Functions with Prescribed Best Linear Approximations

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Abstract

A common problem in applied mathematics is that of finding a function in a Hilbert space with prescribed best approximations from a finite number of closed vector subspaces. In the present paper we study the question of the existence of solutions to such problems. A finite family of subspaces is said to satisfy the *Inverse Best Approximation Property (IBAP)* if there exists a point that admits any selection of points from these subspaces as best approximations. We provide various characterizations of the IBAP in terms of the geometry of the subspaces. Connections between the IBAP and the linear convergence rate of the periodic projection algorithm for solving the underlying affine feasibility problem are also established. The results are applied to investigate problems in harmonic analysis, integral equations, signal theory, and wavelet frames.

1 Introduction

A classical problem arising in areas such as harmonic analysis, optics, and signal theory is that of finding a function $x \in L^2(\mathbb{R}^N)$ with prescribed values on subsets of the space (or time) and Fourier domains [10, 20, 23, 29, 37, 36]. In geometrical terms, this problem can be abstracted into that of finding a function possessing prescribed best approximations from two closed vector subspaces of $L^2(\mathbb{R}^N)$ [39]. More generally, a broad range of problems in applied mathematics can be formulated as follows: given m closed vector subspaces $(U_i)_{1 \leq i \leq m}$ of a (real or complex) Hilbert space $\mathcal{H}$,

$$\text{find } x \in \mathcal{H} \text{ such that } (\forall i \in \{1, \dots, m\}) \quad P_i x = u_i, \tag{1.1}$$

where, for every $i \in \{1, \dots, m\}$, P_i is the (metric) projector onto U_i and $u_i \in U_i$. In connection with (1.1), a central question is whether a solution exists, irrespective of the choice of the prescribed best linear approximations $(u_i)_{1\leq i\leq m}$. The main objective of the present paper is to address this question.

Definition 1.1 Let $(U_i)_{1\leq i\leq m}$ be a family of closed vector subspaces of $\mathcal{H}$ and let $(P_i)_{1\leq i\leq m}$ denote their respective projectors. Then $(U_i)_{1\leq i\leq m}$ satisfies the *inverse best approximation property (IBAP)* if

$$\big(\forall (u_i)_{1\leq i\leq m}\in \times_{i=1}^m U_i\big)\big(\exists\, x\in\mathcal{H}\big)\big(\forall i\in\{1,\ldots,m\}\big)\quad P_i x=u_i. \tag{1.2}$$

Moreover, for every $(u_i)_{1\leq i\leq m}\in \times_{i=1}^m U_i$, we set

$$S(u_1,\ldots,u_m)=\bigcap_{i=1}^m \big\{x\in\mathcal{H}\;\big|\; P_i x=u_i\big\}, \tag{1.3}$$

and, for every $i\in\{0,\ldots,m-1\}$,

$$U_{i+}=\sum_{j=i+1}^m U_j\,,\quad P_{i+}=P_{\overline{U_{i+}}}\,,\quad\text{and}\quad P_{i+}^{\perp}=P_{U_{i+}^{\perp}}. \tag{1.4}$$

The paper is organized as follows. In Section 2, we first show that the linear independence of the subspaces $(U_i)_{1\leq i\leq m}$ is necessary for satisfying the IBAP, but that it is not sufficient in infinite dimensional spaces. The main result of Section 2 is Theorem 2.8, which provides various characterizations of the IBAP. Several corollaries are derived and, in particular, we obtain in Proposition 2.10 conditions for the consistency of affine feasibility problems. In Section 3, we discuss minimum norm solutions and establish connections between the IBAP and the rate of convergence of the periodic projection algorithm for solving (1.1). Finally, Section 4 is devoted to applications to systems of integral equations, constrained moment problems, harmonic analysis, wavelet frames, and signal recovery.

Remark 1.2 Since best approximations are well defined for nonempty closed convex subsets of $\mathcal{H}$, the IBAP could be considered in this more general context. However, useful results can be expected to be scarce, even for two closed convex cones K_1 and K_2. Indeed, denote the projectors onto K_1 and K_2 by P_1 and P_2, respectively. If k_1 is a point on the boundary of K_1 which is not a support point of K_1 (by the Bishop-Phelps theorem [33, Theorem 3.18(i)] support points are dense in the boundary of K_1), then the only point $x\in\mathcal{H}$ such that $P_1x=k_1$ is $x=k_1$. Therefore, there is no point $x\in\mathcal{H}$ such that $P_1x=k_1$ and $P_2x=k_2$ unless $k_2=P_2k_1$, which means that the IBAP does not hold. Let us add that, even if every boundary point of K_1 is a support point (e.g., the interior of K_1 is nonempty or $\mathcal{H}$ is finite dimensional), the IBAP can also trivially fail: take for instance $\mathcal{H}=\mathbb{R}^2$, $K_1=[0,+\infty[\times[0,+\infty[$, $K_2=\big\{(\beta,-\beta)\;\big|\;\beta\in\mathbb{R}\big\}$, $k_1=(0,1)$, and $k_2=(1,-1)$.

Throughout, $\mathcal{H}$ is a real or complex Hilbert space with scalar product $\langle\cdot\mid\cdot\rangle$ and norm $\|\cdot\|$. The distance to a closed affine subspace S of $\mathcal{H}$ is denoted by d_S, and its projector by P_S. Moreover, $(U_i)_{1\leq i\leq m}$ is a fixed family of closed vector subspaces of $\mathcal{H}$ with respective projectors $(P_i)_{1\leq i\leq m}$. Finally, $\mathbb{N}=\{0,1,\ldots\}$ denotes the set of natural numbers.

2 Characterizations of the inverse best approximation property

We first record some useful descriptions of the set of solutions to (1.1).

Proposition 2.1 *Let $(u_i)_{1\leq i\leq m} \in \times_{i=1}^m U_i$. Then the following hold.*

(i) $S(u_1,\ldots,u_m) = \bigcap_{i=1}^m (u_i + U_i^\perp)$.

(ii) *Let $x \in S(u_1,\ldots,u_m)$. Then $S(u_1,\ldots,u_m) = x + \bigcap_{i=1}^m U_i^\perp$.*

Proof. (i): Let $x \in \mathcal{H}$ and $i \in \{1,\ldots,m\}$. The projection theorem asserts that $P_i x = u_i \Leftrightarrow x - u_i \in U_i^\perp \Leftrightarrow x \in u_i + U_i^\perp$. Hence, (1.3) yields $x \in S(u_1,\ldots,u_m) \Leftrightarrow x \in \bigcap_{i=1}^m (u_i + U_i^\perp)$.

(ii): Let $y \in \mathcal{H}$. By the linearity of the operators $(P_i)_{1\leq i\leq m}$, $y \in S(u_1,\ldots,u_m) \Leftrightarrow (\forall i \in \{1,\ldots,m\})\ P_i(y-x) = 0 \Leftrightarrow (\forall i \in \{1,\ldots,m\})\ y - x \in U_i^\perp \Leftrightarrow y \in x + \bigcap_{i=1}^m U_i^\perp$. ☐

The main objective of this section is to provide characterizations of the inverse best approximation property. Let us start with a necessary condition.

Proposition 2.2 *Let $(u_i)_{1\leq i\leq m} \in (\times_{i=1}^m U_i) \smallsetminus \{(0,\ldots,0)\}$ be such that $\sum_{i=1}^m u_i = 0$. Then $S(u_1,\ldots,u_m) = \varnothing$.*

Proof. Suppose that $x \in S(u_1,\ldots,u_m)$. Then, for every $i \in \{1,\ldots,m\}$, $u_i = P_i x$ and therefore $\langle u_i \mid x - u_i\rangle = 0$, i.e., $\|u_i\|^2 = \langle u_i \mid x\rangle$. Hence $0 < \sum_{i=1}^m \|u_i\|^2 = \langle \sum_{i=1}^m u_i \mid x\rangle = 0$, and we reach a contradiction. ☐

Recall that the subspaces $(U_i)_{1\leq i\leq m}$ are linearly independent if [24, Definition 6.6]

$$\big(\forall (u_i)_{1\leq i\leq m} \in \times_{i=1}^m U_i\big) \quad \sum_{i=1}^m u_i = 0 \quad \Rightarrow \quad (\forall i \in \{1,\ldots,m\}) \quad u_i = 0. \tag{2.1}$$

Using the notation introduced in (1.4), this property is characterized by [24, Lemma 6.6]

$$(\forall i \in \{1,\ldots,m-1\}) \quad U_i \cap U_{i+} = \{0\}. \tag{2.2}$$

Corollary 2.3 *Suppose that $(U_i)_{1\leq i\leq m}$ satisfies the inverse best approximation property. Then the subspaces $(U_i)_{1\leq i\leq m}$ are linearly independent.*

As the following example shows, the linear independence of the subspaces $(U_i)_{1\leq i\leq m}$ is not sufficient to guarantee the inverse best approximation property.

Example 2.4 Suppose that $\mathcal{H}$ is separable, let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$, let $(\alpha_n)_{n\in\mathbb{N}}$ be a square-summable sequence in $]0,+\infty[$, and set $(\forall n \in \mathbb{N})\ f_n = (e_{2n} + \alpha_n e_{2n+1})/\sqrt{1+\alpha_n^2}$. Set $m = 2$,

$$U_1 = \overline{\operatorname{span}}\,\{e_{2n}\}_{n\in\mathbb{N}},\ U_2 = \overline{\operatorname{span}}\,\{f_n\}_{n\in\mathbb{N}},\ u_1 = 0, \text{ and } u_2 = \sum_{n\in\mathbb{N}} \alpha_n f_n. \tag{2.3}$$

Then $U_1 \cap U_2 = \{0\}$ and $S(u_1,u_2) = \varnothing$.

Proof. By construction, $(e_{2n})_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ are orthonormal bases of U_1 and U_2, respectively. It follows easily that $U_1 \cap U_2 = \{0\}$. Now suppose that there exists a vector $x \in \mathcal{H}$ such that $P_1 x = u_1$ and $P_2 x = u_2$. Then the identities $\sum_{n\in\mathbb{N}} \langle x \mid e_{2n}\rangle e_{2n} = P_1 x = u_1 = 0$ imply that

$$(\forall n \in \mathbb{N}) \quad \langle x \mid e_{2n}\rangle = 0. \tag{2.4}$$

Hence, it results from the identities $\sum_{n\in\mathbb{N}} \alpha_n f_n = u_2 = P_2 x = \sum_{n\in\mathbb{N}} \langle x \mid f_n\rangle f_n$ that

$$(\forall n \in \mathbb{N}) \quad \alpha_n = \langle x \mid f_n\rangle = \frac{\alpha_n}{\sqrt{1+\alpha_n^2}} \langle x \mid e_{2n+1}\rangle. \tag{2.5}$$

Therefore, $\inf_{n\in\mathbb{N}} \langle x \mid e_{2n+1}\rangle = \inf_{n\in\mathbb{N}} \sqrt{1+\alpha_n^2} = 1$, which is impossible. □

The next result states that linear independence is necessary and sufficient for obtaining an approximate inverse best approximation property.

