(f_i(x_i)-\langle x_i \mid z_i\rangle\big)+g\Bigg(\sum_{i\in I}L_ix_i-r\Bigg)$$

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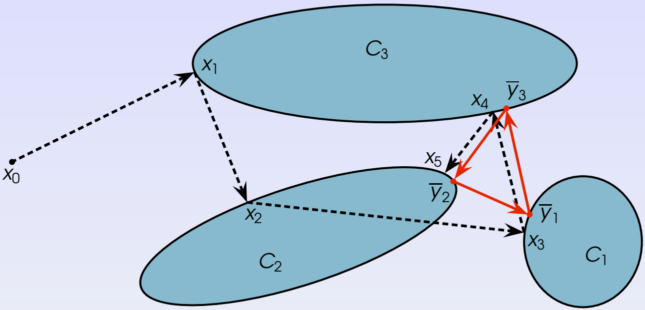
Periodic projection methods: inconsistent case



- Basic feasibility problem: find a common point of nonempty closed convex sets $(C_i)_{1\leqslant i\leqslant m}$ by the method of periodic projections $x_{mn+1} = \text{proj}_1 \cdots \text{proj}_m x_{mn}$
- If the sets turn out not to intersect, the method produces a cycle $(\overline{y}_1, \overline{y}_2, \overline{y}_3)$

Periodic projection methods: inconsistent case

- Denote by $\operatorname{cyc}(C_1,\ldots,C_m)$ is the set of cycles of $(C_1,\ldots,C_m)$, i.e.,

$$\operatorname{cyc}(C_1,\ldots,C_m) = \big\{(\overline{y}_1,\ldots,\overline{y}_m) \in \mathcal{H}^m \;\big|\; \overline{y}_1 = \operatorname{proj}_1\overline{y}_2,\ldots, \overline{y}_{m-1} = \operatorname{proj}_{m-1}\overline{y}_m,\ \overline{y}_m = \operatorname{proj}_m\overline{y}_1\big\}.$$

- **Question (Gurin-Polyak-Raik, 1967):** Let $m \geqslant 3$ be an integer. Does there exist a function $\Phi\colon \mathcal{H}^m \to \mathbb{R}$ such that, for every ordered family of nonempty closed convex subsets $(C_1,\ldots,C_m)$ of $\mathcal{H}$, $\operatorname{cyc}(C_1,\ldots,C_m)$ is the set of solutions to the variational problem

$$\underset{y_1\in C_1,\ldots,\, y_m\in C_m}{\text{minimize}} \quad \Phi(y_1,\ldots,y_m)\,?$$

Cyclic projection methods

- **Theorem (Baillon, PLC, Cominetti, 2012):** Suppose that $\dim \mathcal{H} \geqslant 2$ and let $\mathbb{N} \ni m \geqslant 3$. There exists ***no*** function $\Phi\colon \mathcal{H}^m \to \mathbb{R}$ such that, for every ordered family of nonempty closed convex subsets $(C_1, \ldots, C_m)$ of $\mathcal{H}$, $\operatorname{cyc}(C_1, \ldots, C_m)$ is the set of solutions to the variational problem

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$$\underset{y_1 \in C_1, \ldots, y_m \in C_m}{\text{minimize}} \quad \Phi(y_1, \ldots, y_m).$$

- However, cycles do have a meaning: if we denote by L the circular left shift, they solve the inclusion

$$(0, \ldots, 0) \in \underbrace{N_{C_1 \times \cdots \times C_m}}_{\text{subdifferential}}(y_1, \ldots, y_m) + \underbrace{(\mathrm{Id} - L)}_{\text{not a subdifferential}}(y_1, \ldots, y_m),$$

which involves two maximally monotone operators

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Splitting structured problems: 3 basic methods

$A, B\colon \mathcal{H} \to 2^{\mathcal{H}}$ maximally monotone, solve $0 \in A\overline{x} + B\overline{x}$.

