Approximating Curves for Nonexpansive and Monotone Operators

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Abstract

A classical tool in nonlinear analysis is the notion of an approximating curve, whereby a particular solution to a nonuniquely solvable problem is obtained as the limit of the solutions to uniquely solvable perturbed problems. We introduce and analyze new types of approximating curves for nonexpansive fixed point problems and monotone inclusion problems in Hilbert spaces. The solution attained by these curves solves a strictly monotone variational inequality over the original solution set. Various special cases are discussed.

1 Introduction

In nonlinear analysis, a common approach to solving a problem with multiple solutions is to replace it by a family of perturbed problems admitting a unique solution, and to obtain a particular original solution as the limit of these perturbed solutions as the perturbation vanishes. This principle arises for instance in minimization problems (Tikhonov regularization [2, 26]), in partial differential equations (viscosity solutions [28, Section 33.11]), in monotone inclusions [28, Section 32.18], in variational inequalities [9], in evolution equations (elliptic regularization [19, Chapitre 3]), and in fixed point theory (approximating curves [16]); further examples will be found in [3, 25, 28] and the references therein. For the sake of illustration, let us consider two examples in a Hilbert space $\mathcal{H}$.

- Let T be a nonexpansive operator defined on $\mathcal{H}$, and suppose that the set $\operatorname{Fix} T$ of its fixed points is nonempty. Given $a \in \mathcal{H}$, a classical way to perturb the basic fixed point equation $x = Tx$ is to add to T a viscosity term $\varepsilon(a - T)$, which yields $x_\varepsilon = \varepsilon a + (1-\varepsilon)Tx_\varepsilon$, where $\varepsilon \in \left]0,1\right[$. As the viscosity term vanishes, i.e., as $\varepsilon \to 0$, the approximating curve $(x_\varepsilon)_{\varepsilon\in]0,1[}$ converges strongly to the best approximation x_0 to a from $\operatorname{Fix} T$ [9]. A simple manipulation shows that the same result holds for the approximating curve defined by

$$(\forall \varepsilon \in \left]0,1\right[)\;\; x_\varepsilon = T\big(x_\varepsilon + \varepsilon(a - x_\varepsilon)\big). \tag{1.1}$$

- Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator with zeros. Given $\varepsilon \in \left]0,1\right[$, consider the perturbation $0 \in Ax_\varepsilon + \varepsilon x_\varepsilon$ of the inclusion $0 \in Ax$. Then the approximating curve $(x_\varepsilon)_{\varepsilon\in]0,1[}$ converges strongly to the zero x_0 of A of minimal norm as $\varepsilon \to 0$ [11].

Besides their importance in the problems mentioned above, approximating curves are also relevant to numerical methods since understanding their properties is central in the analysis of parent continuous [3, 21, 23] and discrete [5, 12, 17, 27] dynamical systems (see also [13] for an application of such dynamical systems to concrete problems). The goal of this paper is to analyze the properties of new types of approximating curves for fixed point and monotone inclusion problems. The limit attained by these curves is the solution of the general variational inequality $0 \in N_C x_0 + Bx_0$, where N_C denotes the normal cone operator to the original solution set C and $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ is a suitable strictly monotone operator.

Throughout, $\mathcal{H}$ is a real Hilbert space with scalar product $\langle \cdot \mid \cdot \rangle$, norm $\|\cdot\|$, and identity operator Id. In addition, P_C denotes the projector onto a nonempty closed convex subset C of $\mathcal{H}$, and $N_C\colon \mathcal{H} \to 2^{\mathcal{H}}$ its normal cone operator, i.e.,

$$N_C\colon x \mapsto \begin{cases} \big\{u \in \mathcal{H} \mid (\forall y \in C)\;\; \langle y - x \mid u\rangle \le 0\big\}, & \text{if } x \in C;\\ \varnothing, & \text{otherwise.}\end{cases} \tag{1.2}$$

As is customary, $\to$ and $\rightharpoonup$ denote, respectively, strong and weak convergence.

