FINDING BEST APPROXIMATION PAIRS RELATIVE TO TWO CLOSED CONVEX SETS IN HILBERT SPACES

Heinz H. Bauschke[*], Patrick L. Combettes[†], and D. Russell Luke[‡]

Abstract

We consider the problem of finding a best approximation pair, i.e., two points which achieve the minimum distance between two closed convex sets in a Hilbert space. When the sets intersect, the method under consideration, termed AAR for averaged alternating reflections, is a special instance of an algorithm due to Lions and Mercier for finding a zero of the sum of two maximal monotone operators. We investigate systematically the asymptotic behavior of AAR in the general case when the sets do not necessarily intersect and show that the method produces best approximation pairs provided they exist. Finitely many sets are handled in a product space, in which case the AAR method is shown to coincide with a special case of Spingarn's method of partial inverses.

Keywords: Best approximation pair, convex set, firmly nonexpansive map, Hilbert space, hybrid projection-reflection method, method of partial inverses, normal cone, projection, reflection, weak convergence.

1 Introduction

Throughout this paper,

(1) X is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$,

and

(2) A and B are two nonempty closed convex (possibly non-intersecting) sets in X.

[*]Department of Mathematics and Statistics, University of Guelph, Guelph, Ontario N1G 2W1, Canada. E-mail: `hbauschk@uoguelph.ca`.

[†]Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie – Paris 6, 75005 Paris, France. E-mail: `plc@math.jussieu.fr`.

[‡]The Pacific Institute for the Mathematical Sciences, Simon Fraser University, Burnaby, British Columbia V5A 1S6, Canada. E-mail: `rluke@cecm.sfu.ca`.

Let I denote the identity operator on X, and let P_A and P_B be the *projectors* (best approximation operators) onto A and B, respectively. Given a point $a \in X$, the standard best approximation problem relative to B is to [16]

$$\text{find } b \in B \quad \text{such that} \quad \|a - b\| = \inf \|a - B\|. \tag{3}$$

A natural extension of this problem is to find a *best approximation pair* relative to (A, B), i.e., to

$$\text{find } (a, b) \in A \times B \quad \text{such that} \quad \|a - b\| = \inf \|A - B\|. \tag{4}$$

If $A = \{a\}$, (4) reduces to (3) and its solution is $P_B a$. On the other hand, when the problem is consistent, i.e., $A \cap B \neq \varnothing$, then (4) reduces to the well-known convex feasibility problem for two sets [4, 13] and its solution set is $\big\{(x, x) \in X \times X \colon x \in A \cap B\big\}$. The formulation (4) captures a wide range of problems in applied mathematics and engineering [11, 24, 27, 30, 35].

The *method of alternating projections* applied to the sets A and B is perhaps the most straightforward algorithm to obtain a best approximation pair. It is described by the algorithm

$$\text{Take } x_0 \in X \text{ and set } (\forall n \in \mathbb{N}) \;\; x_n = (P_A P_B)^n x_0. \tag{5}$$

It was shown in [10, Theorem 2] that if A or B is compact, then the sequence $(x_n, P_B x_n)_{n\in\mathbb{N}}$ converges in norm to a best approximation pair. Best approximation pairs may not exist in general. However, if they do, then the sequence generated by (5) solves (4) in the sense that $(x_n, P_B x_n)_{n\in\mathbb{N}}$ converges weakly to some best approximation pair. This happens in particular when one of the sets is bounded [3, 11, 24].

While simple and elegant, the method of alternating projections can suffer from slow convergence, as theoretical [5, 21] and numerical [12] investigations have shown. We analyze an alternative strategy based on reflections rather than projections. Denote the *reflectors* with respect to A and B by $R_A = 2P_A - I$ and $R_B = 2P_B - I$ respectively, and consider the successive approximation method

$$\text{Take } x_0 \in X \text{ and set } (\forall n \in \mathbb{N}) \;\; x_n = T^n x_0, \tag{6}$$

where

$$T = \tfrac{1}{2}\big(R_A R_B + I\big). \tag{7}$$

In this algorithm, the sets A and B are activated in an alternating fashion through reflection operations. Since the update is formed by averaging the current iterate with the composition of the two reflections, we refer to (6)–(7) as the *Averaged Alternating Reflections (AAR)* method.

Let us now motivate this method from four different viewpoints. 1) If $A \cap B \neq \varnothing$, then the AAR method is a special case of a nonlinear variant of the Douglas-Rachford algorithm [17] proposed by Lions and Mercier in [26] to find a zero of the sum of two maximal monotone operators (in our setting, the normal cone maps of A and B). 2) In [7], we have used a relaxed version of (6)–(7), which we called the Hybrid Projection-Reflection (HPR) method, to solve the nonconvex phase retrieval problem in imaging. This algorithm was inspired by our attempt to use reliable convex optimization techniques as a basis to analyze current state-of-the-art techniques in phase retrieval [6]. Projection-type algorithms have been used in computational phase retrieval for over thirty years. During this period, an algorithm that is similar to the AAR method — known in the optics community as Hybrid Input-Output (HIO) — has emerged as the preferred algorithm for iterative phase retrieval; in contrast, alternating projections often do not converge to an acceptable neighborhood of the solution, and, even when they do, it can take over a thousand times the number of iterations required for HIO or other AAR-type algorithms [20]. 3) If B is the Cartesian product of finitely many halfspaces and A is the diagonal subspace of the corresponding product space, then the AAR method coincides with Spingarn's method of partial inverses for solving linear inequalities (see Section 4 for further details). On page 61 of [33], this method is reported to be more advantageous numerically than cyclic projections for certain problems. 4) The following simple example in the Euclidean plane illustrates a convex feasibility problem in which the AAR method exhibits better convergence behavior than the method of alternating projections. Let $A = \{(r,s) \in \mathbb{R}^2 : s \leq 0\}$, $B = \{(r,s) \in \mathbb{R}^2 : r \leq s\}$, and fix $x_0 = (8,4)$ as a starting point for the sequence $(x_n)_{n\in\mathbb{N}}$ generated the AAR method (6)–(7). Then $x_1 = (6,-2)$, $x_2 = (2,-4)$, $x_3 = (-1,-3)$, and $x_4 = x_n = (-2,-2)$, for every $n \geq 4$. Thus the AAR method finds the point $(-2,-2) \in A \cap B$ in four iterations. On the other hand, the sequence generated by the method of alternating projections (5) with the same starting point $(8,4)$ converges to $(0,0) \in A \cap B$, but not in finitely many steps.