- Douglas-Rachford splitting (1979)

$$\left\lfloor \begin{array}{l} y_n = J_{\gamma B} x_n \\ z_n = J_{\gamma A}(2y_n - x_n) \\ x_{n+1} = x_n + z_n - y_n \end{array} \right.$$

- $B\colon \mathcal{H} \to \mathcal{H}$ $1/\beta$-cocoercive: forward-backward splitting (1979+)

$$\left\lfloor \begin{array}{l} 0 < \gamma_n < 2/\beta \\ y_n = x_n - \gamma_n B x_n \\ x_{n+1} = J_{\gamma_n A} y_n \end{array} \right.$$

- $B\colon \mathcal{H} \to \mathcal{H}$ μ-Lipschitzian: forward-backward-forward splitting (2000)

$$\left\lfloor \begin{array}{l} 0 < \gamma_n < 1/\mu \\ y_n = x_n - \gamma_n B x_n \\ z_n = J_{\gamma_n A} y_n \\ r_n = z_n - \gamma_n B z_n \\ x_{n+1} = x_n - y_n + r_n \end{array} \right.$$

Splitting structured problems: 3 basic methods

- A large number of minimization methods are special cases of these monotone operator splitting methods in a suitable setting that may involve
  - product spaces
  - dual spaces
  - primal-dual spaces
  - renormed spaces
  - or a combination thereof
- The simplifying reformulations typically involve monotone operators which are not subdifferentials

Proximal splitting methods in convex optimization

- $f \in \Gamma_0(\mathcal{H})$, $\varphi_k \in \Gamma_0(\mathcal{G}_k)$, $\ell_k \in \Gamma_0(\mathcal{G}_k)$ strongly convex, $L_k\colon \mathcal{H} \to \mathcal{G}_k$ linear bounded, $\|L_k\| = 1$, $h\colon \mathcal{H} \to \mathbb{R}$ convex and smooth:

$$\underset{x \in \mathcal{H}}{\text{minimize}} \;\; f(x) + \sum_{k=1}^{p} (\varphi_k \,\square\, \ell_k)(L_k x - r_k) + h(x)$$

where: $\varphi_k \,\square\, \ell_k \colon x \mapsto \inf_{y \in \mathcal{H}} \big(\varphi_k(y) + \ell_k(x - y)\big)$

- Example: multiview total variation image recovery from observations $r_k = L_k \overline{x} + w_k$:

$$\underset{x \in \mathcal{H}}{\text{minimize}} \;\; \sum_{k \in \mathbb{N}} \phi_k(\langle x \mid e_k \rangle) + \sum_{k=1}^{p-1} \alpha_k \underbrace{d_{C_k}}_{\iota_C \,\square\, \|\cdot\|} (L_k x - r_k) + \beta \|\nabla x\|_{1,2}$$

- A splitting algorithm activates each function and each linear operator individually

Proximal splitting methods in convex optimization

- $A = \partial f,\ \ C = \nabla h,\ B_k = \partial g_k,$ and $D_k = \partial \ell_k$
- $\mathcal{K} = \mathcal{H} \oplus \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_p$
- Subdifferential: $\boldsymbol{M}\colon \mathcal{K} \to 2^{\mathcal{K}}\colon (x, v_1, \ldots, v_p) \mapsto$
$$(-z+Ax) \times (r_1+B_1^{-1}v_1) \times \cdots \times (r_p+B_p^{-1}v_p)$$
- Not a subdifferential: $\boldsymbol{Q}\colon \mathcal{K} \to \mathcal{K}\colon (x, v_1, \ldots, v_p) \mapsto$
$$\Big(Cx + \textstyle\sum_{k=1}^{p} L_k^* v_k, -L_1x+D_1^{-1}v_1, \ldots, -L_px+D_p^{-1}v_p\Big)$$
- $\boldsymbol{M}$ and $\boldsymbol{Q}$ are maximally monotone, $\boldsymbol{Q}$ is Lipschitzian, the zeros of $\boldsymbol{M}+\boldsymbol{Q}$ are primal-dual solutions
- Solve $\mathbf{0} \in \boldsymbol{M}\boldsymbol{x} + \boldsymbol{Q}\boldsymbol{x}$, where $\boldsymbol{x} = (x, v_1, \ldots, v_p)$ via Tseng's forward-backward-forward splitting algorithm

$$\left\lfloor \begin{array}{l} \boldsymbol{y}_n = \boldsymbol{x}_n - \boldsymbol{Q}\boldsymbol{x}_n \\ \boldsymbol{p}_n = (\mathrm{Id} + \boldsymbol{M})^{-1}\, \boldsymbol{y}_n \\ \boldsymbol{q}_n = \boldsymbol{p}_n - \boldsymbol{Q}\boldsymbol{p}_n \\ \boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \boldsymbol{y}_n + \boldsymbol{q}_n \end{array} \right.$$

in $\mathcal{K}$ to get...