2 Nonexpansive fixed point problems

The domain and fixed point set of an operator $T\colon \mathcal{H} \to \mathcal{H}$ are denoted by $\operatorname{dom} T$ and $\operatorname{Fix} T$, respectively. Recall that T is nonexpansive if it is Lipschitz-continuous with constant 1, firmly nonexpansive if $2T - \operatorname{Id}$ is nonexpansive, and a strict contraction if it is Lipschitz-continuous with a constant in $\left[0,1\right[$. It will be convenient to introduce the following notion.

Definition 2.1 *Let $(T_\varepsilon)_{\varepsilon\in]0,1[}$ be a family of operators from $\mathcal{H}$ to $\mathcal{H}$ with domain $\mathcal{H}$ and let $(x_\varepsilon)_{\varepsilon\in]0,1[}$ be a family in $\mathcal{H}$. Then $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is* $\mathfrak{T}$-focused *with respect to $(T_\varepsilon)_{\varepsilon\in]0,1[}$ if, for every $x \in \mathcal{H}$ and every sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $\left]0,1\right[$ such that $\varepsilon_n \downarrow 0$,*

$$\big[\, x_{\varepsilon_n} \rightharpoonup x \quad \textit{and} \quad x_{\varepsilon_n} - T_{\varepsilon_n} x_{\varepsilon_n} \to 0 \,\big] \quad \Rightarrow \quad (\forall \varepsilon \in \left]0,1\right[)\;\; T_\varepsilon x = x. \tag{2.1}$$

Example 2.2 Let $T\colon \operatorname{dom} T = \mathcal{H} \to \mathcal{H}$ be a nonexpansive operator such that $\operatorname{Fix} T \neq \varnothing$, let $(\lambda_\varepsilon)_{\varepsilon\in]0,1[}$ be a family in $]0,1]$ such that $\inf_{\varepsilon\in]0,1[} \lambda_\varepsilon > 0$, set $(\forall \varepsilon \in\,]0,1[)\ T_\varepsilon = \operatorname{Id} + \lambda_\varepsilon(T - \operatorname{Id})$, and take $(x_\varepsilon)_{\varepsilon\in]0,1[}$ in $\mathcal{H}$. Then $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is $\mathfrak{T}$-focused with respect to $(T_\varepsilon)_{\varepsilon\in]0,1[}$.

Proof. Suppose that $]0,1[\, \ni \varepsilon_n \downarrow 0$, $x_{\varepsilon_n} \rightharpoonup x$, and $x_{\varepsilon_n} - T_{\varepsilon_n} x_{\varepsilon_n} \to 0$. Then, since $\inf_{\varepsilon\in]0,1[} \lambda_\varepsilon > 0$, we obtain $x_{\varepsilon_n} - T x_{\varepsilon_n} \to 0$ and the demiclosed principle [10, Lemma 2] yields $x \in \operatorname{Fix} T \equiv \operatorname{Fix} T_\varepsilon$. ☐

Our first result concerns the convergence of a generalization of (1.1).

Theorem 2.3 *Let $(T_\varepsilon)_{\varepsilon\in]0,1[}$ and $(S_\varepsilon)_{\varepsilon\in]0,1[}$ be families of nonexpansive operators from $\mathcal{H}$ to $\mathcal{H}$ with domain $\mathcal{H}$, let $Q\colon \operatorname{dom} Q = \mathcal{H} \to \mathcal{H}$ be a strict contraction, and suppose that $C = \bigcap_{\varepsilon\in]0,1[} \operatorname{Fix} T_\varepsilon \neq \varnothing$. Then there exists a unique point $x_0 \in C$ such that $x_0 = P_C(Q x_0)$. Now set*

$$(\forall \varepsilon \in\,]0,1[)\ \ x_\varepsilon = T_\varepsilon\big(x_\varepsilon + \varepsilon(QS_\varepsilon x_\varepsilon - x_\varepsilon)\big). \tag{2.2}$$