Proposition 2.5 *The following are equivalent.*

(i) *The subspaces $(U_i)_{1\leq i\leq m}$ are linearly independent.*

(ii) *For every $(u_i)_{1\leq i\leq m} \in \times_{i=1}^m U_i$ and every $\varepsilon \in \,]0,+\infty[$, there exists $x \in \mathcal{H}$ such that*

$$\max_{1\leq i\leq m} \|P_i x - u_i\| \leq \varepsilon. \tag{2.6}$$

Proof. Set $V = \big\{(P_i x)_{1\leq i\leq m} \;\big|\; x \in \mathcal{H}\big\}$ and let W be the orthogonal complement of V in the Hilbert direct sum $\bigoplus_{i=1}^m U_i$.

(i)⇒(ii): Take $(u_i)_{1\leq i\leq m} \in W$ and set $x = \sum_{i=1}^m u_i$. Then $\sum_{i=1}^m \langle u_i \mid x\rangle = \sum_{i=1}^m \langle u_i \mid P_i x\rangle = 0$, which implies that $\|x\|^2 = \sum_{i=1}^m \langle u_i \mid x\rangle = 0$. Hence $x = 0$ and, in view of the assumption of independence, we conclude that $(\forall i \in \{1,\ldots,m\})\ u_i = 0$. Therefore, V is dense in $\bigoplus_{i=1}^m U_i$.

(ii)⇒(i): Take $(u_i)_{1\leq i\leq m} \in \times_{i=1}^m U_i$ such that $\sum_{i=1}^m u_i = 0$, take $\varepsilon \in \,]0,+\infty[$, and take $x \in \mathcal{H}$ such that (2.6) holds. Then $\sum_{i=1}^m \langle u_i \mid P_i x\rangle = \sum_{i=1}^m \langle u_i \mid x\rangle = 0$ and therefore

$$\begin{aligned}\sum_{i=1}^m \|u_i\|^2 &= \sum_{i=1}^m \|u_i - P_i x\|^2 + 2\mathrm{Re}\sum_{i=1}^m \langle u_i - P_i x \mid P_i x\rangle + \sum_{i=1}^m \|P_i x\|^2\\ &= \sum_{i=1}^m \|u_i - P_i x\|^2 - \sum_{i=1}^m \|P_i x\|^2\\ &\leq m\varepsilon^2. \end{aligned} \tag{2.7}$$

Hence, $(\forall i \in \{1,\ldots,m\})\ u_i = 0$. □

In order to provide characterizations of the inverse best approximation property, we require the following tools.

Definition 2.6 [18, Definition 9.4] Let U and V be closed vector subspaces of $\mathcal{H}$. The angle determined by U and V is the real number in $[0,\pi/2]$ the cosine of which is given by

$$\mathsf{c}(U,V) = \sup\big\{|\langle x \mid y\rangle| \;\big|\; x \in U \cap (U\cap V)^\perp,\ y \in V \cap (U\cap V)^\perp,\ \|x\| \leq 1,\ \|y\| \leq 1\big\}. \tag{2.8}$$

Lemma 2.7 *Let U and V be closed vector subspaces of $\mathcal{H}$, let $u \in U$, let $v \in V$, and set $S = (u + U^{\perp}) \cap (v + V^{\perp})$. Then the following hold.*

(i) *Let $x \in S$. Then $S = P_{\overline{U+V}}\, x + (U^{\perp} \cap V^{\perp})$.*

(ii) *Suppose that $\|P_U P_V\| < 1$ and set*

$$z = \overline{u} + \overline{v}, \quad \text{where} \quad \begin{cases} \overline{u} = (\mathrm{Id} - P_U P_V)^{-1}(u - P_U v) \\ \overline{v} = (\mathrm{Id} - P_V P_U)^{-1}(v - P_V u). \end{cases} \tag{2.9}$$

Then the following hold.

(a) *$S \neq \varnothing$.*

(b) *$z = P_S\, 0$.*

Proof. (i): As in Proposition 2.1, we can write $S = x + (U^{\perp} \cap V^{\perp})$. Hence, since $(\overline{U+V})^{\perp} = (U+V)^{\perp} = U^{\perp} \cap V^{\perp}$, we get $S = x + (U^{\perp} \cap V^{\perp}) = P_{(U^{\perp} \cap V^{\perp})^{\perp}} x + (U^{\perp} \cap V^{\perp}) = P_{\overline{U+V}}\, x + (U^{\perp} \cap V^{\perp})$.

(ii): These properties are known (see for instance [23, Item 3.B) p. 91] and [23, Section 5 on pp. 92–93], respectively); we provide short alternative proofs for completeness.

(ii)(a): Let $u \in U$ and $v \in V$. Since P_U and P_V are self-adjoint, $\|P_V P_U\| = \|(P_V P_U)^*\| = \|P_U^* P_V^*\| = \|P_U P_V\| < 1$, and the vectors $\overline{u}$ and $\overline{v}$ are therefore well defined. Moreover, it follows from the identity $\overline{u} = \sum_{j \in \mathbb{N}} (P_U P_V)^j (u - P_U v)$ that $\overline{u} \in U$ and therefore that $P_U \overline{u} = \overline{u}$. On the other hand, the second equality in the right-hand side of (2.9) yields

$$\begin{aligned} P_U \overline{v} &= P_U \Bigg(\sum_{j \in \mathbb{N}} (P_V P_U)^j (v - P_V u) \Bigg) \\ &= (\mathrm{Id} - P_U P_V)^{-1} (P_U v - P_U P_V u) \\ &= (\mathrm{Id} - P_U P_V)^{-1} \big((\mathrm{Id} - P_U P_V) u - (u - P_U v) \big) \\ &= u - \overline{u}. \end{aligned} \tag{2.10}$$

Thus, $P_U z = P_U(\overline{u} + \overline{v}) = \overline{u} + P_U \overline{v} = u$. Likewise, $P_V \overline{v} = \overline{v}$ and $P_V \overline{u} = v - \overline{v}$, which implies that $P_V z = P_V(\overline{u} + \overline{v}) = P_V \overline{u} + \overline{v} = v$. Altogether, $z \in S$.

(ii)(b): As seen above, $z \in S$, $\overline{u} \in U$, and $\overline{v} \in V$. Now let $x \in S$. As in Proposition 2.1(ii), we can write $x = z + w = \overline{u} + \overline{v} + w$, for some $w \in U^{\perp} \cap V^{\perp}$. Hence, $\|x\|^2 = \|z\|^2 + 2\mathrm{Re}\langle \overline{u} \mid w \rangle + 2\mathrm{Re}\langle \overline{v} \mid w \rangle + \|w\|^2 = \|z\|^2 + \|w\|^2 \geq \|z\|^2$. □

We can now provide various characterizations of the inverse best approximation property (the notation (1.4) will be used repeatedly).

Theorem 2.8 *The following are equivalent.*

(i) *$(U_i)_{1 \leq i \leq m}$ satisfies the inverse best approximation property.*

(ii) *$(\forall i \in \{1, \ldots, m-1\})(\forall u_i \in U_i)(\exists\, x \in \mathcal{H})\ u_i = P_i x$ and $(\forall j \in \{i+1, \ldots, m\})\ P_j x = 0$.*

(iii) $(\forall i \in \{1,\dots,m-1\})$ $P_i(U_{i+}^\perp) = U_i$.

(iv) $(\forall i \in \{1,\dots,m-1\})$ $U_i^\perp + U_{i+}^\perp = \mathcal{H}$.

(v) *The subspaces* $(U_i)_{1\le i\le m}$ *are linearly independent and* $(\forall i \in \{1,\dots,m-1\})(\exists\, \gamma_i \in \left]0,+\infty\right[)$ $d_{U_i^\perp \cap U_{i+}^\perp} \le \gamma_i\big(d_{U_i^\perp} + d_{U_{i+}^\perp}\big)$.

(vi) *The subspaces* $(U_i)_{1\le i\le m}$ *are linearly independent and* $(\forall i \in \{1,\dots,m-1\})$ $U_i + U_{i+}$ *is closed.*

(vii) *The subspaces* $(U_i)_{1\le i\le m}$ *are linearly independent and, for every* $i \in \{1,\dots,m-1\}$, $\mathsf{c}(U_i, U_{i+}) < 1$.

(viii) $(\forall i \in \{1,\dots,m-1\})(\exists\, \gamma_i \in \left[1,+\infty\right[)(\forall u_i \in U_i)$ $\|u_i\| \le \gamma_i \|P_{i+}^\perp u_i\|$.

(ix) $(\forall i \in \{1,\dots,m-1\})(\exists\, \gamma_i \in \left[2,+\infty\right[)(\forall x \in \mathcal{H})$ $\|x\| \le \gamma_i(\|P_i^\perp x\| + \|P_{i+}^\perp x\|)$.

(x) $(\forall i \in \{1,\dots,m-1\})$ $\|P_i P_{i+}\| < 1$.

Proof. (i)⇒(ii): Clear.

(ii)⇒(iii): Let $i \in \{1,\dots,m-1\}$. It is clear that $P_i(U_{i+}^\perp) \subset U_i$. Conversely, let $u_i \in U_i$. By assumption, there exists $x \in \bigcap_{j=i+1}^m U_j^\perp = U_{i+}^\perp$ such that $u_i = P_i x$. In other words, $U_i \subset P_i(U_{i+}^\perp)$. Altogether, $P_i(U_{i+}^\perp) = U_i$.

(iii)⇒(iv): Let $i \in \{1,\dots,m-1\}$. We have

$$\mathcal{H} = U_i^\perp + U_i = U_i^\perp + P_i(U_{i+}^\perp) = U_i^\perp + \bigcup_{v \in U_{i+}^\perp} (v - P_{U_i^\perp} v) = U_i^\perp + \bigcup_{v \in U_{i+}^\perp} \{v\} = U_i^\perp + U_{i+}^\perp. \tag{2.11}$$

(iv)⇒(v): Let $i \in \{1,\dots,m-1\}$. We have

$$U_i \cap U_{i+} = (U_i^\perp + U_{i+}^\perp)^\perp = \mathcal{H}^\perp = \{0\}. \tag{2.12}$$

As seen in (2.2), this shows the independence claim. Moreover, since $U_i^\perp + U_{i+}^\perp = \mathcal{H}$ is closed, the inequality on the distance functions follows from [8, Corollaire II.9].

(v)⇒(vi): Let $i \in \{1,\dots,m-1\}$. It follows from [8, Remarque 7 p. 22] (see also [3, Proposition 5.16]) that $U_i^\perp + U_{i+}^\perp$ is closed. In turn, since [8, Théorème II.15] asserts that $U_i^{\perp\perp} + U_{i+}^{\perp\perp}$ is closed, we deduce that

$$U_i + \overline{U_{i+}} \text{ is closed.} \tag{2.13}$$

It remains to show that U_{i+} is closed. If $i = m-1$, $U_{i+} = U_m$ is closed. On the other hand, if $i \in \{2,\dots,m-1\}$ and U_{i+} is closed, we deduce from (2.13) that $U_{(i-1)+} = U_i + U_{i+} = U_i + \overline{U_{i+}}$ is closed.