We now recall the known convergence results for the AAR method.

Fact 1.1 *Suppose that $A \cap B \neq \varnothing$ and let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence generated by* (6)–(7)*. Then the following hold.*

(i) *$(x_n)_{n\in\mathbb{N}}$ converges weakly to some fixed point x of T and $P_B x \in A \cap B$.*

(ii) *The "shadow" sequence $(P_B x_n)_{n\in\mathbb{N}}$ is bounded and each of its weak cluster points belongs to $A \cap B$.*

Proof. See [26, Theorem 1] (specialized to the normal cone maps of A and B), or the more direct proof of [6, Fact 5.9]. ☐

The aim of this paper is to analyze completely the asymptotic behavior of the AAR method (6)–(7), covering in particular the case when $A \cap B = \varnothing$. In addition, we shall briefly explore extensions of our main results to the setting of finitely many sets.

The paper is organized as follows. We provide basic facts on the geometry of two closed convex sets in Section 2. In Section 3, we show that, for any sequence $(x_n)_{n\in\mathbb{N}}$ generated by (6)–(7), either $\|P_B x_n\| \to +\infty$ and (4) has no solution, or $\big((P_A R_B x_n, P_B x_n)\big)_{n\in\mathbb{N}}$ is bounded and its weak cluster points are solutions of (4). Additional results are presented for the case when A is a linear subspace. This is of relevance in Section 4, where finitely many sets are handled in a product space. We conclude by establishing a connection with Spingarn's method of partial inverses [32, 33, 34].

Notation. The closure of a set $C \subset X$ is denoted by $\overline{C}$ and its interior by $\operatorname{int} C$; its recession cone is $\operatorname{rec}(C) = \big\{x \in X : x + C \subset C\big\}$ (note that $\operatorname{rec}\varnothing = X$) and its normal cone map is given by

$$N_C : x \mapsto \begin{cases} \big\{u \in X : (\forall c \in C)\ \langle c - x, u\rangle \leq 0\big\}, & \text{if } x \in C; \\ \varnothing, & \text{otherwise.} \end{cases}$$

If C is a convex cone, its polar cone is $C^{\ominus} = \big\{x \in X : (\forall c \in C)\ \langle c, x\rangle \leq 0\big\}$ and $C^{\oplus} = -C^{\ominus}$. The range of an operator T is denoted by $\operatorname{ran} T$ (with closure $\overline{\operatorname{ran}}\, T$) and its fixed point set by $\operatorname{Fix} T$. Finally, $\rightharpoonup$ denotes weak convergence and $\mathbb{N}$ is the set of nonnegative integers.

2 The geometry of two closed convex sets

Recall (see [22, Theorem 12.1]) that an operator $\widetilde{T}$ from X to X is *firmly nonexpansive*, i.e.,

$$(8) \qquad (\forall x \in X)(\forall y \in X)\ \ \|\widetilde{T}x - \widetilde{T}y\|^2 + \|(I - \widetilde{T})x - (I - \widetilde{T})y\|^2 \leq \|x - y\|^2,$$

if and only if $\widetilde{R} = 2\widetilde{T} - I$ is *nonexpansive*, i.e.,

$$(9) \qquad (\forall x \in X)(\forall y \in X)\ \ \|\widetilde{R}x - \widetilde{R}y\| \leq \|x - y\|.$$

Fact 2.1 *Suppose that C is a nonempty closed convex set in X. Then, for every point $x \in X$, there exists a unique point $P_C x \in C$ such that $\|x - P_C x\| = \inf \|x - C\|$. The point $P_C x$ is characterized by*

$$(10) \qquad P_C x \in C \quad \text{and} \quad (\forall c \in C) \quad \langle c - P_C x, x - P_C x\rangle \leq 0.$$

The operator $P_C : X \to C : x \mapsto P_C x$ is called the projector *onto C; it is firmly nonexpansive and consequently, the reflector $R_C = 2P_C - I$ is nonexpansive.*

Proof. See [16, Theorems 4.1 and 5.5], [22, Chapter 12], [23, Propositions 3.5 and 11.2], or [36, Lemma 1.1]. ☐

Fact 2.2 *Suppose that C is a nonempty closed convex set in X. Then $\overline{\operatorname{ran}}\,(I - P_C) = \big(\operatorname{rec}(C)\big)^{\ominus}$.*

Proof. See [36, Theorem 3.1]. ☐

In order to study the geometry of the given two closed convex sets A and B, it is convenient to introduce the following objects, which we use throughout the paper:

$$(11) \qquad D = \overline{B - A}, \quad v = P_D(0), \quad E = A \cap (B - v), \quad \text{and} \quad F = (A + v) \cap B.$$

It follows at once from (10) that

$$(12) \qquad -v \in N_D(v).$$

Note also that if $A \cap B \neq \varnothing$, then $E = F = A \cap B$. However, even when $A \cap B = \varnothing$, the sets E and F may be nonempty and they serve as substitutes for the intersection. Indeed, $\|v\|$ measures the "gap" between the sets A and B.

Fact 2.3

(i) $\|v\| = \inf \|A - B\|$, *and the infimum is attained if and only if* $v \in B - A$.

(ii) $E = \mathrm{Fix}(P_A P_B)$ *and* $F = \mathrm{Fix}(P_B P_A)$.

(iii) $E + v = F$.

(iv) *If* $e \in E$ *and* $f \in F$, *then* $P_B e = P_F e = e + v$ *and* $P_A f = P_E f = f - v$.

(v) E *and* F *are nonempty provided one of the following conditions holds:*

- (a) $A \cap B \neq \varnothing$.
- (b) $B - A$ *is closed.*
- (c) A *or* B *is bounded.*
- (d) A *and* B *are polyhedral sets (intersections of finitely many halfspaces).*
- (e) $\mathrm{rec}(A) \cap \mathrm{rec}(B)$ *is a linear subspace, and* A *or* B *is locally compact.*

Proof. See [2, Section 5] and [3, Section 2]. ☐

Proposition 2.4 *Suppose that* $f \in F$ *and* $y \in N_D(v)$, *and set* $e = f - v \in E$. *Then the following hold.*

(i) $N_D(v) = N_B(f) \cap \big(-N_A(e)\big)$.