Then $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is uniquely defined. In addition, if $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is $\mathfrak{T}$-focused with respect to $(T_\varepsilon)_{\varepsilon\in]0,1[}$, $C \subset \bigcap_{\varepsilon\in]0,1[} \operatorname{Fix} S_\varepsilon$, and, for every $x \in \mathcal{H}$ and every sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $]0,1[$ such that $\varepsilon_n \downarrow 0$,

$$\big[\, x_{\varepsilon_n} \to x \in C \quad \textit{and} \quad x_{\varepsilon_n} - T_{\varepsilon_n} x_{\varepsilon_n} \to 0 \,\big] \quad \Rightarrow \quad S_{\varepsilon_n} x_{\varepsilon_n} \to x, \tag{2.3}$$

then $x_\varepsilon \to x_0$ as $\varepsilon \to 0$.

Proof. Let $\varepsilon \in\,]0,1[$. Since T_ε is nonexpansive, $\operatorname{Fix} T_\varepsilon$ is closed and convex [16, Proposition 1.5.3] and, therefore, C is a nonempty closed convex set. As a result, since P_C is nonexpansive and Q is a strict contraction, $P_C Q$ is a strict contraction, and it follows from the standard Banach-Picard theorem that the point x_0 is uniquely defined. Likewise, since S_ε is nonexpansive, the composition QS_ε is a strict contraction. In turn, $\varepsilon QS_\varepsilon + (1-\varepsilon)\operatorname{Id}$ is a strict contraction and so is $T_\varepsilon\big(\varepsilon QS_\varepsilon + (1-\varepsilon)\operatorname{Id}\big)$. Hence, the point x_ε is uniquely defined in (2.2).

To show the last assertion, let $\theta \in [0,1[$ be the Lipschitz constant of Q and let x be a point in C. Then we deduce from (2.2) that

$$\begin{aligned}(\forall \varepsilon \in\,]0,1[)\ \ \|x_\varepsilon - x\| &= \|T_\varepsilon\big(\varepsilon QS_\varepsilon x_\varepsilon + (1-\varepsilon)x_\varepsilon\big) - T_\varepsilon x\| \\ &\leq \|\varepsilon QS_\varepsilon x_\varepsilon + (1-\varepsilon)x_\varepsilon - x\| \\ &= \|\varepsilon(QS_\varepsilon x_\varepsilon - QS_\varepsilon x) + (1-\varepsilon)(x_\varepsilon - x) + \varepsilon(Qx - x)\| \\ &\leq \varepsilon\theta\|S_\varepsilon x_\varepsilon - S_\varepsilon x\| + (1-\varepsilon)\|x_\varepsilon - x\| + \varepsilon\|Qx - x\| \\ &\leq (1 - \varepsilon + \varepsilon\theta)\|x_\varepsilon - x\| + \varepsilon\|Qx - x\|.\end{aligned} \tag{2.4}$$

Hence,

$$(\forall \varepsilon \in\,]0,1[)\ \ \|x_\varepsilon - x\| \leq \frac{\|Qx - x\|}{1-\theta}. \tag{2.5}$$

Consequently, $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is bounded and, since

$$(\forall \varepsilon \in\,]0,1[)\ \ \|QS_\varepsilon x_\varepsilon - x_\varepsilon\| \leq \|QS_\varepsilon x_\varepsilon - QS_\varepsilon x\| + \|x_\varepsilon - Qx\| \leq \theta\|x_\varepsilon - x\| + \|x_\varepsilon - Qx\|, \tag{2.6}$$

we obtain

$$\beta = \sup_{\varepsilon\in]0,1[} \|QS_\varepsilon x_\varepsilon - x_\varepsilon\| < +\infty. \tag{2.7}$$