(vi)⇒(vii): Let $i \in \{1,\dots,m-1\}$. Then U_{i+} and $U_i + U_{i+}$ are closed and it follows from [18, Theorem 9.35] that $\mathsf{c}(U_i, U_{i+}) < 1$.

(vii)⇒(viii): Let $i \in \{1,\ldots,m-1\}$ and let $u_i \in U_i$. Then (2.8) yields

$$\begin{aligned}\|u_i\|^2 &= \|P_{i+}^{\perp}u_i\|^2 + \|P_{i+}u_i\|^2\\ &= \|P_{i+}^{\perp}u_i\|^2 + \langle u_i \mid P_{i+}u_i\rangle\\ &\leq \|P_{i+}^{\perp}u_i\|^2 + \mathsf{c}(U_i,U_{i+})\|u_i\|\,\|P_{i+}u_i\|\\ &\leq \|P_{i+}^{\perp}u_i\|^2 + \mathsf{c}(U_i,U_{i+})\|u_i\|^2. \end{aligned}\tag{2.14}$$

Hence, $\|P_{i+}^{\perp}u_i\|^2 \geq (1-\mathsf{c}(U_i,U_{i+}))\|u_i\|^2$.

(viii)⇒(ix): Let $i \in \{1,\ldots,m-1\}$ and let $x \in \mathcal{H}$. There exists $\gamma \in [1,+\infty[$ such that

$$\begin{aligned}\|x\| &\leq \|P_ix\| + \|P_i^{\perp}x\|\\ &\leq \gamma\|P_{i+}^{\perp}P_ix\| + \|P_i^{\perp}x\|\\ &\leq \gamma(\|P_{i+}^{\perp}x\| + \|P_{i+}^{\perp}P_i^{\perp}x\|) + \|P_i^{\perp}x\|\\ &\leq \gamma\|P_{i+}^{\perp}x\| + (1+\gamma)\|P_i^{\perp}x\|. \end{aligned}\tag{2.15}$$

(ix)⇒(x): Let $i \in \{1,\ldots,m-1\}$ and let $x \in \mathcal{H}$. There exists $\gamma \in [2,+\infty[$ such that

$$\begin{aligned}\|P_ix\|^2 &= \|P_{i+}P_ix\|^2 + \|P_{i+}^{\perp}P_ix\|^2\\ &= \|P_{i+}P_ix\|^2 + (\|P_{i+}^{\perp}P_ix\| + \|P_i^{\perp}P_ix\|)^2\\ &\geq \|P_{i+}P_ix\|^2 + \gamma^{-2}\|P_ix\|^2. \end{aligned}\tag{2.16}$$

Therefore $\|P_{i+}P_ix\|^2 \leq (1-\gamma^{-2})\|P_ix\|^2 \leq (1-\gamma^{-2})\|x\|^2$. Hence $\|P_{i+}P_i\| < 1$ and, in turn, $\|P_iP_{i+}\| = \|P_i^*P_{i+}^*\| = \|(P_{i+}P_i)^*\| = \|P_{i+}P_i\| < 1$.

(x)⇒(i): Fix $(u_i)_{1\leq i\leq m} \in \bigtimes_{i=1}^m U_i$ and set $(\forall i \in \{0,\ldots,m-1\})$ $S_i = \bigcap_{j=i+1}^m (u_j + U_j^{\perp})$. Let us show by induction that

$$(\forall i \in \{0,\ldots,m-2\}) \quad S_i \neq \varnothing \quad \text{and} \quad (\forall x_i \in S_i) \quad S_i = P_{i+}x_i + U_{i+}^{\perp}. \tag{2.17}$$

First, let us set $i = m-2$. Since, by assumption $\|P_{m-1}P_m\| < 1$, it follows from Lemma 2.7(ii)(a) that $S_{m-2} \neq \varnothing$. Moreover, we deduce from Lemma 2.7(i) that, for every $x_{m-2} \in S_{m-2}$,

$$S_{m-2} = P_{\overline{U_{m-1}+U_m}}x_{m-2} + (U_{m-1}^{\perp} \cap U_m^{\perp}) = P_{(m-2)+}x_{m-2} + U_{(m-2)+}^{\perp}. \tag{2.18}$$

Next, suppose that (2.17) is true for some $i \in \{1,\ldots,m-2\}$ and let $x_i \in S_i$. Then, using Lemma 2.7(i), we obtain

$$S_{i-1} = (u_i + U_i^{\perp}) \cap S_i = (u_i + U_i^{\perp}) \cap \big(P_{i+}x_i + U_{i+}^{\perp}\big). \tag{2.19}$$

Since, by assumption $\|P_iP_{i+}\| < 1$, it follows from Lemma 2.7(ii)(a) that $S_{i-1} \neq \varnothing$. Now, let $x_{i-1} \in S_{i-1}$. Combining (2.19) and Lemma 2.7 (i), we obtain

$$S_{i-1} = P_{\overline{U_i+U_{i+}}}x_{i-1} + \Big(U_i^{\perp} \cap U_{i+}^{\perp}\Big) = P_{(i-1)+}x_{i-1} + U_{(i-1)+}^{\perp}. \tag{2.20}$$

This proves by induction that (2.17) is true. For $i = 0$, we thus obtain $S_0 = \bigcap_{j=1}^m (u_j + U_j^{\perp}) \neq \varnothing$. In view of Proposition 2.1(i), the proof is complete. ☐

An immediate application of Theorem 2.8 concerns the area of affine feasibility problems [4, 9, 10, 14, 28, 36]. Given a family of closed affine subspaces $(S_i)_{1\leq i\leq m}$ of $\mathcal{H}$, the problem is to

$$\text{find } \; x \in \bigcap_{i=1}^{m} S_i. \tag{2.21}$$

In applications, a key issue is whether this problem is consistent in the sense that it admits a solution. Our next proposition gives a sufficient condition for consistency. First, we recall a standard fact.

Lemma 2.9 *Let S be a closed affine subspace of $\mathcal{H}$, let $V = S - S$ be the closed vector subspace parallel to S, and let $y \in S$. Then $S = y + V$ and $(\forall x \in \mathcal{H})\ P_S x = y + P_V(x - y)$.*

Proposition 2.10 *Let $(S_i)_{1\leq i\leq m}$ be closed affine subspaces of $\mathcal{H}$ and suppose that $(U_i)_{1\leq i\leq m}$ are the orthogonal complements of their respective parallel vector subspaces. If $(U_i)_{1\leq i\leq m}$ satisfies the inverse best approximation property (in particular, if any of properties (ii)–(x) in Theorem 2.8 holds), then the affine feasibility problem (2.21) is consistent.*

Proof. For every $i \in \{1,\ldots,m\}$, let $a_i \in S_i$, and set $V_i = S_i - S_i$ and $u_i = P_i a_i$. Then, by Lemma 2.9, $(\forall i \in \{1,\ldots,m\})\ S_i = a_i + V_i = a_i + U_i^{\perp} = u_i + U_i^{\perp}$. Thus,

$$\bigcap_{i=1}^{m} S_i = \bigcap_{i=1}^{m} (u_i + U_i^{\perp}), \tag{2.22}$$

and it follows from Proposition 2.1(i) that (2.21) is consistent if $(U_i)_{1\leq i\leq m}$ satisfies the IBAP. ☐

Remark 2.11 The converse to Proposition 2.10 fails. For instance, let S_1 and S_2 be distinct intersecting lines in $\mathcal{H} = \mathbb{R}^3$. Then $U_1 = (S_1 - S_1)^{\perp}$ and $U_2 = (S_2 - S_2)^{\perp}$ are two-dimensional planes and they are therefore linearly dependent. Hence, the IBAP cannot hold by virtue of Corollary 2.3.

In the case of two subspaces, Theorem 2.8 yields simpler conditions.

Corollary 2.12 *The following are equivalent.*

(i) (U_1, U_2) *satisfies the inverse best approximation property.*

(ii) $(\forall u_1 \in U_1)\ S(u_1, 0) \neq \varnothing$.

(iii) $P_1(U_2^{\perp}) = U_1$.

(iv) $U_1^{\perp} + U_2^{\perp} = \mathcal{H}$.

(v) $U_1 \cap U_2 = \{0\}$ *and* $(\exists\, \gamma \in \,]0,+\infty[)\ d_{U_1^{\perp}\cap U_2^{\perp}} \leq \gamma\big(d_{U_1^{\perp}} + d_{U_2^{\perp}}\big)$.

(vi) $U_1 \cap U_2 = \{0\}$ *and* $U_1 + U_2$ *is closed.*

(vii) $U_1 \cap U_2 = \{0\}$ *and* $\mathsf{c}(U_1, U_2) < 1$.

(viii) $(\exists\, \gamma \in [1, +\infty[)(\forall u_1 \in U_1)\ \|u_1\| \leq \gamma \|P_2^\perp u_1\|$.

(ix) $(\exists\, \gamma \in [2, +\infty[)(\forall x \in \mathcal{H})\ \|x\| \leq \gamma(\|P_1^\perp x\| + \|P_2^\perp x\|)$.

(x) $\|P_1 P_2\| < 1$.

Remark 2.13 Corollary 2.12 provides necessary and sufficient conditions for the existence of solutions to (1.1) when $m = 2$. The implication (ix)⇒(i) appears in [23, Item 3.B) p. 91], the equivalences (vi)⇔(viii)⇔(ix)⇔(x) appear in [23, Item 1.A) p. 88], and the equivalences (iii)⇔(iv)⇔(x) appear in [31, Lemma on p. 201].

As consequences of Theorem 2.8, we can now describe scenarii in which the necessary condition established in Corollary 2.3 is also sufficient.

Corollary 2.14 *Suppose that the closed vector subspaces $(U_i)_{1\leq i\leq m}$ are linearly independent, that $\|P_{m-1}P_m\| < 1$ and that, for every $i \in \{1, \ldots, m-2\}$, U_i is finite dimensional or finite codimensional. Then $(U_i)_{1\leq i\leq m}$ satisfies the inverse best approximation property.*

Proof. In view of the equivalence (i)⇔(vii) in Theorem 2.8, it is enough to show that $(\forall i \in \{1, \ldots, m-1\})\ \mathsf{c}(U_i, U_{i+}) < 1$. For $i = m-1$, since $\|P_i P_{i+}\| = \|P_{m-1}P_m\| < 1$, we derive from the implication (x)⇒(vii) in Corollary 2.12 that $\mathsf{c}(U_i, U_{i+}) < 1$. Now suppose that, for some $i \in \{2, \ldots, m-1\}$, $\mathsf{c}(U_i, U_{i+}) < 1$. Using to the implication (vii)⇒(vi) in Corollary 2.12, we deduce that $U_{(i-1)+} = U_i + U_{i+}$ is closed. In turn, since U_{i-1} is finite or cofinite dimensional, it follows from [18, Corollary 9.37] that $\mathsf{c}(U_{(i-1)}, U_{(i-1)+}) < 1$, which completes the proof by induction. □

Corollary 2.15 *Suppose that the closed vector subspaces $(U_i)_{1\leq i\leq m}$ are linearly independent and that, for every $i \in \{1, \ldots, m-1\}$, U_i is finite dimensional or finite codimensional. Then $(U_i)_{1\leq i\leq m}$ satisfies the inverse best approximation property.*

Proof. Since U_{m-1} is finite dimensional or finite codimensional, it follows from [18, Corollary 9.37] and the implication (vii)⇒(x) in Corollary 2.12 that $\|P_{m-1}P_m\| < 1$. Hence, the claim follows from Corollary 2.14. □

Example 2.16 Let V be a closed vector subspace of $\mathcal{H}$ and let $(v_i)_{1\leq i\leq m-1}$ be linearly independent vectors such that $V^\perp \cap \operatorname{span}\{v_i\}_{1\leq i\leq m-1} = \{0\}$. Then, for every $(\eta_i)_{1\leq i\leq m-1} \in \mathbb{C}^{m-1}$, the constrained moment problem

$$x \in V \quad \text{and} \quad (\forall i \in \{1, \ldots, m-1\}) \quad \langle x \mid v_i \rangle = \eta_i \tag{2.23}$$

admits a solution.