(ii) $P_B(f + y) = f$.

(iii) $P_A(e - y) = e$.

Proof. (i) follows from (11). (ii) and (ii) follow from (i) and (10). ☐

Proposition 2.5 *Suppose that* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *are sequences in* A *and* B, *respectively. Then*

$$b_n - a_n \to v \quad \Leftrightarrow \quad \|b_n - a_n\| \to \|v\|. \tag{13}$$

Now assume that $b_n - a_n \to v$. *Then the following hold.*

(i) $b_n - P_A b_n \to v$ *and* $P_B a_n - a_n \to v$.

(ii) *The weak cluster points of* $(a_n)_{n\in\mathbb{N}}$ *and* $(P_A b_n)_{n\in\mathbb{N}}$ *(resp.* $(b_n)_{n\in\mathbb{N}}$ *and* $(P_B a_n)_{n\in\mathbb{N}}$*) belong to* E *(resp.* F*). Consequently, the weak cluster points of the sequences*

$$\big((a_n, b_n)\big)_{n\in\mathbb{N}},\ \big((a_n, P_B a_n)\big)_{n\in\mathbb{N}},\ \big((P_A b_n, b_n)\big)_{n\in\mathbb{N}}$$

are best approximation pairs relative to (A, B).

(iii) *If* $E = \varnothing$ *(or, equivalently,* $F = \varnothing$*), then* $\min\big\{\|a_n\|, \|P_A b_n\|, \|b_n\|, \|P_B a_n\|\big\} \to +\infty$.

Proof. The implication "$\Rightarrow$" is clear. Conversely, let $(\forall n \in \mathbb{N})\ d_n = b_n - a_n \in B - A \subset \overline{B - A} = D$. It follows from (10) that $(\forall n \in \mathbb{N})\ \langle d_n - v, v\rangle \geq 0$. Hence

$$(\forall n \in \mathbb{N})\ \ \|d_n\|^2 - \|v\|^2 = \|d_n - v\|^2 + 2\langle d_n - v, v\rangle \geq \|d_n - v\|^2, \tag{14}$$

which proves (13). Assume for the remainder of the proof that $b_n - a_n \to v$ or, equivalently, $\|b_n - a_n\| \to \|v\|$. Since

$$\begin{aligned}(\forall n \in \mathbb{N})\ \ \|b_n - a_n\| &\geq \max\big\{\|b_n - P_A b_n\|, \|P_B a_n - a_n\|\big\}\\ &\geq \min\big\{\|b_n - P_A b_n\|, \|P_B a_n - a_n\|\big\}\\ &\geq \|v\|,\end{aligned}$$

we conclude that $\big(\|b_n - P_A b_n\|\big)_{n\in\mathbb{N}}$ and $\big(\|P_B a_n - a_n\|\big)_{n\in\mathbb{N}}$ both converge to $\|v\|$. As just proved, this now yields $b_n - P_A b_n \to v$ and $P_B a_n - a_n \to v$. Hence (i) holds. Let $a \in A$ be a weak cluster point of $(a_n)_{n\in\mathbb{N}}$, say $a_{k_n} \rightharpoonup a$. Then $b_{k_n} \rightharpoonup v + a \in B \cap (v + A) = F$. Hence $a \in A \cap (B - v) = E$. The remaining three sequences are treated similarly and thus (ii) is verified. Finally, (iii) is a direct consequence of (ii). □

Remark 2.6 Sequences conforming to the assumptions described in Proposition 2.5 can be generated by (5), upon rewriting it as

$$\text{Take } b_{-1} \in B \text{ and set } (\forall n \in \mathbb{N})\ \ a_n = P_A b_{n-1} \ \text{ and } \ b_n = P_B a_n. \tag{15}$$

Indeed, [3, Theorem 4.8] implies that $b_n - a_n \to v$ (see also [11]). This happens also for the iterates generated by *Dykstra's algorithm* [3, Theorem 3.8]. In Theorem 3.13 below, we shall see that the AAR method also gives rise to sequences with this behavior.

Corollary 2.7 $v \in \overline{(P_B - I)(A)} \cap \overline{(I - P_A)(B)} \subset (\operatorname{rec} B)^{\oplus} \cap (\operatorname{rec} A)^{\ominus}$.

Proof. In view of Proposition 2.5(i) and Remark 2.6,

$$v \in \overline{(P_B - I)(A)} \cap \overline{(I - P_A)(B)} \subset \overline{\operatorname{ran}}\,(P_B - I) \cap \overline{\operatorname{ran}}\,(I - P_A). \tag{16}$$

Now apply Fact 2.2. □

Remark 2.8 Corollary 2.7 can be refined in certain cases.

(i) First assume that $A = a + K$ and $B = b + L$, where K and L are closed convex cones. Then $\operatorname{rec}(A) = K$ and $\operatorname{rec}(B) = L$. Hence, by Corollary 2.7, $v \in L^{\oplus} \cap K^{\ominus}$. In fact, [2, Ex. 2.2] shows that

$$v = P_{L^{\oplus} \cap K^{\ominus}}(b - a).$$

(ii) Now assume that A is a closed *affine subspace*, say $A = a + K$, where K is a closed linear subspace. Then $K = A - A$ and hence

$$v \in (A - A)^{\perp}.$$

3 The Averaged Alternating Reflections (AAR) method

Let us start with a key observation concerning the operator $T = (R_A R_B + I)/2$.

Proposition 3.1 *T is firmly nonexpansive and defined on X.*

Proof. By Fact 2.1, the projectors P_A and P_B are firmly nonexpansive. As pointed out in the beginning of Section 2, the corresponding reflectors R_A and R_B are nonexpansive. It follows that $R_A R_B$ is nonexpansive as well and, hence, that T is firmly nonexpansive. ☐

Several fundamental results on firmly nonexpansive maps have been discovered over the past four decades. Specializing these to T, we obtain the following.

Fact 3.2 *Let $x_0 \in X$. Then:*

(i) $(T^n x_0 - T^{n+1} x_0)_{n\in\mathbb{N}}$ *converges in norm to the unique element of minimum norm in* $\overline{\operatorname{ran}}\,(I-T)$*;*

(ii) $\operatorname{Fix} T \neq \varnothing \Leftrightarrow (T^n x_0)_{n\in\mathbb{N}}$ *converges weakly to some point in* $\operatorname{Fix} T$*;*

(iii) $\operatorname{Fix} T = \varnothing \Leftrightarrow \|T^n x_0\| \to +\infty$.