Proximal splitting methods in convex optimization

- Algorithm:

$$
\begin{array}{l}
\text{for } n = 0, 1, \ldots \\
\left\lfloor
\begin{array}{l}
y_{1,n} = x_n - \left(\nabla h(x_n) + \sum_{k=1}^{m} L_k^* v_{k,n}\right) \\
p_{1,n} = \operatorname{prox}_f y_{1,n} \\
\text{For } \ k = 1, \ldots, p \\
\left\lfloor
\begin{array}{l}
y_{2,k,n} = v_{k,n} + \left(L_k x_n - \nabla \ell_k^*(v_{k,n})\right) \\
p_{2,k,n} = \operatorname{prox}_{g_k^*}(y_{2,k,n} - r_k) \\
q_{2,k,n} = p_{2,k,n} + \left(L_k p_{1,n} - \nabla \ell_k^*(p_{2,k,n})\right) \\
v_{k,n+1} = v_{k,n} - y_{2,k,n} + q_{2,k,n}
\end{array}
\right. \\
q_{1,n} = p_{1,n} - \left(\nabla h(p_{1,n}) + \sum_{k=1}^{m} L_k^* p_{2,k,n}\right) \\
x_{n+1} = x_n - y_{1,n} + q_{1,n}
\end{array}
\right.
\end{array}
$$

- $(x_n)_{n\in\mathbb{N}}$ converges weakly to a solution (PLC, 2013)

Some limitations of the state-of-the-art

We present a new framework that circumvents simultaneously the limitations of current methods, which require:

- inversions of linear operators or knowledge of bounds on norms of all the L_{ki}
- the proximal parameters must be the same for all the subdifferential operators
- activation of the proximal operators of all the functions: impossible in huge-scale problems
- synchronicity: all proximity operator evaluations must be computed and used during the current iteration

and, in general,

- converge only weakly

Composite convex optimization problem

- Let $\boldsymbol{F}$ be the set of solutions to the problem

$$\underset{x_i \in \mathcal{H}_i,\, i \in I}{\text{minimize}} \sum_{i \in I} \big(f_i(x_i) - \langle x_i \mid z_i^* \rangle \big) + \sum_{k \in K} g_k \Big(\sum_{i \in I} L_{ki} x_i - r_k \Big)$$

where $f_i \in \Gamma_0(\mathcal{H}_i)$, $g_k \in \Gamma_0(\mathcal{G}_k)$, $L_{ki} \in \mathscr{B}(\mathcal{H}_i, \mathcal{G}_k)$

Composite convex optimization problem

- Let $\boldsymbol{F}$ be the set of solutions to the problem

$$\underset{x_i \in \mathcal{H}_i,\, i \in I}{\text{minimize}} \sum_{i \in I} \big(f_i(x_i) - \langle x_i \mid z_i^* \rangle\big) + \sum_{k \in K} g_k\bigg(\sum_{i \in I} L_{ki} x_i - r_k\bigg)$$

where $f_i \in \Gamma_0(\mathcal{H}_i)$, $g_k \in \Gamma_0(\mathcal{G}_k)$, $L_{ki} \in \mathscr{B}(\mathcal{H}_i, \mathcal{G}_k)$

- Let $\boldsymbol{F}^*$ be the set of solutions to the dual problem

$$\underset{v_k^* \in \mathcal{G}_k,\, k \in K}{\text{minimize}} \sum_{i \in I} f_i^*\bigg(z_i^* - \sum_{k \in K} L_{ki}^* v_k^*\bigg) + \sum_{k \in K} \big(g_k^*(v_k^*) + \langle v_k^* \mid r_k \rangle\big)$$