Proof. This is a special case of Corollary 2.15, where $U_m = V^\perp$, $u_m = 0$, and, for every $i \in \{1, \ldots, m-1\}$, $U_i = \operatorname{span}\{v_i\}$ and $u_i = \eta_i v_i/\|v_i\|^2$. □

Corollary 2.17 *Suppose that the subspaces $(U_i)_{1\leq i\leq m}$ are linearly independent and that $\mathcal{H}$ is finite dimensional. Then $(U_i)_{1\leq i\leq m}$ satisfies the inverse best approximation property.*

The above results pertain to the existence of solutions to (1.1). We conclude this section with a uniqueness result that follows at once from Proposition 2.1(ii).

Proposition 2.18 *Let* $(u_i)_{1\leq i\leq m} \in \bigtimes_{i=1}^{m} U_i$. *Then* (1.1) *has at most one solution if and only if* $\bigcap_{i=1}^{m} U_i^{\perp} = \{0\}$.

Combining Theorem 2.8 and Proposition 2.18 yields conditions for the existence of unique solutions to (1.1). Here is an example in which $m = 2$.

Example 2.19 The following are equivalent.

(i) For every $u_1 \in U_1$ and $u_2 \in U_2$, $S(u_1, u_2)$ is a singleton.

(ii) $U_1^{\perp} + U_2^{\perp} = \mathcal{H}$ and $U_1^{\perp} \cap U_2^{\perp} = \{0\}$.

Proof. Existence follows from the implication (iv)⇒(i) in Corollary 2.12, and uniqueness from Proposition 2.18. □

3 IBAP and the periodic projection algorithm

If $(U_i)_{1\leq i\leq m}$ satisfies the IBAP, then (1.1) will in general admit infinitely many solutions (see Proposition 2.18) and it is of interest to identify specific solutions such as those of minimum norm.

Proposition 3.1 *Suppose that* $(U_i)_{1\leq i\leq m}$ *satisfies the inverse best approximation property, let* $(u_i)_{1\leq i\leq m} \in \bigtimes_{i=1}^{m} U_i$, *and, for every* $i \in \{1, \ldots, m-1\}$, *set*

$$\begin{aligned} T_i \colon U_{i+} &\to U_i + U_{i+} \\ v &\mapsto (\operatorname{Id} - P_i P_{i+})^{-1}(u_i - P_i v) + (\operatorname{Id} - P_{i+} P_i)^{-1}(v - P_{i+} u_i). \end{aligned} \tag{3.1}$$

Define recursively $\overline{x}_m = u_m$ *and* $(\forall i \in \{m-1, \ldots, 1\})$ $\overline{x}_i = T_i \overline{x}_{i+1}$. *Then, for every* $i \in \{1, \ldots, m\}$,

$$\overline{x}_i = P_{S_i} 0, \quad \textit{where} \quad S_i = \bigcap_{j=i}^{m} (u_j + U_j^{\perp}). \tag{3.2}$$

In particular, $\overline{x}_1 = P_{S(u_1,\ldots,u_m)} 0$ *is the minimal norm solution to* (1.1).

Proof. Let $i \in \{1, \ldots, m-1\}$. We first observe that the operator T_i is well defined since the implication (i)⇒(x) in Theorem 2.8 yields $\|P_i P_{i+}\| = \|P_{i+} P_i\| < 1$. Moreover, the expansions $(\operatorname{Id} - P_i P_{i+})^{-1} = \sum_{j\in\mathbb{N}} (P_i P_{i+})^j$ and $(\operatorname{Id} - P_{i+} P_i)^{-1} = \sum_{j\in\mathbb{N}} (P_{i+} P_i)^j$ imply that its range is indeed contained in $U_i + U_{i+}$. Thus, $\overline{x}_i$ is a well defined point in $U_i + U_{i+} = U_{(i-1)+}$.

To prove (3.2), we proceed by induction. First, for $i = m$, since $u_m \in U_m$, we obtain at once

$$\overline{x}_i = u_m = P_{(u_m + U_m^{\perp})} 0 = P_{S_i} 0. \tag{3.3}$$

Now, suppose that (3.2) is true for some $i \in \{2,\ldots,m\}$. By definition,

$$\overline{x}_{i-1} = (\mathrm{Id} - P_{i-1}P_{(i-1)+})^{-1}(u_{i-1} - P_{i-1}\overline{x}_i) + (\mathrm{Id} - P_{(i-1)+}P_{i-1})^{-1}(\overline{x}_i - P_{(i-1)+}u_{i-1}). \tag{3.4}$$

Since $\overline{x}_i \in U_{(i-1)+}$ and $u_{i-1} \in U_{i-1}$, Lemma 2.7(ii)(b) asserts that $\overline{x}_{i-1}$ is the element of minimal norm in $(u_{i-1} + U_{i-1}^{\perp}) \cap (\overline{x}_i + U_{(i-1)+}^{\perp})$. On the other hand since, by (3.2), $\overline{x}_i \in \bigcap_{j=i}^{m}(u_j + U_j^{\perp})$, we derive from (1.4) that, as in Proposition 2.1,

$$\overline{x}_i + U_{(i-1)+}^{\perp} = \overline{x}_i + \left(\sum_{j=i}^{m} U_j\right)^{\perp} = \overline{x}_i + \bigcap_{j=i}^{m} U_j^{\perp} = \bigcap_{j=i}^{m}(u_j + U_j^{\perp}). \tag{3.5}$$

As a result, $\overline{x}_{i-1}$ is the element of minimum norm in

$$(u_{i-1} + U_{i-1}^{\perp}) \cap \bigcap_{j=i}^{m}(u_j + U_j^{\perp}). \tag{3.6}$$

In other words, $\overline{x}_{i-1} = P_{S_{i-1}}0$, which completes the proof. $\square$

Conceptually, Proposition 3.1 provides a finite recursion for computing the minimal norm solution $\overline{x}_1$ to (1.1) for a given selection of vectors $(u_i)_{1\leq i\leq m} \in \bigtimes_{i=1}^{m} U_i$. This scheme is in general not of direct numerical use since it requires the inversion of operators in (3.1). However, minimal norm solutions and, more generally, best approximations from the solution set of (1.1) can be computed iteratively via projection methods. Indeed, for every $r \in \mathcal{H}$ and $(u_i)_{1\leq i\leq m} \in \bigtimes_{i=1}^{m} U_i$, let us denote by $B(r; u_1,\ldots,u_m)$ the best approximation to r from $S(u_1,\ldots,u_m)$, i.e., by Proposition 2.1(i),

$$B(r; u_1,\ldots,u_m) = P_{S(u_1,\ldots,u_m)}r = P_{\bigcap_{i=1}^{m}(u_i + U_i^{\perp})}r. \tag{3.7}$$

A standard numerical method for computing $B(r; u_1,\ldots,u_m)$ is the periodic projection algorithm

$$x_0 = r \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad x_{n+1} = Q_1 \cdots Q_m x_n \tag{3.8}$$

where, for every $i \in \{1,\ldots,m\}$, Q_i is the projector onto $u_i + U_i^{\perp}$, i.e.,

$$Q_i = P_{u_i + U_i^{\perp}} \colon x \mapsto u_i + x - P_i x. \tag{3.9}$$

This algorithm is rooted in the classical work of Kaczmarz [26] and von Neumann [40]. Although it has been generalized in various directions [3, 4, 9, 15], it is still widely used due to its simplicity and ease of implementation. If $S(u_1,\ldots,u_m) \neq \varnothing$, the sequence $(x_n)_{n\in\mathbb{N}}$ generated by (3.8) converges strongly to $B(r; u_1,\ldots,u_m)$. If $u_i \equiv 0$, this result was first established by von Neumann [40] for $m = 2$ and extended by Halperin [22] for $m > 2$. Strong convergence to $B(r; u_1,\ldots,u_m)$ in the general affine case ($u_i \not\equiv 0$) is a routine modification of Halperin's proof via Lemma 2.9 (see [18] for a detailed account). Interestingly, if the projectors are not activated periodically in (3.8) but in a more chaotic fashion, only weak convergence has been established [1] and it is still an open question whether strong convergence holds.

In connection with (3.8), an important question is whether the convergence of $(x_n)_{n\in\mathbb{N}}$ to $B(r; u_1,\ldots,u_m)$ occurs at a linear rate. The answer is negative and it has actually been shown that arbitrarily slow convergence may occur [5] in the sense that, for every sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $]0,1[$ such that $\alpha_n \downarrow 0$, there exists $r \in \mathcal{H}$ such that

$$(\forall n \in \mathbb{N}) \quad \|x_n - B(r; u_1, \ldots, u_m)\| \geq \alpha_n. \tag{3.10}$$

On the other hand, several conditions have been found [3, 5, 6, 17, 19, 27] that guarantee that, if (1.1) admits a solution for some $(u_i)_{1\leq i\leq m} \in \times_{i=1}^m U_i$, then, for every $r \in \mathcal{H}$, the sequence $(x_n)_{n\in\mathbb{N}}$ generated by (3.8) converges uniformly linearly to $B(r; u_1, \ldots, u_m)$ in the sense that there exists $\alpha \in [0,1[$ such that [19, Section 4]

$$(\forall n \in \mathbb{N}) \quad \|x_n - B(r; u_1, \ldots, u_m)\| \leq \alpha^n \|r - B(r; u_1, \ldots, u_m)\|. \tag{3.11}$$

The next result states that the IBAP implies uniform linear convergence of the periodic projection algorithm for solving the underlying affine feasibility problem (1.1) for every $(u_i)_{1\leq i\leq m} \in \times_{i=1}^m U_i$ and every $r \in \mathcal{H}$. In other words, if (1.1) admits a solution for every $(u_i)_{1\leq i\leq m} \in \times_{i=1}^m U_i$, then uniform linear convergence always occurs in (3.8).