Proof. (See also [9].) (i): [1, Corollary 2.3] and [29, Corollary 2] (ii): [28, Theorem 3]. (iii): [1, Corollary 2.2]. ☐

The following identities will be useful later.

Proposition 3.3 *Let $x \in X$. Then:*

(i) $x - Tx = P_B x - P_A R_B x$*;*

(ii) $\|x - Tx\|^2 = \|x - P_B x\|^2 + \langle x - P_A R_B x, R_B x - P_A R_B x\rangle$.

Proof. Indeed,

$$\begin{aligned} x - Tx &= x - \tfrac{1}{2}(R_A R_B x + x) = \tfrac{1}{2}(x - 2P_A R_B x + R_B x) = \tfrac{1}{2}(x - 2P_A R_B x + 2P_B x - x) \\ &= P_B x - P_A R_B x. \end{aligned}$$

Hence (i) holds, and we obtain further

$$\begin{aligned} \|x - Tx\|^2 &= \|P_B x - P_A R_B x\|^2 \\ &= \|P_B x - x\|^2 + \|x - P_A R_B x\|^2 + 2\langle x - P_A R_B x, P_B x - x\rangle \\ &= \|P_B x - x\|^2 + \|x - P_A R_B x\|^2 + \langle x - P_A R_B x, R_B x - x\rangle \\ &= \|P_B x - x\|^2 + \|x - P_A R_B x\|^2 + \langle x - P_A R_B x, (R_B x - P_A R_B x) - (x - P_A R_B x)\rangle \\ &= \|P_B x - x\|^2 + \langle x - P_A R_B x, R_B x - P_A R_B x\rangle, \end{aligned}$$

as announced in (ii). $\square$

Theorem 3.4 *The unique element of minimum norm in* $\overline{\operatorname{ran}}\,(I-T)$ *is* v.

Proof. It follows from Fact 3.2(i) that $\overline{\operatorname{ran}}\,(I-T)$ possesses a unique element of minimum norm, say w. We shall show that $w = v$. On the one hand, by Proposition 3.3(i), we have $\operatorname{ran}(I-T) \subset B-A$ and hence $w \in \overline{B-A} = D$. On the other hand, it follows from Proposition 3.3(ii) and (10) that, for every $a \in A$,

$$\|w\|^2 \leq \|a - Ta\|^2 = \|P_B a - a\|^2 + \langle a - P_A R_B a, R_B a - P_A R_B a\rangle \leq \|P_B a - a\|^2 = \inf \|B - a\|^2.$$

Hence $\|w\| \leq \inf \|B - A\|$ and, therefore, $w = P_D 0 = v$. $\square$

Theorem 3.5 *The set* $\operatorname{Fix}(T+v)$ *is closed and convex. Moreover,*

$$F + N_D(v) \subset \operatorname{Fix}(T+v) \subset v + F + N_D(v). \tag{17}$$

Proof. Since T is firmly nonexpansive, so is $T+v$. Hence $\operatorname{Fix}(T+v)$ is closed and convex (see, for instance, [22, Lemma 3.4] or [23, Proposition 5.3]). Now pick $f \in F$, $y \in N_D(v)$, and set $x = f + y$. By Proposition 2.4(ii), we have $P_B x = f$. Hence $R_B x = 2P_B x - x = 2f - (f+y) = f - y$. Now, let $e = f - v$. It follows from (12) that $y - v \in N_D(v)$. Therefore, using Proposition 2.4(iii), we obtain $P_A R_B x = P_A(f-y) = P_A(e-(y-v)) = e = f - v$. Hence $P_B x - P_A R_B x = f - (f-v) = v$. By Proposition 3.3(i), $x - Tx = P_B x - P_A R_B x = v$ and, in turn, $x = Tx + v$, i.e., $x \in \operatorname{Fix}(T+v)$. Thus,

$$F + N_D(v) \subset \operatorname{Fix}(T+v). \tag{18}$$

To establish the remaining inclusion, pick $x \in \operatorname{Fix}(T+v)$. Then $x - Tx = v$ or, equivalently (see Proposition 3.3), $P_B x - P_A R_B x = v$. Let $f = P_B x = v + P_A R_B x$ and $y = x - v - f$. Then $f \in B \cap (A+v) = F$ and $x = v + f + y$. It now suffices to show that $y \in N_D(v)$. To see this, pick $a \in A$ and $b \in B$. On the one hand, since $f = P_B x$, Fact 2.1 results in $\langle b - f, x - f\rangle \leq 0$. Using the definition of y, we write the last inequality equivalently as

$$\langle b - f, y + v\rangle \leq 0. \tag{19}$$

On the other hand, $P_A(2f - x) = P_A(2P_B x - x) = P_A R_B x = f - v$. Again using Fact 2.1, we deduce $\langle a - f + v, -y\rangle = \langle a - (f-v), (2f-x) - (f-v)\rangle \leq 0$. Hence

$$\langle f - a - v, y\rangle \leq 0. \tag{20}$$

Adding (19) and (20), we obtain $\langle b-a-v, y\rangle + \langle b-f, v\rangle \leq 0$. This inequality, (12), Proposition 2.4(ii), and Fact 2.1 now yield $\langle b-a-v, y\rangle \leq \langle b-f, -v\rangle = \langle b-f, (f-v)-f\rangle \leq 0$. We conclude that $y \in N_D(v)$. $\square$

Remark 3.6 A little care with (17) shows that $\operatorname{rec}(F) + N_D(v) \subset \operatorname{rec}(\operatorname{Fix}(T+v))$. In particular, if $F \neq \varnothing$, then $-v \in \operatorname{rec}(\operatorname{Fix}(T+v))$ (use (12)).

The next two examples illustrate that the bracketing given for $\operatorname{Fix}(T+v)$ in Theorem 3.5 is tight.

Example 3.7 Let $X = \mathbb{R}$, $A = \{0\}$, and $B = [1, +\infty[$. Then $D = B$, $v = 1$, $F = \{1\}$, and $\operatorname{Fix}(T + v) = F + N_D(v)$.