Composite convex optimization problem

- Let $\boldsymbol{F}$ be the set of solutions to the problem

$$\underset{x_i \in \mathcal{H}_i,\, i \in I}{\text{minimize}} \sum_{i \in I} \big(f_i(x_i) - \langle x_i \mid z_i^* \rangle \big) + \sum_{k \in K} g_k \Big(\sum_{i \in I} L_{ki} x_i - r_k \Big)$$

where $f_i \in \Gamma_0(\mathcal{H}_i)$, $g_k \in \Gamma_0(\mathcal{G}_k)$, $L_{ki} \in \mathscr{B}(\mathcal{H}_i, \mathcal{G}_k)$

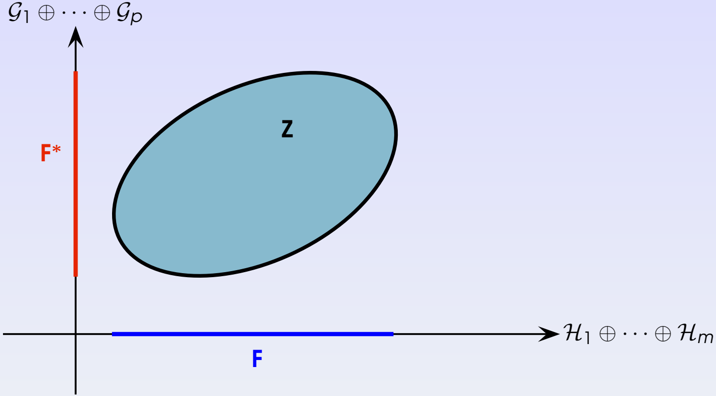
- Let $\boldsymbol{F}^*$ be the set of solutions to the dual problem

$$\underset{v_k^* \in \mathcal{G}_k,\, k \in K}{\text{minimize}} \sum_{i \in I} f_i^* \Big(z_i^* - \sum_{k \in K} L_{ki}^* v_k^* \Big) + \sum_{k \in K} \big(g_k^*(v_k^*) + \langle v_k^* \mid r_k \rangle \big)$$

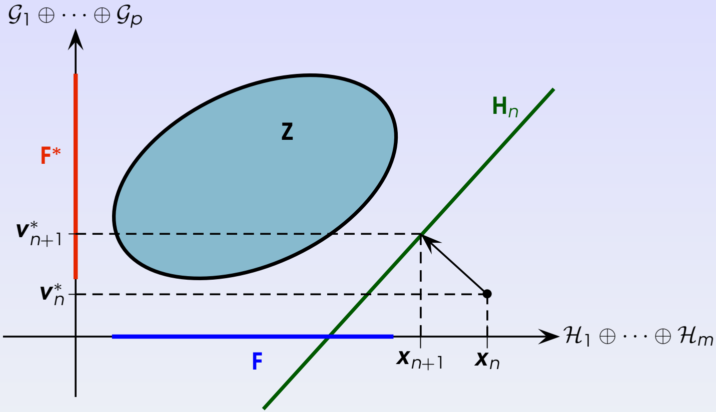
- Associated Kuhn-Tucker set

$$\boldsymbol{Z} = \Big\{ \big((\overline{x}_i)_{i \in I}, (\overline{v}_k^*)_{k \in K} \big) \;\Big|\; \overline{x}_i \in \mathcal{H}_i \;\text{ and }\; z_i^* - \sum_{k \in K} L_{ki}^* \overline{v}_k^* \in \partial f_i(\overline{x}_i),$$
$$\overline{v}_k^* \in \mathcal{G}_k \;\text{ and }\; \sum_{i \in I} L_{ki} \overline{x}_i - r_k \in \partial g_k^*(\overline{v}_k^*) \Big\}$$

Underlying geometry: The Kuhn-Tucker set



Underlying geometry: The Kuhn-Tucker set



- Choose suitable points in the graphs of $(\partial f_i)_{i \in I}$ and $(\partial g_k)_{k \in K}$ to construct a half-space H_n containing Z
- Algorithm: $(x_{n+1}, v^*_{n+1}) = P_{\mathsf{H}_n}(x_n, v^*_n) \rightharpoonup (x, v^*) \in \mathsf{Z} \subset \mathsf{F} \times \mathsf{F}^*$