Proposition 3.2 *Suppose that $(U_i)_{1\leq i\leq m}$ satisfies the inverse best approximation property and set*

$$\alpha = \sqrt{1 - \prod_{i=1}^{m-1} \Big(1 - \mathsf{c}(U_i^\perp, U_{i+}^\perp)^2\Big)}. \tag{3.12}$$

Then $\alpha \in [0,1[$ and, for every $r \in \mathcal{H}$ and every $(u_i)_{1\leq i\leq m} \in \times_{i=1}^m U_i$, the sequence $(x_n)_{n\in\mathbb{N}}$ generated by (3.8) satisfies (3.11).

Proof. We first deduce from the implication (i)⇒(vii) in Theorem 2.8 that $(\forall i \in \{1, \ldots, m-1\})$ $\mathsf{c}(U_i, U_{i+}) < 1$. Hence, it follows from [18, Theorem 9.35] that $(\forall i \in \{1, \ldots, m-1\})$ $\mathsf{c}(U_i^\perp, U_{i+}^\perp) < 1$. In turn, (3.12) and (1.4) imply that

$$\alpha = \sqrt{1 - \prod_{i=1}^{m-1} \left(1 - \mathsf{c}\Big(U_i^\perp, \bigcap_{j=i+1}^{m} U_j^\perp\Big)^2\right)} \in [0,1[\,. \tag{3.13}$$

Now let $(u_i)_{1\leq i\leq m} \in \times_{i=1}^m U_i$. Since the IBAP holds, we have

$$S(u_1, \ldots, u_m) \neq \varnothing. \tag{3.14}$$

Altogether, it follows from (3.13), (3.14), and [18, Corollary 9.34] applied to $(U_i^\perp)_{1\leq i\leq m}$ that (3.11) holds. ☐

In the case when $m = 2$, the above result admits a partial converse based on a result of [5].

Proposition 3.3 *Suppose that $U_1 \cap U_2 = \{0\}$, that (U_1, U_2) does not satisfy the IBAP, and that $(u_1, u_2) \in U_1 \times U_2$ satisfies $S(u_1, u_2) \neq \varnothing$. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $]0,1[$ such that $\alpha_n \downarrow 0$. Then there exists $r \in \mathcal{H}$ such that the sequence $(x_n)_{n\in\mathbb{N}}$ generated by (3.8) with $m = 2$ satisfies*

$$(\forall n \in \mathbb{N}) \quad \|x_n - B(r; u_1, u_2)\| \geq \alpha_n. \tag{3.15}$$

Proof. It follows from our hypotheses and the equivalence (i)⇔(vi) in Corollary 2.12 that U_1+U_2 is not closed. In turn, we derive from [5, Theorem 1.4(2)] that there exists $y_0 \in \mathcal{H}$ such that the sequence $(y_n)_{n\in\mathbb{N}}$ generated by the alternating projection algorithm

$$(\forall n \in \mathbb{N}) \quad y_{n+1} = P_{U_1^\perp} P_{U_2^\perp} y_n \tag{3.16}$$

satisfies

$$(\forall n \in \mathbb{N}) \quad \|y_n - P_{U_1^\perp \cap U_2^\perp} y_0\| \geq \alpha_n. \tag{3.17}$$

Now let $y \in S(u_1, u_2)$ and set $r = y + y_0$. It follows from Proposition 2.1(ii) that $S(u_1, u_2) = y + (U_1^\perp \cap U_2^\perp)$. Hence, it follows from (3.7) and Lemma 2.9 that

$$B(r; u_1, u_2) = y + P_{U_1^\perp \cap U_2^\perp}(r - y) = y + P_{U_1^\perp \cap U_2^\perp} y_0. \tag{3.18}$$

On the other hand, $x_0 - y = y_0$ and, using Lemma 2.9, (3.8) with $m = 2$ and (3.18) yield

$$(\forall n \in \mathbb{N}) \quad x_{n+1} - y = P_{u_1+U_1^\perp} P_{u_2+U_2^\perp} x_n - y = P_{y+U_1^\perp} P_{y+U_2^\perp} x_n - y = P_{U_1^\perp} P_{U_2^\perp}(x_n - y). \tag{3.19}$$

This and (3.16) imply by induction that $(\forall n \in \mathbb{N})\ x_n - y = y_n$. In turn, we derive from (3.18) and (3.17) that

$$(\forall n \in \mathbb{N}) \quad \|x_n - B(r; u_1, u_2)\| = \|(y_n + y) - (y + P_{U_1^\perp \cap U_2^\perp} y_0)\| = \|y_n - P_{U_1^\perp \cap U_2^\perp} y_0\| \geq \alpha_n, \tag{3.20}$$

which completes the proof. ☐

4 Applications

In this section, we present several applications of Theorem 2.8. As usual, $L^2(\mathbb{R}^N)$ is the space of real- or complex-valued absolutely square-integrable functions on the N-dimensional Euclidean space $\mathbb{R}^N$, $\widehat{x}$ denotes the Fourier transform of a function $x \in L^2(\mathbb{R}^N)$ and $\operatorname{supp}\widehat{x}$ the support of $\widehat{x}$. Moreover, if $A \subset \mathbb{R}^N$, 1_A denotes the characteristic function of A and $\complement A$ the complement of A. Finally, μ designates the Lebesgue measure on $\mathbb{R}^N$, $\operatorname{ran} T$ the range of an operator T and $\overline{\operatorname{ran}}\, T$ is the closure of $\operatorname{ran} T$.

The following lemma and its subsequent refinement will be used on several occasions.

Lemma 4.1 [2, Proposition 8], [7, Corollary 1] *Let A and B be measurable subsets of $\mathbb{R}^N$ of finite Lebesgue measure, and let $x \in L^2(\mathbb{R}^N)$ be such that $x1_{\complement A} = 0$ and $\widehat{x}1_{\complement B} = 0$. Then $x = 0$.*

Lemma 4.2 [2, p. 264], [21, Theorem 8.4] *Let A and B be measurable subsets of $\mathbb{R}^N$ of finite Lebesgue measure. Set $U = \{x \in L^2(\mathbb{R}^N) \mid x1_{\complement A} = 0\}$ and $V = \{x \in L^2(\mathbb{R}^N) \mid \widehat{x}1_{\complement B} = 0\}$. Then $\|P_U P_V\| < 1$.*

4.1 Systems of linear equations

Going back to Definition 1.1, we can say that $(U_i)_{1\leq i\leq m}$ satisfies the IBAP if for every $(u_i)_{1\leq i\leq m} \in \times_{i=1}^m \operatorname{ran} P_i$ there exists $x \in \mathcal{H}$ such that $(\forall i \in \{1,\ldots,m\})\ P_i x = u_i$. As we have shown, this property holds if (iv) in Theorem 2.8 is satisfied, i.e., if $(\forall i \in \{1,\ldots,m-1\})\ \ker P_i + \bigcap_{j=i+1}^{m} \ker P_j = \mathcal{H}$. In the following proposition, we show that such surjectivity results remain valid if projectors are replaced by more general linear operators.

Proposition 4.3 *For every $i \in \{1,\ldots,m\}$, let $\mathcal{G}_i$ be a normed vector space and let $T_i\colon \mathcal{H} \to \mathcal{G}_i$ be linear and bounded. Suppose that*

$$(\forall i \in \{1,\ldots,m-1\}) \quad \ker T_i + \bigcap_{j=i+1}^{m} \ker T_j = \mathcal{H}. \tag{4.1}$$

Then, for every $(y_i)_{1\leq i\leq m} \in \times_{i=1}^{m} \operatorname{ran} T_i$, there exists $x \in \mathcal{H}$ such that

$$(\forall i \in \{1,\ldots,m\}) \quad T_i x = y_i. \tag{4.2}$$

Proof. For every $i \in \{1,\ldots,m\}$, let $y_i \in \operatorname{ran} T_i$, set $U_i = (\ker T_i)^{\perp}$, and let $u_i \in U_i$ be such that $T_i u_i = y_i$. Now let $x \in \mathcal{H}$. Then x solves (4.2) $\Leftrightarrow$ $(\forall i \in \{1,\ldots,m\})\ T_i x = T_i u_i \Leftrightarrow (\forall i \in \{1,\ldots,m\})\ T_i(x-u_i) = 0 \Leftrightarrow (\forall i \in \{1,\ldots,m\})\ x - u_i \in \ker T_i = U_i^{\perp} \Leftrightarrow (\forall i \in \{1,\ldots,m\})$ $P_i x = u_i$. We thus recover an instance of problem (1.1) and, in view of the equivalence between items (i) and (iv) in Theorem 2.8, we obtain the existence of solutions to (4.2) if, for every $i \in \{1,\ldots,m-1\}$, $U_i^{\perp} + U_{i+}^{\perp} = \mathcal{H}$, i.e., if (4.1) holds. ☐

We now give an application of Proposition 4.3 to systems of integral equations.

Proposition 4.4 *For every $i \in \{1,\ldots,m\}$, let v_i, w_i, and y_i be functions in $L^2(\mathbb{R}^N)$ such that there exists $x_i \in L^2(\mathbb{R}^N)$ that satisfies $\int_{\mathbb{R}^N} x_i(s)v_i(s)w_i(t-s)ds = y_i(t)$ μ-a.e. on $\mathbb{R}^N$. Moreover, suppose that there exist measurable sets $(A_i)_{1\leq i\leq m}$ in $\mathbb{R}^N$ such that*

$$(\forall i \in \{1,\ldots,m\}) \quad \mu\big((A_i + \operatorname{supp} \widehat{v_i}) \cap \operatorname{supp} \widehat{w_i}\big) = 0 \tag{4.3}$$

and

$$(\forall i \in \{1,\ldots,m-1\}) \quad A_i \cup \bigcap_{j=i+1}^{m} A_j = \mathbb{R}^N. \tag{4.4}$$

Then there exists $x \in L^2(\mathbb{R}^N)$ such that

$$(\forall i \in \{1,\ldots,m\}) \quad \int_{\mathbb{R}^N} x(s)v_i(s)w_i(t-s)ds = y_i(t) \ \ \mu\text{-a.e. on } \mathbb{R}^N. \tag{4.5}$$

Proof. The result is an application of Proposition 4.3 in $\mathcal{H} = L^2(\mathbb{R}^N)$. To see this, denote by $\star$ the N-dimensional convolution operation and, for every $i \in \{1,\ldots,m\}$ and every $x \in \mathcal{H}$, set $T_i x = (xv_i) \star w_i$. Then $(T_i)_{1\leq i\leq m}$ are bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$ since, by [8, Théorème IV.15],

$$\big(\forall i \in \{1,\ldots,m\}\big)\big(\forall x \in \mathcal{H}\big) \quad \|T_i x\| = \|(xv_i) \star w_i\| \leq \|xv_i\|_{L^1}\|w_i\| \leq \|x\|\,\|v_i\|\,\|w_i\|. \tag{4.6}$$