Example 3.8 Let $X = \mathbb{R}$, $A = [1, +\infty[$, and $B = \{0\}$. Then $D =]-\infty, -1]$, $v = -1$, $F = B$, and $\operatorname{Fix}(T + v) = v + F + N_D(v)$.

The following result, which improves upon [6, Fact A1], gives a complete description of $\operatorname{Fix} T$ in the consistent case.

Corollary 3.9 *Suppose that* $A \cap B \neq \varnothing$. *Then* $\operatorname{Fix} T = (A \cap B) + N_D(0)$ *and* $P_B(\operatorname{Fix} T) = A \cap B$.

Proof. Since $A \cap B \neq \varnothing$, we have $v = 0$ and $F = A \cap B$. The formula for $\operatorname{Fix} T$ (resp. $P_B(\operatorname{Fix} T)$) follows from Theorem 3.5 (resp. Proposition 2.4(ii)). ☐

Remark 3.10 We show that if the sets A and B do not "overlap sufficiently", then $\operatorname{Fix} T$ may be strictly larger than $A \cap B$. Indeed, let $X = \mathbb{R}$, $A = \{0\}$, and $B = [0, +\infty[$. Then $D = B$, $v = 0$, $F = \{0\} = A \cap B$, yet $\operatorname{Fix} T =]-\infty, 0]$. This simple example shows that iterating T alone may not yield a point in $A \cap B$. Hence it is important to monitor the "shadow sequence" $(P_B T^n x_0)_{n \in \mathbb{N}}$; see Fact 1.1 and Theorem 3.13 below.

Remark 3.11 If $0 \in \operatorname{int}(B - A)$ (*a fortiori* if the Slater-type condition $\big(A \cap \operatorname{int}(B)\big) \cup \big(B \cap \operatorname{int}(A)\big) \neq \varnothing$ holds), then $N_D(0) = \{0\}$ and consequently (Corollary 3.9) $\operatorname{Fix} T = A \cap B$.

Lemma 3.12 *Suppose that* $F \neq \varnothing$, *let* $y_0 \in \operatorname{Fix}(T + v)$ *and set* $y_n = T^n y_0$, *for all* $n \in \mathbb{N}$. *Then* $(y_n)_{n \in \mathbb{N}} = (y_0 - nv)_{n \in \mathbb{N}}$ *lies in* $\operatorname{Fix}(T + v)$. *Moreover,*

$$(21) \qquad (\forall n \in \mathbb{N}) \quad \|x_{n+1} - y_0 + (n + 1)v\|^2 + \|x_n - x_{n+1} - v\|^2 \leq \|x_n - y_0 + nv\|^2.$$

Proof. The proof proceeds by induction on n. Clearly, $y_0 - 0v = y_0 \in \operatorname{Fix}(T + v)$. Now assume that $y_n = y_0 - nv \in \operatorname{Fix}(T+v)$, for some $n \in \mathbb{N}$. Then $y_0 - nv = y_n = (T+v)(y_n) = Ty_n + v = y_{n+1} + v$ and hence $y_{n+1} = y_0 - (n+1)v$. Moreover, (17) is precisely what is needed to show that $y_{n+1} \in \operatorname{Fix}(T+v)$. Hence the claims regarding $(y_n)_{n \in \mathbb{N}}$ are proven. Next, (21) follows from the firm nonexpansiveness of T (Proposition 3.1) applied to x_n and $y_n = y_0 - nv$. ☐

Theorem 3.13 (Averaged Alternating Reflections (AAR) method) *Let* $x_0 \in X$ *and set* $x_n = T^n x_0$, *for all* $n \in \mathbb{N}$. *Then the following hold.*

(i) $x_n - x_{n+1} = P_B x_n - P_A R_B x_n \to v$ *and* $P_B x_n - P_A P_B x_n \to v$.

(ii) *If* $A \cap B \neq \varnothing$, *then* $(x_n)_{n \in \mathbb{N}}$ *converges weakly to a point in* $\operatorname{Fix}(T) = (A \cap B) + N_D(0)$; *otherwise,* $\|x_n\| \to +\infty$.

(iii) *Exactly one of the following two alternatives holds.*

(a) $F = \varnothing$, $\|P_B x_n\| \to +\infty$, *and* $\|P_A P_B x_n\| \to +\infty$.

(b) $F \neq \varnothing$, *the sequences* $(P_B x_n)_{n\in\mathbb{N}}$ *and* $(P_A P_B x_n)_{n\in\mathbb{N}}$ *are bounded, and their weak cluster points belong to* F *and* E*, respectively; in fact, the weak cluster points of*

$$\big((P_A R_B x_n, P_B x_n)\big)_{n\in\mathbb{N}} \quad \textit{and} \quad \big((P_A P_B x_n, P_B x_n)\big)_{n\in\mathbb{N}} \tag{22}$$

are best approximation pairs relative to (A, B).

Proof. (i): On the one hand, Proposition 3.3(i) yields

$$(\forall n \in \mathbb{N})\ \ x_n - x_{n+1} = x_n - Tx_n = P_B x_n - P_A R_B x_n. \tag{23}$$

On the other hand, Fact 3.2(i) and Theorem 3.4 imply

$$x_n - x_{n+1} = T^n x_0 - T^{n+1} x_0 \to v. \tag{24}$$

Altogether, we obtain the first claim and, by Proposition 2.5(i), $P_B x_n - P_A P_B x_n \to v$. (ii): This follows immediately from Fact 3.2(ii)&(iii) and Corollary 3.9. (iii): If $F = \varnothing$, then (i) and Proposition 2.5(iii) yield $\|P_B x_n\| \to +\infty$ and $\|P_A P_B x_n\| \to +\infty$. Now assume that $F \neq \varnothing$. We claim that $(P_B x_n)_{n\in\mathbb{N}}$ is bounded. Indeed, fix $f \in F \subset \operatorname{Fix}(T + v)$ (see Theorem 3.5). Repeated use of (21) (with $y_0 = f$) in Lemma 3.12 yields $\|x_n - (f - nv)\| \le \|x_0 - f\|$, for all $n \in \mathbb{N}$. Also, since $P_B(f - nv) = f$ (Proposition 2.4(ii)) and P_B is nonexpansive (Fact 2.1), we have

$$(\forall n \in \mathbb{N})\ \ \|P_B x_n - f\| = \|P_B x_n - P_B(f - nv)\| \le \|x_n - (f - nv)\| \le \|x_0 - f\|.$$

Hence $(P_B x_n)_{n\in\mathbb{N}}$ is bounded. The remaining statements regarding the weak cluster points now follow from (i) and Proposition 2.5(ii). ☐

Remark 3.14 The conclusions of Theorem 3.13 can be strengthened provided A or B has additional properties.