Now set $(\forall\varepsilon \in\,]0,1[)\ y_\varepsilon = x_\varepsilon + \varepsilon(QS_\varepsilon x_\varepsilon - x_\varepsilon)$. Then (2.2) yields

$$\begin{aligned}(\forall y \in C)(\forall\varepsilon \in\,]0,1[)\ \ \varepsilon^2\|QS_\varepsilon x_\varepsilon - x_\varepsilon\|^2 &+ 2\varepsilon \left\langle QS_\varepsilon x_\varepsilon - x_\varepsilon \mid x_\varepsilon - y\right\rangle \\ &= \|y_\varepsilon - T_\varepsilon y_\varepsilon\|^2 + 2\left\langle y_\varepsilon - T_\varepsilon y_\varepsilon \mid T_\varepsilon y_\varepsilon - y\right\rangle \\ &= \|y_\varepsilon - y\|^2 - \|T_\varepsilon y_\varepsilon - y\|^2 \\ &\geq 0.\end{aligned} \tag{2.8}$$

Therefore, by (2.7),

$$(\forall y \in C)(\forall\varepsilon \in\,]0,1[)\ \ \left\langle x_\varepsilon - QS_\varepsilon x_\varepsilon \mid x_\varepsilon - y\right\rangle \leq \frac{\varepsilon}{2}\|QS_\varepsilon x_\varepsilon - x_\varepsilon\|^2 \leq \frac{\varepsilon\beta^2}{2}. \tag{2.9}$$

Hence, using Cauchy-Schwarz, we obtain

$$\begin{aligned}(\forall y \in C)(\forall\varepsilon \in\,]0,1[)\ \ (1-\theta)\|x_\varepsilon - y\|^2 &\leq \|x_\varepsilon - y\|^2 - \|x_\varepsilon - y\|\cdot\|QS_\varepsilon x_\varepsilon - QS_\varepsilon y\| \\ &\leq \|x_\varepsilon - y\|^2 - \left\langle x_\varepsilon - y \mid QS_\varepsilon x_\varepsilon - QS_\varepsilon y\right\rangle \\ &= \left\langle (\mathrm{Id} - QS_\varepsilon)x_\varepsilon - (\mathrm{Id} - QS_\varepsilon)y \mid x_\varepsilon - y\right\rangle \\ &\leq \frac{\varepsilon\beta^2}{2} + \left\langle x_\varepsilon - y \mid Qy - y\right\rangle.\end{aligned} \tag{2.10}$$

Next, we derive from (2.2) and (2.7) that

$$\begin{aligned}(\forall\varepsilon \in\,]0,1[)\ \ \|x_\varepsilon - T_\varepsilon x_\varepsilon\| &= \|T_\varepsilon\big(x_\varepsilon + \varepsilon(QS_\varepsilon x_\varepsilon - x_\varepsilon)\big) - T_\varepsilon x_\varepsilon\| \\ &\leq \varepsilon\|QS_\varepsilon x_\varepsilon - x_\varepsilon\| && (2.11)\\ &\leq \varepsilon\beta. && (2.12)\end{aligned}$$

Thus,

$$\lim_{\varepsilon\to 0} \|x_\varepsilon - T_\varepsilon x_\varepsilon\| = 0. \tag{2.13}$$

To complete the proof, let $(\varepsilon_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in $]0,1[$ such that $\varepsilon_n \downarrow 0$. Then it is enough to show that $x_{\varepsilon_n} \to x_0$. Let w be a weak cluster point of $(x_{\varepsilon_n})_{n\in\mathbb{N}}$, say $x_{\varepsilon_{k_n}} \rightharpoonup w$. Then it follows from (2.13) and (2.1) that $w \in C$. Therefore, (2.10) yields

$$(\forall n \in \mathbb{N})\ \ (1-\theta)\|x_{\varepsilon_n} - w\|^2 \leq \frac{\varepsilon_n\beta^2}{2} + \left\langle x_{\varepsilon_n} - w \mid Qw - w\right\rangle, \tag{2.14}$$

which implies that $x_{\varepsilon_{k_n}} \to w$. Consequently, by (2.13) and (2.3), we obtain $S_{\varepsilon_{k_n}} x_{\varepsilon_{k_n}} \to w$ and, therefore, (2.9) results in