Now fix $i \in \{1,\ldots,m-1\}$. Since (4.5) can be written as (4.2), Proposition 4.3 asserts that it suffices to show that

$$\ker T_i + \bigcap_{j=i+1}^{m} \ker T_j = \mathcal{H}. \tag{4.7}$$

To this end, let $z \in \mathcal{H}$. It follows from (4.4) that we can write $z = z_1 + z_2$, where $\widehat{z_1} = \widehat{z}\,1_{A_i}$ and $\widehat{z_2} = \widehat{z}\,1_{\complement A_i}$. We have

$$\widehat{T_i z_1} = \big[(z_1 v_i) \star w_i\big]^{\wedge} = (\widehat{z_1} \star \widehat{v_i})\widehat{w_i} = \big((\widehat{z}\,1_{A_i}) \star \widehat{v_i}\big)\widehat{w_i} \tag{4.8}$$

and

$$\operatorname{supp}\big((\widehat{z}\,1_{A_i}) \star \widehat{v_i}\big) \subset \operatorname{supp}(\widehat{z}\,1_{A_i}) + \operatorname{supp}\widehat{v_i} \subset A_i + \operatorname{supp}\widehat{v_i}. \tag{4.9}$$

Therefore, we derive from (4.8) and (4.3) that

$$\mu\big(\operatorname{supp}\widehat{T_i z_1}\big) = \mu\Big(\operatorname{supp}\big((\widehat{z}\,1_{A_i}) \star \widehat{v_i}\big) \cap \operatorname{supp}\widehat{w_i}\Big) = 0. \tag{4.10}$$

This shows that $z_1 \in \ker T_i$. Now fix $j \in \{i+1, \ldots, m\}$. Then it remains to show that $z_2 \in \ker T_j$. Since (4.4) yields $\complement A_i = \bigcap_{k=i+1}^{m} A_k \subset A_j$, arguing as above, we get

$$\operatorname{supp}\widehat{T_j z_2} = \operatorname{supp}\big(((\widehat{z}\,1_{\complement A_i}) \star \widehat{v_j})\widehat{w_j}\big) \subset \big(\complement A_i + \operatorname{supp}\widehat{v_j}\big) \cap \operatorname{supp}\widehat{w_j} \subset \big(A_j + \operatorname{supp}\widehat{v_j}\big) \cap \operatorname{supp}\widehat{w_j}. \tag{4.11}$$

In turn, we deduce from (4.3) that $\mu(\operatorname{supp}\widehat{T_j z_2}) = 0$ and therefore that $z_2 \in \ker T_j$. ☐

We now give an example in which the hypotheses of Proposition 4.4 are satisfied with $m = 3$.

Example 4.5 Let $\{\alpha, \beta, \gamma\} \subset \mathbb{R}$ and let $\{v_1, v_2, v_3, w_1, w_2, w_3\} \subset L^2(\mathbb{R})$. Suppose that $0 < \gamma < 2\alpha$ and that

$$\begin{cases} \operatorname{supp}\widehat{v_1} \subset [\beta, \beta+\gamma]\,,\ \operatorname{supp}\widehat{v_2} \subset [\alpha, +\infty[\,,\ \operatorname{supp}\widehat{v_3} \subset\,]-\infty, -\alpha] \\ \operatorname{supp}\widehat{w_1} \subset [-\alpha+\beta+\gamma, \alpha+\beta]\,,\ \operatorname{supp}\widehat{w_2} \subset\,]-\infty, 0]\,,\ \operatorname{supp}\widehat{w_3} \subset [0, +\infty[\,. \end{cases} \tag{4.12}$$

Now set $A_1 =]-\infty, -\alpha] \cup [\alpha, +\infty[$, $A_2 = [-\alpha, +\infty[$, and $A_3 =]-\infty, \alpha]$. Then (4.4) is satisfied and, since $A_1 + \operatorname{supp}\widehat{v_1} \subset]-\infty, -\alpha+\beta+\gamma] \cup [\alpha+\beta, +\infty[$, $A_2 + \operatorname{supp}\widehat{v_2} \subset [0, \infty[$, and $A_3 + \operatorname{supp}\widehat{v_3} \subset]-\infty, 0]$, so is (4.3).

Next, we consider a moment problem with wavelet frames [12, 13, 16].

Proposition 4.6 *Let ψ be a band-limited function in $L^2(\mathbb{R})$, say $\operatorname{supp}\widehat{\psi} \subset [-\rho, \rho]$ for some $\rho \in]0, +\infty[$. Suppose that $(\psi_{j,k})_{(j,k)\in\mathbb{Z}^2}$, where $\psi_{j,k} \colon t \mapsto 2^{j/2}\psi(2^j t - k)$, is a frame for $L^2(\mathbb{R})$, i.e., there exist constants α and β in $]0, +\infty[$ such that*

$$\big(\forall x \in L^2(\mathbb{R})\big) \quad \alpha\|x\|^2 \leq \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle x \mid \psi_{j,k}\rangle|^2 \leq \beta\|x\|^2, \tag{4.13}$$

and, moreover, that $(\psi_{j,k})_{(j,k)\in\mathbb{Z}^2}$ admits a lower Riesz bound $\gamma \in]0, +\infty[$, i.e.,

$$\big(\forall (c_{j,k})_{(j,k)\in\mathbb{Z}^2} \in \ell^2(\mathbb{Z}^2)\big) \quad \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |c_{j,k}|^2 \leq \gamma \Bigg\| \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} c_{j,k}\psi_{j,k} \Bigg\|^2. \tag{4.14}$$

Let A be a measurable subset of $\mathbb{R}$ such that $0 < \mu(A) < +\infty$, let $J \in \mathbb{Z}$, and set

$$\Lambda = \big\{(j,k) \in \mathbb{Z} \times \mathbb{Z} \;\big|\; j \leq J\big\}. \tag{4.15}$$

Then, for every function $y \in L^2(A)$ and every sequence $(\eta_{j,k})_{(j,k)\in\Lambda} \in \ell^2(\Lambda)$, there exists $x \in L^2(\mathbb{R})$ such that

$$x|_A = y \quad \text{and} \quad (\forall (j,k) \in \Lambda) \quad \langle x \mid \psi_{j,k}\rangle = \eta_{j,k}. \tag{4.16}$$

Proof. Set $\mathcal{H} = L^2(\mathbb{R})$, $\mathcal{G}_1 = L^2(A)$, and $\mathcal{G}_2 = \ell^2(\Lambda)$, and define bounded linear operators

$$T_1 \colon \mathcal{H} \to \mathcal{G}_1 \colon x \mapsto x|_A \quad \text{and} \quad T_2 \colon \mathcal{H} \to \mathcal{G}_2 \colon x \mapsto (\langle x \mid \psi_{j,k} \rangle)_{(j,k)\in\Lambda}. \tag{4.17}$$

Then $\operatorname{ran} T_1 = \mathcal{G}_1$ and, on the other hand, it follows from [11, Lemma 2.2(ii)] and (4.14) that $\operatorname{ran} T_2 = \mathcal{G}_2$. Hence, in view of (4.16), we must show that, for every $y_1 \in \operatorname{ran} T_1$ and every $y_2 \in \operatorname{ran} T_2$, there exists $x \in \mathcal{H}$ such that $T_1 x = y_1$ and $T_2 x = y_2$. Appealing to Proposition 4.3, it is enough to show that $\ker T_1 + \ker T_2 = \mathcal{H}$ or, equivalently, that

$$U_1^{\perp} + U_2^{\perp} = \mathcal{H}, \quad \text{where} \quad U_1 = \overline{\operatorname{ran}}\, T_1^* \quad \text{and} \quad U_2 = \overline{\operatorname{ran}}\, T_2^*. \tag{4.18}$$

Set $U = \big\{x \in L^2(\mathbb{R}) \;\big|\; x1_{\complement A} = 0\big\}$, $B = [-2^J\rho, 2^J\rho]$, and $V = \big\{x \in L^2(\mathbb{R}) \;\big|\; \widehat{x}1_{\complement B} = 0\big\}$. By Lemma 4.1, $U \cap V = \{0\}$ and it therefore follows from [18, Lemma 9.5] and Lemma 4.2 that

$$\mathsf{c}(U, V) = \|P_U P_V - P_{U\cap V}\| = \|P_U P_V\| < 1. \tag{4.19}$$

On the other hand, it follows from (4.17) that $T_1^* \colon \mathcal{G}_1 \to \mathcal{H}$ satisfies

$$(\forall y \in \mathcal{G}_1)(\forall t \in \mathbb{R}) \quad (T_1^* y)(t) = \begin{cases} y(t), & \text{if } t \in A;\\ 0, & \text{otherwise} \end{cases} \tag{4.20}$$

and that

$$T_2^* \colon \mathcal{G}_2 \to \mathcal{H} \colon (\eta_{j,k})_{(j,k)\in\Lambda} \mapsto \sum_{(j,k)\in\Lambda} \eta_{j,k}\psi_{j,k}. \tag{4.21}$$

Since $\bigcup_{(j,k)\in\Lambda} \operatorname{supp} \widehat{\psi_{j,k}} \subset B$, we have

$$U_1 \subset U \quad \text{and} \quad U_2 \subset V. \tag{4.22}$$

Hence, $U_1 \cap U_2 = \{0\}$ and (4.19) yields

$$\mathsf{c}(U_1, U_2) \leq \mathsf{c}(U, V) < 1. \tag{4.23}$$

In view of the implication (vii)⇒(iv) in Corollary 2.12, we conclude that (4.18) holds. ☐

4.2 Subspaces spanned by nearly pairwise bi-orthogonal sequences

The following proposition provides a wide range of applications of Theorem 2.8 with $m = 3$.

Proposition 4.7 *Let $(u_{1,k})_{k\in\mathbb{Z}}$, $(u_{2,k})_{k\in\mathbb{Z}}$, and $(u_{3,k})_{k\in\mathbb{Z}}$ be orthonormal sequences in $\mathcal{H}$ such that*

$$(\forall k \in \mathbb{Z})(\forall i \in \{1,2\})(\forall j \in \{i+1, 3\})(\forall l \in \mathbb{Z} \smallsetminus \{k\}) \quad u_{i,k} \perp u_{j,l}. \tag{4.24}$$

Moreover, suppose that

$$\sup_{k\in\mathbb{Z}} \sqrt{|\langle u_{1,k} \mid u_{2,k}\rangle|} + \sup_{k\in\mathbb{Z}} \sqrt{|\langle u_{2,k} \mid u_{3,k}\rangle|} + \sup_{k\in\mathbb{Z}} \sqrt{|\langle u_{1,k} \mid u_{3,k}\rangle|} < 1. \tag{4.25}$$

Then, for all sequences $(\alpha_{1,k})_{k\in\mathbb{Z}}$, $(\alpha_{2,k})_{k\in\mathbb{Z}}$, and $(\alpha_{3,k})_{k\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$, there exists $x \in \mathcal{H}$ such that

$$(\forall k \in \mathbb{Z}) \quad \alpha_{1,k} = \langle x \mid u_{1,k}\rangle,\ \alpha_{2,k} = \langle x \mid u_{2,k}\rangle,\ \text{and}\ \alpha_{3,k} = \langle x \mid u_{3,k}\rangle. \tag{4.26}$$

Proof. For every $i \in \{1,2,3\}$, set $U_i = \overline{\operatorname{span}}\{u_{i,k}\}_{k\in\mathbb{Z}}$ and observe that $(\forall x \in \mathcal{H})$ $P_i x = \sum_{k\in\mathbb{Z}} \langle x \mid u_{i,k}\rangle u_{i,k}$. Accordingly, we have to show that (U_1, U_2, U_3) satisfies the IBAP. Using the equivalence (i)⇔(x) in Theorem 2.8, this amounts to showing that $\|P_1 P_{1+}\| < 1$ and $\|P_2 P_3\| < 1$.