Asynchronous block-iterative proximal splitting (PLC, Eckstein, 2018)

$$
\begin{array}{l}
\text{for } n = 0, 1, \ldots \\
\quad \text{for every } i \in I_n \\
\quad\quad l_{i,n}^* = \sum_{k\in K} L_{ki}^* v_{k,c_i(n)}^* \\
\quad\quad (a_{i,n}, a_{i,n}^*) = \Big(\operatorname{prox}_{\gamma_{i,c_i(n)} f_i}\big(x_{i,c_i(n)} + \gamma_{i,c_i(n)}(z_i - l_{i,n}^*)\big), \gamma_{i,c_i(n)}^{-1}(x_{i,c_i(n)} - a_{i,n}) - l_{i,n}^*\Big) \\
\quad \text{for every } i \in I \smallsetminus I_n \\
\quad\quad (a_{i,n}, a_{i,n}^*) = (a_{i,n-1}, a_{i,n-1}^*) \\
\quad \text{for every } k \in K_n \\
\quad\quad l_{k,n} = \sum_{i\in I} L_{ki} x_{i,d_k(n)} \\
\quad\quad (b_{k,n}, b_{k,n}^*) = \Big(r_k + \operatorname{prox}_{\mu_{k,d_k(n)} g_k}\big(l_{k,n} + \mu_{k,d_k(n)} v_{k,d_k(n)}^* - r_k\big), v_{k,d_k(n)}^* + \mu_{k,d_k(n)}^{-1}(l_{k,n} - b_{k,n})\Big) \\
\quad \text{for every } k \in K \smallsetminus K_n \\
\quad\quad (b_{k,n}, b_{k,n}^*) = (b_{k,n-1}, b_{k,n-1}^*) \\
\quad \big((t_{i,n}^*)_{i\in I}, (t_{k,n})_{k\in K}\big) = \big((a_{i,n}^* + \sum_{k\in K} L_{ki}^* b_{k,n}^*)_{i\in I}, (b_{k,n} - \sum_{i\in I} L_{ki} a_{i,n})_{k\in K}\big) \\
\quad \tau_n = \sum_{i\in I} \|t_{i,n}^*\|^2 + \sum_{k\in K} \|t_{k,n}\|^2 \\
\quad \text{if } \tau_n > 0 \\
\quad\quad \theta_n = \dfrac{\lambda_n}{\tau_n} \max\Big\{0, \sum_{i\in I} \big(\langle x_{i,n} \mid t_{i,n}^*\rangle - \langle a_{i,n} \mid a_{i,n}^*\rangle\big) + \sum_{k\in K} \big(\langle t_{k,n} \mid v_{k,n}^*\rangle - \langle b_{k,n} \mid b_{k,n}^*\rangle\big)\Big\} \\
\quad \text{else } \theta_n = 0 \\
\quad \text{for every } i \in I \\
\quad\quad x_{i,n+1} = x_{i,n} - \theta_n t_{i,n}^* \\
\quad \text{for every } k \in K \\
\quad\quad v_{k,n+1}^* = v_{k,n}^* - \theta_n t_{k,n}
\end{array}
$$

Numerical example

The problem is to

$$\underset{x\in C}{\text{minimize}}\ \ 6\|\nabla x\|_{1,2} + 5d_D^2(x) + 10\|H_1x - y_1\|_2^2 + 10\|H_2x - y_2\|_2^2,$$

where

- $C = [0, 255]^N$, $N = 128 \times 128$
- $\|\nabla\|_{1,2} \colon \mathbb{R}^N \to \mathbb{R}$ is the total variation
- $D = \big\{x \in \mathbb{R}^N \mid \widehat{x}1_K = \widehat{\overline{x}}1_K\big\}$ where the set K contains the frequencies in $\{0, \ldots, \sqrt{N}/8 - 1\}^2$ (+ symmetries)
- H_1 and H_2 model convolution blurs of size 3×11 and 7×5, y_1 and y_2 are noisy observations