(i) Best approximation pairs exist and can be found as described in Theorem 3.13(iii)(b) whenever (at least) one of the conditions listed in Fact 2.3(v) is satisfied.

(ii) Suppose that best approximation pairs relative to (A, B) exist, i.e., $F \neq \varnothing$. If P_B is weakly continuous (as is the case when X is finite-dimensional or B is a closed affine subspace), then $\big((P_A R_B x_n, P_B x_n)\big)_{n\in\mathbb{N}}$ and $\big((P_A P_B x_n, P_B x_n)\big)_{n\in\mathbb{N}}$ both converge weakly to such a pair.

We shall discuss the important case when A is an affine or linear subspace in Theorem 3.17 and Proposition 3.19 below.

Remark 3.15 If $x_0 \in X$ and $y_0 \in \operatorname{Fix}(T + v)$, then (21) implies that $\big(\|T^n x_0 + nv - y_0\|\big)_{n\in\mathbb{N}}$ is decreasing. Consequently, $(T^n x_0 + nv)_{n\in\mathbb{N}}$ is *Fejér monotone* with respect to $\operatorname{Fix}(T + v)$. In certain settings, Fejér monotonicity sheds further light on the behavior of the sequence $(T^n x_0 + nv)_{n\in\mathbb{N}}$. For instance, if $\operatorname{int}\operatorname{Fix}(T + v) \neq \varnothing$, then $(T^n x_0 + nv)_{n\in\mathbb{N}}$ must converge in norm. See [4, 14] for this and further properties.

Remark 3.16 Pick $x_0 \in X$ and set $x_n = T^n x_0$, for every $n \in \mathbb{N}$.

(i) Theorem 3.13(i) states that $P_B x_n - P_A P_B x_n \to v$. Hence, using Fact 2.3(i), $(\delta_n)_{n\in\mathbb{N}} = \big(\|P_B x_n - P_A P_B x_n\|^2\big)_{n\in\mathbb{N}}$ converges to $\|v\|^2 = \inf \|A - B\|^2$, the (squared) gap between A and B. In [7, Section 4], a normalized version of δ_n was employed as a stopping criterion and error measure in an application of the AAR method to image processing.

(ii) By [29, Corollary 2], $x_n/n \to -v$. Hence, one can monitor the value of $\|x_n/n\|$ during the execution of the AAR method as an approximation of the gap $\|v\|$.

Theorem 3.17 (when A is an affine subspace) *Suppose that A is a closed affine subspace and $x_0 \in X$. Let $x_n = T^n x_0$, for all $n \in \mathbb{N}$. Then*

$$P_B x_n - P_A x_n \to v. \tag{25}$$

If $F \neq \varnothing$, then $(P_A x_n)_{n\in\mathbb{N}}$ is bounded and its weak cluster points belong to E. If furthermore $A \cap B \neq \varnothing$, then $(x_n)_{n\in\mathbb{N}}$ converges weakly to some point $x \in (A \cap B) + N_D(0)$. Moreover, $(P_A x_n)_{n\in\mathbb{N}}$ and $(P_B x_n)_{n\in\mathbb{N}}$ converge weakly to $P_A x \in A \cap B$.

Proof. Since A is an affine subspace, P_A is an affine operator. It follows that $P_A R_B = P_A(P_B + P_B - I) = P_A P_B + P_A P_B - P_A = 2P_A P_B - P_A$ and hence $P_B - P_A R_B = P_B + P_A - 2P_A P_B = 2(P_B - P_A P_B) + P_A - P_B$. In turn, this implies

$$P_B - P_A = 2(P_B - P_A P_B) - (P_B - P_A R_B). \tag{26}$$

Now apply (26) to $(x_n)_{n\in\mathbb{N}}$, invoke Theorem 3.13(i), and deduce that $P_B x_n - P_A x_n \to v$. If $F \neq \varnothing$, then $(P_B x_n)_{n\in\mathbb{N}}$ is bounded (Theorem 3.13(iii)(b)). Consequently, (25) implies that $(P_A x_n)_{n\in\mathbb{N}}$ is bounded and that every weak cluster point of $(P_A x_n)_{n\in\mathbb{N}}$ belongs to E (Proposition 2.5(ii)). Now assume that $A \cap B \neq \varnothing$, whence $v = 0$ and $E = F = A \cap B$. It follows from Theorem 3.13(ii) that $x_n \rightharpoonup x \in (A \cap B) + N_D(0)$. Since P_A is weakly continuous, we have $P_A x_n \rightharpoonup P_A x$. By (25) and the weak closedness of B, we conclude that $P_B x_n \rightharpoonup P_A x \in A \cap B$. ☐

Remark 3.18 The convergence statement (25) need not hold if A is not an affine subspace: indeed, if $x_0 = 0$ in Example 3.8, then $P_B x_n - P_A x_n = -\max\{1, n\} \to -\infty$.

When A is a linear subspace, an additional property complements the results of Theorem 3.17.

Proposition 3.19 (when A is a linear subspace) *Suppose that A is a closed linear subspace. Then $P_A\big(\operatorname{Fix}(T + v)\big) = E$. If $A \cap B \neq \varnothing$, then $P_A\big(\operatorname{Fix}(T)\big) = A \cap B$.*

Proof. In view of Theorem 3.5 and Fact 2.3(iii), we may assume that $E \neq \varnothing$. Pick $e \in E$. Adding $A^\perp$ to (17) yields

$$F + N_D(v) + A^\perp \subset \operatorname{Fix}(T + v) + A^\perp \subset v + F + N_D(v) + A^\perp. \tag{27}$$

On the other hand, $v \in A^\perp$ (Remark 2.8(ii)) and $N_D(v) \subset -N_A(e) = A^\perp$ (Proposition 2.4(i)). Hence (27) implies $F + A^\perp = \mathrm{Fix}(T+v)$. Together with Fact 2.3(iv), this yields $P_A\big(\mathrm{Fix}(T+v)\big) = E$. Now suppose $A \cap B \neq \varnothing$. By (11), $v = 0$ and $E = A \cap B$. Therefore, $P_A\big(\mathrm{Fix}(T+v)\big) = E$ becomes $P_A\big(\mathrm{Fix}(T)\big) = A \cap B$. ☐

We conclude this section with another special case.