First, let us fix $i \in \{1,2\}$ and $j \in \{i+1, 3\}$, and let us show that

$$\sup_{k\in\mathbb{Z}} |\langle u_{i,k} \mid u_{j,k}\rangle|^2 \leq \|P_i P_j\| \leq \sup_{k\in\mathbb{Z}} |\langle u_{i,k} \mid u_{j,k}\rangle| . \tag{4.27}$$

In view of (4.24), we have

$$(\forall x \in \mathcal{H}) \quad P_i P_j x = \sum_{l\in\mathbb{Z}} \langle x \mid u_{j,l}\rangle\langle u_{j,l} \mid u_{i,l}\rangle u_{i,l}. \tag{4.28}$$

Hence, for every $k \in \mathbb{Z}$, $P_i P_j u_{i,k} = |\langle u_{i,k} \mid u_{j,k}\rangle|^2 u_{i,k}$ and therefore $\|P_i P_j\| \geq \|P_i P_j u_{i,k}\| = |\langle u_{i,k} \mid u_{j,k}\rangle|^2$. This proves the first inequality in (4.27). On the other hand, it follows from (4.28) that

$$(\forall x \in \mathcal{H}) \quad \|P_i P_j x\|^2 = \sum_{l\in\mathbb{Z}} |\langle x \mid u_{j,l}\rangle\langle u_{j,l} \mid u_{i,l}\rangle|^2 \leq \sup_{l\in\mathbb{Z}} |\langle u_{j,l} \mid u_{i,l}\rangle|^2 \|x\|^2. \tag{4.29}$$

This proves the second inequality in (4.27).

Since (4.25) and (4.27) imply that $\|P_2 P_3\| < 1$, it remains to show that $\|P_1 P_{1+}\| < 1$. We derive from (4.25) and (4.27) that

$$\begin{aligned}(\sqrt{\|P_1P_2\|} + \sqrt{\|P_1P_3\|})(1 + \sqrt{\|P_2P_3\|}) &= \sqrt{\|P_1P_2\|} + \sqrt{\|P_1P_3\|} - \|P_2P_3\| \\ &\quad + (\sqrt{\|P_1P_2\|} + \sqrt{\|P_1P_3\|} + \sqrt{\|P_2P_3\|})\sqrt{\|P_2P_3\|} \\ &< 1 - \|P_2P_3\|. \end{aligned} \tag{4.30}$$

For every $k \in \mathbb{Z}$, let $P_{3,k}^{\perp}$ denote the projector onto $\{u_{3,k}\}^{\perp}$ and set

$$v_{2,k} = \frac{P_{3,k}^{\perp} u_{2,k}}{\|P_{3,k}^{\perp} u_{2,k}\|} = \frac{u_{2,k} - \langle u_{2,k} \mid u_{3,k}\rangle u_{3,k}}{\sqrt{1 - |\langle u_{2,k} \mid u_{3,k}\rangle|^2}}, \tag{4.31}$$

which is well defined since (4.25) guarantees that $|\langle u_{2,k} \mid u_{3,k}\rangle| < 1$. Let us note that (4.31) yields

$$\begin{aligned} u_{3,k} - \frac{\langle u_{3,k} \mid u_{2,k}\rangle v_{2,k}}{\sqrt{1 - |\langle u_{2,k} \mid u_{3,k}\rangle|^2}} &= u_{3,k} - \frac{\langle u_{3,k} \mid u_{2,k}\rangle u_{2,k}}{1 - |\langle u_{2,k} \mid u_{3,k}\rangle|^2} + \frac{|\langle u_{2,k} \mid u_{3,k}\rangle|^2 u_{3,k}}{1 - |\langle u_{2,k} \mid u_{3,k}\rangle|^2} \\ &= \frac{1}{1 - |\langle u_{2,k} \mid u_{3,k}\rangle|^2}\big(u_{3,k} - \langle u_{3,k} \mid u_{2,k}\rangle u_{2,k}\big). \end{aligned} \tag{4.32}$$

On the other hand, it follows from (4.24) and (4.31) that $\{v_{2,k}\}_{k\in\mathbb{Z}} \cup \{u_{3,k}\}_{k\in\mathbb{Z}}$ is an orthonormal set and that

$$\overline{\operatorname{span}}\big(\{v_{2,k}\}_{k\in\mathbb{Z}} \cup \{u_{3,k}\}_{k\in\mathbb{Z}}\big) = \overline{\operatorname{span}}\big(\{P_{3,k}^{\perp} u_{2,k}\}_{k\in\mathbb{Z}} \cup \{u_{3,k}\}_{k\in\mathbb{Z}}\big) = \overline{U_2 + U_3} = \overline{U_{1+}}. \tag{4.33}$$

To compute $\|P_1 P_{1+}\|$, let $x \in \mathcal{H}$ and let $k \in \mathbb{Z}$. We derive from (4.33) that

$$P_{1+} x = \sum_{l\in\mathbb{Z}} \langle x \mid v_{2,l}\rangle v_{2,l} + \sum_{l\in\mathbb{Z}} \langle x \mid u_{3,l}\rangle u_{3,l}. \tag{4.34}$$

Hence, using (4.24), (4.31), and (4.32), we obtain

$$\begin{aligned}\langle P_{1+}x \mid u_{1,k}\rangle &= \langle x \mid v_{2,k}\rangle\langle v_{2,k} \mid u_{1,k}\rangle + \langle x \mid u_{3,k}\rangle\langle u_{3,k} \mid u_{1,k}\rangle \\ &= \langle x \mid u_{2,k}\rangle \frac{\langle v_{2,k} \mid u_{1,k}\rangle}{\sqrt{1-|\langle u_{2,k} \mid u_{3,k}\rangle|^2}} \\ &\quad + \langle x \mid u_{3,k}\rangle\left(\langle u_{3,k} \mid u_{1,k}\rangle - \frac{\langle u_{3,k} \mid u_{2,k}\rangle\langle v_{2,k} \mid u_{1,k}\rangle}{\sqrt{1-|\langle u_{2,k} \mid u_{3,k}\rangle|^2}}\right) \\ &= \langle x \mid u_{2,k}\rangle\beta_k + \langle x \mid u_{3,k}\rangle\gamma_k, \end{aligned} \tag{4.35}$$

where

$$\beta_k = \frac{\langle u_{2,k} \mid u_{1,k}\rangle - \langle u_{2,k} \mid u_{3,k}\rangle\langle u_{3,k} \mid u_{1,k}\rangle}{1-|\langle u_{2,k} \mid u_{3,k}\rangle|^2} \tag{4.36}$$

and

$$\gamma_k = \frac{\langle u_{3,k} \mid u_{1,k}\rangle - \langle u_{3,k} \mid u_{2,k}\rangle\langle u_{2,k} \mid u_{1,k}\rangle}{1-|\langle u_{2,k} \mid u_{3,k}\rangle|^2}. \tag{4.37}$$

We note that (4.27) yields

$$|\beta_k| \leq \frac{|\langle u_{1,k} \mid u_{2,k}\rangle| + |\langle u_{2,k} \mid u_{3,k}\rangle|\,|\langle u_{1,k} \mid u_{3,k}\rangle|}{1-|\langle u_{2,k} \mid u_{3,k}\rangle|^2} \leq \frac{\sqrt{\|P_1P_2\|} + \sqrt{\|P_2P_3\|}\,\sqrt{\|P_1P_3\|}}{1-\|P_2P_3\|} \tag{4.38}$$

and, likewise,

$$|\gamma_k| \leq \frac{\sqrt{\|P_1P_3\|} + \sqrt{\|P_2P_3\|}\,\sqrt{\|P_1P_2\|}}{1-\|P_2P_3\|}. \tag{4.39}$$

Thus, we obtain

$$\begin{aligned}(\forall x \in \mathcal{H}) \quad \|P_1P_{1+}x\| &= \sqrt{\sum_{k\in\mathbb{Z}} |\langle P_{1+}x \mid u_{1,k}\rangle|^2} \\ &\leq \sqrt{\sum_{k\in\mathbb{Z}} |\langle x \mid u_{2,k}\rangle\beta_k|^2} + \sqrt{\sum_{k\in\mathbb{Z}} |\langle x \mid u_{3,k}\rangle\gamma_k|^2} \\ &\leq \left(\sup_{k\in\mathbb{Z}}|\beta_k| + \sup_{k\in\mathbb{Z}}|\gamma_k|\right)\|x\| \\ &\leq \frac{(\sqrt{\|P_1P_2\|} + \sqrt{\|P_1P_3\|})(1+\sqrt{\|P_2P_3\|})}{1-\|P_2P_3\|}\|x\|. \end{aligned} \tag{4.40}$$

Appealing to (4.30), we conclude that $\|P_1P_{1+}\| < 1$. □

Remark 4.8 A concrete example of subspaces satisfying the hypotheses of Proposition 4.7 can be constructed from an orthonormal wavelet basis. Take $\psi \in L^2(\mathbb{R})$ such that the functions $(\psi_{k,l})_{(k,l)\in\mathbb{Z}^2}$, where $\psi_{k,l}\colon t \mapsto 2^{k/2}\psi(2^kt-l)$, form an orthonormal basis of $L^2(\mathbb{R})$ [16]. For every $i \in \{1,2,3\}$ let, for every $k \in \mathbb{Z}$, $(\eta_{i,k,l})_{l\in\mathbb{Z}}$ be a sequence in $\ell^2(\mathbb{Z})$ such that $\sum_{l\in\mathbb{Z}}|\eta_{i,k,l}|^2 = 1$ and define

$$U_i = \overline{\operatorname{span}}\,\{u_{i,k}\}_{k\in\mathbb{Z}}, \quad \text{where} \quad (\forall k \in \mathbb{Z}) \quad u_{i,k} = \sum_{l\in\mathbb{Z}} \eta_{i,k,l}\psi_{k,l}. \tag{4.41}$$

Then $(u_{1,k})_{k\in\mathbb{Z}}$, $(u_{2,k})_{k\in\mathbb{Z}}$, and $(u_{3,k})_{k\in\mathbb{Z}}$ are orthonormal sequences in $L^2(\mathbb{R})$ that satisfy (4.24). Moreover since, for every i and j in $\{1,2,3\}$ and every $k \in \mathbb{Z}$, $\langle u_{i,k} \mid u_{j,k}\rangle = \sum_{l\in\mathbb{Z}} \eta_{i,k,l}\overline{\eta_{j,k,l}}$, the main hypothesis (4.25) is equivalent to

$$\sup_{k\in\mathbb{Z}} \sqrt{\left|\sum_{l\in\mathbb{Z}} \eta_{1,k,l}\overline{\eta_{2,k,l}}\right|} + \sup_{k\in\mathbb{Z}} \sqrt{\left|\sum_{l\in\mathbb{Z}} \eta_{2,k,l}\overline{\eta_{3,k,l}}\right|} + \sup_{k\in\mathbb{Z}} \sqrt{\left|\sum_{l\in\mathbb{Z}} \eta_{1,k,l}\overline{\eta_{3,k,l}}\right|} < 1. \tag{4.42}$$

4.3 Harmonic analysis and signal recovery

Many problems arising in areas such as harmonic analysis [2, 7, 21, 23, 25, 29], signal theory [10, 32, 39], image processing [14, 36], and optics [30, 37] involve imposing known values of an ideal function in the time (or spatial) and Fourier domains. In this section, we describe applications of Theorem 2.8 to such problems.