Fully proximal implementations of splitting algorithms

- The problem

$$\underset{x \in C}{\text{minimize}} \; 6\|\nabla x\|_{1,2} + 5d_D^2(x) + 10\|H_1 x - y_1\|^2 + 10\|H_2 x - y_2\|_2^2$$

  contains 3 smooth terms

- However each prox_{g_k} in the $g_k \circ L_k$ terms has an explicit prox
- Although some of the primal dual FB and FBF (see below) algorithms can exploit smoothness, a fully proximal implementation turned out to be faster

We apply these algorithms with

- $f = \iota_C$
- $g_1 = 6\|\cdot\|_{1,2}$ and $L_1 = \nabla$
- $g_2 = 5d_D^2$ and $L_2 = \mathrm{Id}$
- $g_3 = 10\|H_1 \cdot - y_1\|_2^2$ and $L_3 = \mathrm{Id}$
- $g_4 = 10\|H_2 \cdot - y_2\|_2^2$ and $L_4 = \mathrm{Id}$

and

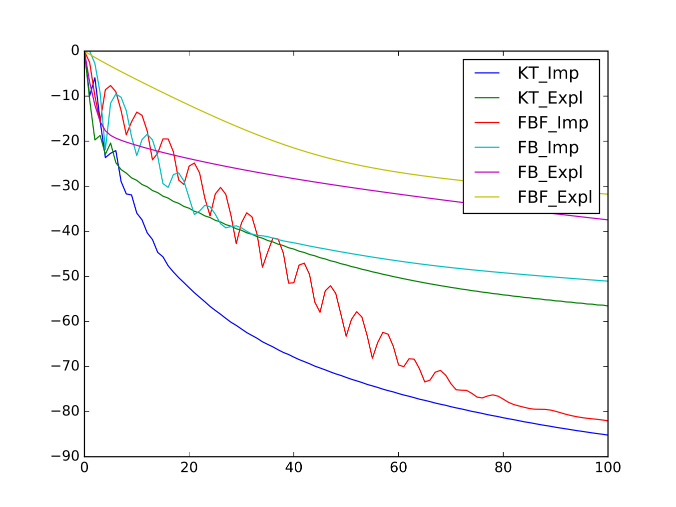
- FBF_Imp, FB_Imp, and KT_Imp: $I = \{1, 2, 3, 4\}$ and $J = \varnothing$
- FBF_Expl, FB_Expl, and KT_Expl: $I = \{1\}$ and $J = \{2, 3, 4\}$

Parameters

Set $\beta = \sqrt{\sum_{k\in K} \|L_k\|^2}$ and:

- FBF_Imp: $\gamma_n \equiv 0.99\beta$
- FBF_Expl: $\gamma_n \equiv 0.99\beta$
- FB_Imp: $\sigma_{1,n} \equiv 3/(2\beta)$, $\sigma_{2,n} \equiv 3/(2\beta)$, $\sigma_{3,n} \equiv 1/(10\beta)$, $\sigma_{4,n} \equiv 1/(10\beta)$, $\tau_n \equiv 1/\beta$, and $\lambda_n \equiv 1$
- FB_Expl: $\sigma_{1,n} \equiv 3/(2\beta)$, $\tau_n \equiv 1/(10\beta)$, and $\lambda_n \equiv 1$
- KT_Imp : $\gamma_n \equiv 0.4$, $\mu_{1,n} \equiv 1$, $\mu_{2,n} \equiv 1$, $\mu_{3,n} \equiv 1$,$\mu_{4,n} \equiv 3/2$, and $\lambda_n \equiv 1$
- KT_Expl : $\gamma_n \equiv 1.5$, $\mu_{1,n} \equiv 0.04$, $\mu_{2,n} \equiv 0.04$, $\mu_{3,n} \equiv 0.09$, $\mu_{4,n} \equiv 0.5$, and $\lambda_n \equiv 1$

Numerical results



Distance to respective solution

Numerical results

Original

Degraded 1

Degraded 2

Restored

Outlook

- Just like in the early 1960s the frontier separating linear from noninear problems was not a useful one, the current dichotomy between the class of convex/monotone problems and its complement (“everything else”) is not pertinent.
- One must define a structured extension of the remarkably efficient convexity/nonexpansiveness/monotonicity trio that would ideally enjoy similar rich connections. This is an extrememely challenging task.

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