Remark 3.20 Suppose that A is an *obtuse cone*, i.e., $A^\oplus \subset A$. Pick $x_0 \in A$ and set $x_n = T^n x_0$, for all $n \in \mathbb{N}$. Since $\mathrm{ran}\, R_A = A$ [8], the entire sequence $(x_n)_{n\in\mathbb{N}}$ lies in A.

4 Finitely many sets

In this final section, we show how one can adapt the two-set results of Section 3 to problems with finitely many sets. We assume that

(28) $$C_1, \ldots, C_J \text{ are finitely many nonempty closed convex sets in } X.$$

The following product space technique was first introduced in [31]. Pick $(\lambda_j)_{1\leq j\leq J}$ in $]0,1]$ such that $\sum_{j=1}^J \lambda_j = 1$ and denote by $\mathbf{X}$ the Hilbert space obtained by equipping the Cartesian product X^J with the inner product $\big((x_j)_{1\leq j\leq J}, (y_j)_{1\leq j\leq J}\big) \mapsto \sum_{j=1}^J \lambda_j \langle x_j, y_j\rangle$. Let

(29) $$\mathbf{A} = \big\{(x,\ldots,x) \in \mathbf{X} \colon x \in X\big\} \quad \text{and} \quad \mathbf{B} = C_1 \times \cdots \times C_J.$$

Then the set $\bigcap_{j=1}^J C_j$ in X corresponds to the set $\mathbf{A} \cap \mathbf{B}$ in $\mathbf{X}$. Moreover, the projections of $\mathbf{x} = (x_j)_{1\leq j\leq J} \in \mathbf{X}$ onto $\mathbf{A}$ and $\mathbf{B}$ are given by

(30) $$P_{\mathbf{A}}\mathbf{x} = \big(\textstyle\sum_{j=1}^J \lambda_j x_j, \ldots, \sum_{j=1}^J \lambda_j x_j\big) \quad \text{and} \quad P_{\mathbf{B}}\mathbf{x} = (P_{C_1}x_1, \ldots, P_{C_J}x_J),$$

respectively. By analogy with (11), we now set

(31) $$\mathbf{D} = \overline{\mathbf{B} - \mathbf{A}}, \quad \mathbf{v} = P_{\mathbf{D}}(\mathbf{0}), \quad \mathbf{E} = \mathbf{A} \cap (\mathbf{B} - \mathbf{v}), \quad \text{and} \quad \mathbf{F} = (\mathbf{A} + \mathbf{v}) \cap \mathbf{B}.$$

Then a point $(e,\ldots,e) \in \mathbf{X}$ belongs to $\mathbf{E}$ if and only if e minimizes the proximity function

(32) $$x \mapsto \textstyle\sum_{j=1}^J \lambda_j \|x - P_{C_j}x\|^2$$

or, equivalently, if $e \in \mathrm{Fix} \sum_{j=1}^J \lambda_j P_{C_j}$ (see [3, 11, 15] for details). Further, let

(33) $$\mathbf{T} = \tfrac{1}{2}(R_{\mathbf{A}} R_{\mathbf{B}} + \mathbf{I}),$$

fix $\mathbf{x}_0 \in \mathbf{A}$, and set $\mathbf{x}_n = \mathbf{T}^n \mathbf{x}_0$, for all $n \in \mathbb{N}$. Then we obtain the AAR method in $\mathbf{X}$ for the two sets $\mathbf{A}$ and $\mathbf{B}$ and, as seen in Remark 3.10, the pertinent sequence to monitor is the "shadow sequence" $(P_{\mathbf{B}}\mathbf{x}_n)_{n\in\mathbb{N}}$. The results of Section 3 can be applied to this product space setting which, in turn, yield new convergence results for algorithms operating in the original space X via (30). Rather than detailing these counterparts, we shall bring to light a particularly interesting connection with Spingarn's method of partial inverses [32] (see also [18, 19, 25]).

Remark 4.1 (Spingarn's method of partial inverses) Since $\mathbf{A}$ is a closed linear subspace, Theorem 3.17 is applicable and one can thus monitor the sequence $(P_{\mathbf{B}}\mathbf{x}_n)_{n\in\mathbb{N}}$ or the sequence $(P_{\mathbf{A}}\mathbf{x}_n)_{n\in\mathbb{N}}$. The latter corresponds precisely to Spingarn's *method of partial inverses* for finding a zero of $\sum_{j=1}^J \lambda_j N_{C_j} = \sum_{j=1}^J N_{C_j}$, i.e., for finding a point in $\bigcap_{j=1}^J C_j$; see [32, Section 6]. It is noteworthy that the main convergence result of Spingarn [32, Corollary 5.1] in this setting can also be deduced from Theorem 3.13 and Proposition 3.19.

Spingarn analyzed further the case when X is a Euclidean space and each set C_j is a halfspace in [33] and [34]. Specifically, he proved that $\mathbf{F} \neq \varnothing$ (this can also be deduced from Fact 2.3(v)(d)), that $(P_{\mathbf{A}}\mathbf{x}_n)_{n\in\mathbb{N}}$ converges linearly to some point in $\mathbf{E}$ [34, Theorems 1 and 2], and that convergence occurs in finitely many steps provided that $\operatorname{int}\bigcap_{j=1}^J C_j \neq \varnothing$ [33, Theorem 2].

Acknowledgments

H. H. Bauschke's work was supported in part by the Natural Sciences and Engineering Research Council of Canada and by a NATO grant. The work of P. L. Combettes was supported in part by a NATO grant. D. R. Luke's work was supported in part by a Postdoctoral Fellowship with the Pacific Institute for the Mathematical Sciences, and in part by the Institute for Numerical and Applied Mathematics at the University of Göttingen.

References

[1] J. B. Baillon, R. E. Bruck, and S. Reich, On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces, *Houston J. Math.,* vol. 4, pp. 1–9, 1978.

[2] H. H. Bauschke and J. M. Borwein, On the convergence of von Neumann's alternating projection algorithm for two sets, *Set-Valued Anal.,* vol. 1, pp. 185–212, 1993.