The following lemma will be required.

Lemma 4.9 *Let U, V, and W be closed vector subspaces of $\mathcal{H}$ such that $W \subset V$. Then $\|P_U P_W\| \leq \|P_U P_V\|$.*

Proof. Set $B = \big\{x \in \mathcal{H} \;\big|\; \|x\| \leq 1\big\}$. Then $P_W(B) \subset B$. In turn, since $W \subset V$, $P_W(B) = P_V(P_W(B)) \subset P_V(B)$ and hence $P_U(P_W(B)) \subset P_U(P_V(B))$. Consequently, $\|P_U P_W\| = \sup\big\{\|P_U P_W x\| \;\big|\; x \in B\big\} \leq \sup\big\{\|P_U P_V x\| \;\big|\; x \in B\big\} = \|P_U P_V\|$. □

The scenario of the next proposition has a simple interpretation in signal recovery [14, 36]: an N-dimensional square-summable signal has known values over certain domains of the spatial and frequency domains and, in addition, $m-2$ scalar linear measurements of it are available.

Proposition 4.10 *Let A and B be measurable subsets of $\mathbb{R}^N$ of finite Lebesgue measure, and suppose that $m \geq 3$. Moreover, let $(v_i)_{1\leq i\leq m-2}$ be functions in $L^2(\mathbb{R}^N)$ with disjoint supports $(C_i)_{1\leq i\leq m-2}$ such that*

$$(\forall i \in \{1,\ldots,m-2\}) \quad \mu(C_i) < +\infty \quad \text{and} \quad \mu(C_i \cap \complement A) > 0. \tag{4.43}$$

Then, for all functions v_m and v_{m-1} in $L^2(\mathbb{R}^N)$ and every $(\eta_i)_{1\leq i\leq m-2} \in \mathbb{R}^{m-2}$, there exists a function $x \in L^2(\mathbb{R}^N)$ such that

$$(\forall i \in \{1,\ldots,m-2\}) \quad \int_{C_i} x(t)\overline{v_i(t)}dt = \eta_i, \;\; x|_A = v_{m-1}|_A, \;\; \text{and} \;\; \widehat{x}|_B = \widehat{v}_m|_B. \tag{4.44}$$

Proof. We first observe that the problem under consideration is a special case of (1.1) with $\mathcal{H} = L^2(\mathbb{R}^N)$,

$$\begin{cases} U_i = \operatorname{span}\{v_i\} & \text{and} \quad u_i = \eta_i v_i/\|v_i\|^2, \quad 1 \leq i \leq m-2;\\ U_{m-1} = \big\{x \in \mathcal{H} \;\big|\; x 1_{\complement A} = 0\big\} & \text{and} \quad u_{m-1} = v_{m-1}1_A;\\ U_m = \big\{x \in \mathcal{H} \;\big|\; \widehat{x} 1_{\complement B} = 0\big\} & \text{and} \quad \widehat{u_m} = \widehat{v_m}1_B. \end{cases} \tag{4.45}$$

It follows from Lemma 4.2 that $\|P_{m-1}P_m\| < 1$. Hence, in view of Corollary 2.14, it suffices to show that the closed vector subspaces $(U_i)_{1\leq i\leq m}$ are linearly independent. Since the supports $(C_i)_{1\leq i\leq m-2}$ are disjoint, the subspaces $(U_i)_{1\leq i\leq m-2}$ are independent. Therefore, if we set $U = \sum_{i=1}^{m-2} U_i$, it is enough to show that U, U_{m-1}, and U_m are independent. To this end, take $(y, y_{m-1}, y_m) \in U \times U_{m-1} \times U_m$ such that

$$y + y_{m-1} + y_m = 0, \tag{4.46}$$

and set $C = \bigcup_{i=1}^{m-2} C_i$. We have $(y + y_{m-1})1_{\complement(A\cup C)} = 0$, $\mu(A \cup C) < +\infty$, $\widehat{y}_m 1_{\complement B} = 0$, and $\mu(B) < +\infty$. Hence, it follows from (4.46) and Lemma 4.1 that

$$y + y_{m-1} = 0 \;\text{ and }\; y_m = 0. \tag{4.47}$$

It remains to show that $y = 0$. Since $y \in U$, there exist $(\alpha_i)_{1\leq i\leq m-2} \in \mathbb{C}^{m-2}$ such that $y = \sum_{i=1}^{m-2} \alpha_i v_i$. However, since the supports $(C_i)_{1\leq i\leq m-2}$ are disjoint,

$$\|y\|^2 = \left\| \sum_{i=1}^{m-2} \alpha_i v_i \right\|^2 = \sum_{i=1}^{m-2} |\alpha_i|^2 \|v_i\|^2. \tag{4.48}$$

On the other hand, (4.43) implies that, for every $i \in \{1, \ldots, m-2\}$,

$$\|v_i\|^2 = \int_{C_i\cap A} |v_i(t)|^2 dt + \int_{C_i\cap\complement A} |v_i(t)|^2 dt > \int_{C_i\cap A} |v_i(t)|^2 dt = \|v_i 1_A\|^2. \tag{4.49}$$

At the same time, we derive from (4.47) that $y = -y_{m-1} \in U_{m-1}$ and therefore from (4.45) that $y1_{\complement A} = 0$. Consequently, (4.48) yields

$$\sum_{i=1}^{m-2} |\alpha_i|^2 \|v_i\|^2 = \|y\|^2 = \|y1_A\|^2 = \left\| \sum_{i=1}^{m-2} \alpha_i v_i 1_A \right\|^2 = \sum_{i=1}^{m-2} |\alpha_i|^2 \|v_i 1_A\|^2. \tag{4.50}$$

In view of (4.49), we conclude that $(\forall i \in \{1, \ldots, m-2\})\ \alpha_i = 0$. □

Remark 4.11 In connection with Proposition 4.10, let us make a few comments on the following classical problem: given measurable subsets A and B of $\mathbb{R}^N$ such that $\mu(A) > 0$ and $\mu(B) > 0$, and functions a and b in $L^2(\mathbb{R}^N)$, is there a function $x \in L^2(\mathbb{R}^N)$ such that

$$x|_A = a|_A \quad \text{and} \quad \widehat{x}|_B = b|_B\ ? \tag{4.51}$$

To answer this question, let us set

$$\begin{cases} U_1 = \big\{x \in L^2(\mathbb{R}^N) \mid x1_{\complement A} = 0\big\} & \text{and} \quad u_1 = a1_A, \\ U_2 = \big\{x \in L^2(\mathbb{R}^N) \mid \widehat{x}1_{\complement B} = 0\big\} & \text{and} \quad \widehat{u_2} = b1_B. \end{cases} \tag{4.52}$$

Thus, the problem reduces to an instance of (1.1) in which $m = 2$.

- If $\mu(A) < +\infty$ and $\mu(B) < +\infty$, it follows from Lemma 4.2, (4.52), and the implication (x)⇒(i) in Corollary 2.12 that the answer is affirmative (see also [23, Corollary 5.B p. 100]).
- If $\mu(\complement A) < +\infty$ and $\mu(\complement B) < +\infty$, it follows from (4.52), Proposition 2.18, and Lemma 4.1 (applied to $U_1^\perp$ and $U_2^\perp$) that (4.51) has at most one solution.

- Suppose that A is bounded and that $\mu(\complement B) > 0$, and let $\varepsilon \in]0, +\infty[$. Then there exists $x \in L^2(\mathbb{R}^N)$ such that

$$\int_A |x(t) - a(t)|^2 dt + \int_B |\widehat{x}(\xi) - b(\xi)|^2 d\xi < \varepsilon. \tag{4.53}$$

  To show this, we first observe that $U_1 \cap U_2 = \{0\}$. Indeed, let $y \in U_1 \cap U_2$. Then $\widehat{y}$ can be extended to an entire function on $\mathbb{C}^N$ (see [35, Theorem 7.23] or [38, Theorem III.4.9]) and, at the same time, $\widehat{y}1_{\complement B} = 0$, which implies that $\widehat{y} = 0$ [34, Theorem I.3.7]. Hence, applying Proposition 2.5 with $m = 2$, we obtain the existence of $x \in L^2(\mathbb{R}^N)$ such that

$$\|P_1 x - u_1\|^2 + \|P_2 x - u_2\|^2 < \varepsilon, \tag{4.54}$$

  which yields (4.53). In the case when $\complement B$ is a ball centered at the origin and $b = 0$, (4.53) provides the following approximate band-limited extrapolation result: there exists $x \in L^2(\mathbb{R}^N)$ which approximates a on A and such that $\widehat{x}$ nearly vanishes for high frequencies.

The following example describes a situation in which the IBAP fails.

Example 4.12 The following example is from [30]. Let $C = [-1/2, 1/2] \times [-1/2, 1/2]$ and set $\mathcal{H} = L^2(C)$. Moreover, define

$$(\forall (m,n) \in \mathbb{Z}^2) \quad \widehat{x}(m,n) = \int_C x(s,t) \exp(-i2\pi(ms + nt)) ds dt, \tag{4.55}$$

set $A = [0, 1/2] \times [0, 1/2]$, and set $B = F \cup \big\{(m,0) \bigm| m \in \mathbb{Z}\big\}$, where F is a nonempty finite subset of $\mathbb{Z} \times \mathbb{Z}$. The problem amounts to finding functions with prescribed best approximations from the closed vector subspaces

$$\begin{cases} U_1 = \big\{x \in \mathcal{H} \bigm| x1_{\complement A} = 0\big\} \\ U_2 = \big\{x \in \mathcal{H} \bigm| x(s,t) = x(-s,t) \text{ a.e. on } C\big\} \\ U_3 = \big\{x \in \mathcal{H} \bigm| \widehat{x}1_B = 0\big\}. \end{cases} \tag{4.56}$$

Since $U_1^\perp + U_2^\perp + U_3^\perp = \{0\}$ [30], it follows from Proposition 2.18 that the problem has at most one solution. However, the subspaces are not independent. Indeed, given a finite subset I of $\mathbb{Z}$ such that $(0,n) \notin F$ whenever $n \in I$ and complex numbers $(c_n)_{n \in I}$, the trigonometric polynomial

$$(s,t) \mapsto \sum_{n \in I} c_n e^{i2\pi nt} \tag{4.57}$$

is in $U_2 \cap U_3$. Therefore, in the light of Corollary 2.3, the IBAP does not hold.

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