[3] H. H. Bauschke and J. M. Borwein, Dykstra's alternating projection algorithm for two sets, *J. Approx. Theory,* vol. 79, pp. 418–443, 1994.

[4] H. H. Bauschke and J. M. Borwein, On projection algorithms for solving convex feasibility problems, *SIAM Rev.,* vol. 38, pp. 367–426, 1996.

[5] H. H. Bauschke, J. M. Borwein, and A. S. Lewis, The method of cyclic projections for closed convex sets in Hilbert space, in *Recent Developments in Optimization Theory and Nonlinear Analysis*, pp. 1–38, Contemp. Math., vol. 204, Amer. Math. Soc., Providence, RI, 1997.

[6] H. H. Bauschke, P. L. Combettes, and D. R. Luke, Phase retrieval, error reduction algorithm, and Fienup variants: A view from convex optimization, *J. Opt. Soc. Amer. A,* vol. 19, pp. 1334–1345, 2002.

[7] H. H. Bauschke, P. L. Combettes, and D. R. Luke, Hybrid projection-reflection method for phase retrieval, *J. Opt. Soc. Amer. A,* vol. 20, pp. 1025–1034, 2003.

[8] H. H. Bauschke and S. G. Kruk, Reflection-projection method for convex feasibility problems with an obtuse cone, *J. Optim. Theory Appl.,* to appear.

[9] R. E. Bruck and S. Reich, Nonexpansive projections and resolvents of accretive operators in Banach spaces, *Houston J. Math.,* vol. 3, pp. 459–470, 1977.

[10] W. Cheney and A. A. Goldstein, Proximity maps for convex sets, *Proc. Amer. Math. Soc.,* vol. 10, pp. 448–450, 1959.

[11] P. L. Combettes, Inconsistent signal feasibility problems: Least-squares solutions in a product space, *IEEE Trans. Signal Process.,* vol. 42, pp. 2955–2966, 1994.

[12] P. L. Combettes, Convex set theoretic image recovery by extrapolated iterations of parallel subgradient projections, *IEEE Trans. Image Process.*, vol. 6, pp. 493–506, 1997.

[13] P. L. Combettes, Hilbertian convex feasibility problem: Convergence of projection methods, *Appl. Math. Optim.* vol. 35, pp. 311–330, 1997.

[14] P. L. Combettes, Quasi-Fejérian analysis of some optimization algorithms, in *Inherently Parallel Algorithms for Feasibility and Optimization* (D. Butnariu, Y. Censor, and S. Reich, Eds.), pp. 115–152. New York: Elsevier, 2001.

[15] A. R. De Pierro and A. N. Iusem, A parallel projection method for finding a common point of a family of convex sets, *Pesquisa Operacional,* vol. 5, pp. 1-20, 1985.

[16] F. Deutsch, *Best Approximation in Inner Product Spaces*, New York: Springer-Verlag, 2001.

[17] J. Douglas and H. H. Rachford, On the numerical solution of heat conduction problems in two or three space variables, *Trans. Amer. Math. Soc.,* vol. 82, pp. 421–439, 1956.

[18] J. Eckstein, *Splitting Methods for Monotone Operators with Applications to Parallel Optimization*, PhD thesis, Department of Civil Engineering, Massachusetts Institute of Technology, 1989.

[19] J. Eckstein and D. P. Bertsekas, On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators, *Math. Programming (Series A),* vol. 55, pp. 293–318, 1992.

[20] J. R. Fienup, Phase retrieval algorithms: a comparison, *Appl. Opt.,* vol. 21, pp. 2758–2769, 1982.

[21] C. Franchetti and W. Light, On the von Neumann alternating algorithm in Hilbert space, *J. Math. Anal. Appl.,* vol. 114, pp. 305–314, 1986.

[22] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory,* Cambridge: Cambridge University Press, 1990.

[23] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings,* New York: Marcel Dekker, 1984.

[24] M. Goldburg and R. J. Marks II, Signal synthesis in the presence of an inconsistent set of constraints, *IEEE Trans. Circuits Syst.,* vol. 32, pp. 647–663, 1985.

[25] B. Lemaire, The proximal algorithm, in *New methods in Optimization and Their Industrial Uses,* (J. P. Penot, Ed.), International Series of Numerical Mathematics, vol. 87, pp. 73–87, Boston, MA: Birkhäuser, 1989.

[26] P. L. Lions and B. Mercier, Splitting algorithms for the sum of two nonlinear operators, *SIAM J. Numer. Anal.,* vol. 16, pp. 964–979, 1979.

[27] B. Mercier, *Inéquations Variationnelles de la Mécanique* (Publications Mathématiques d'Orsay, no. 80.01). Orsay, France, Université de Paris-XI, 1980.

[28] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.,* vol. 73, pp. 591-597, 1967.

[29] A. Pazy, Asymptotic behavior of contractions in Hilbert space, *Israel J. Math.,* vol. 9, pp. 235–240, 1971.

[30] J. C. Pesquet and P. L. Combettes, Wavelet synthesis by alternating projections, *IEEE Trans. Signal Process.,* vol. 44, pp. 728–732, 1996.

[31] G. Pierra, Eclatement de contraintes en parallèle pour la minimisation d'une forme quadratique, in *Lecture Notes in Computer Science*, vol. 41, Springer-Verlag, New York, pp. 200–218, 1976.

[32] J. E. Spingarn, Partial inverse of a monotone operator, *Appl. Math. Optim.,* vol. 10, pp. 247–265, 1983.

[33] J. E. Spingarn, A primal-dual projection method for solving systems of linear inequalities, *Linear Algebra Appl.*, vol. 65, pp. 45–62, 1985.

[34] J. E. Spingarn, A projection method for least-squares solutions to overdetermined systems of linear inequalities, *Linear Algebra Appl.*, vol. 86, pp. 211–236, 1987.

[35] D. C. Youla and V. Velasco, Extensions of a result on the synthesis of signals in the presence of inconsistent constraints, *IEEE Trans. Circuits Syst.,* vol. 33, pp. 465–468, 1986.

[36] E. H. Zarantonello, Projections on convex sets in Hilbert space and spectral theory, in *Contributions to Nonlinear Functional Analysis,* (E. H. Zarantonello, Ed.) pp. 237–424. New York: Academic Press, 1971.