$$\begin{aligned}(\forall y \in C)\ \ \left\langle x_{\varepsilon_{k_n}} - QS_{\varepsilon_{k_n}} x_{\varepsilon_{k_n}} \,\middle|\, x_{\varepsilon_{k_n}} - y\right\rangle &\to \left\langle w - Qw \mid w - y\right\rangle \\ &\leq \varlimsup_{n\to+\infty} \left\langle x_{\varepsilon_n} - QS_{\varepsilon_n} x_{\varepsilon_n} \mid x_{\varepsilon_n} - y\right\rangle \leq 0.\end{aligned} \tag{2.15}$$

We thus obtain $\sup_{y\in C} \left\langle w - Qw \mid w - y\right\rangle \leq 0$, i.e., $w = P_C(Qw)$. Since x_0 is the unique fixed point of $P_C Q$, we have $w = x_0$. Accordingly, the bounded sequence $(x_{\varepsilon_n})_{n\in\mathbb{N}}$ admits x_0 as its unique weak cluster point, whence $x_{\varepsilon_n} \rightharpoonup x_0$. In turn, it follows from (2.14) that $x_{\varepsilon_n} \to x_0$. □

Example 2.4 Using the standard characterization of the projection onto a convex set, the limit x_0 of the approximating curve $(x_\varepsilon)_{\varepsilon\in]0,1[}$ in Theorem 2.3 is the solution to the variational inequality

$$x_0 \in C \quad \text{and} \quad (\forall y \in C) \ \ \langle y - x_0 \mid Qx_0 - x_0 \rangle \leq 0. \tag{2.16}$$

Here are some specific examples, where $0 < \alpha \leq \beta < +\infty$.

(i) Suppose that $B\colon \operatorname{dom} B = \mathcal{H} \to \mathcal{H}$ is α-strongly monotone (i.e., $B - \alpha\,\mathrm{Id}$ is monotone) and Lipschitz-continuous with constant β, and let $\gamma \in \left]0, 2\alpha/\beta^2\right[$. Then $Q = \mathrm{Id} - \gamma B$ is a strict contraction with constant $\theta = \sqrt{1 - \gamma(2\alpha - \gamma\beta^2)}$ and x_0 is the unique solution to the variational inequality

$$x_0 \in C \quad \text{and} \quad (\forall y \in C) \ \ \langle y - x_0 \mid Bx_0 \rangle \geq 0. \tag{2.17}$$

(ii) Suppose that $B\colon \operatorname{dom} B = \mathcal{H} \to \mathcal{H}$ is α-strongly monotone and that B/β is firmly nonexpansive, and let $\gamma \in \left]0, 2/\beta\right[$. Then $Q = \mathrm{Id} - \gamma B$ is a strict contraction with constant $\theta = \sqrt{1 - \alpha\gamma(2 - \beta\gamma)}$ (this constant is smaller than that given in [14, Theorem 2]). Indeed, for every x and y in $\mathcal{H}$, we have

$$\begin{aligned} \|Qx - Qy\|^2 &= \|x - y\|^2 - 2\gamma \langle x - y \mid Bx - By \rangle + \gamma^2 \|Bx - By\|^2 \\ &\leq \|x - y\|^2 - \gamma(2 - \beta\gamma) \langle x - y \mid Bx - By \rangle \\ &\leq \big(1 - \alpha\gamma(2 - \beta\gamma)\big) \|x - y\|^2. \end{aligned} \tag{2.18}$$

Here, x_0 is the unique solution to (2.17).

(iii) Suppose that $\varphi\colon \mathcal{H} \to \mathbb{R}$ is convex and differentiable, and that $\nabla\varphi$ is α-strongly monotone and Lipschitz-continuous with constant β. Then it follows from [4, Corollaire 10] that $\nabla\varphi/\beta$ is firmly nonexpansive. Hence, we deduce from (ii) that $Q = \mathrm{Id} - \gamma\nabla\varphi$ is a strict contraction for $\gamma \in \left]0, 2/\beta\right[$. In this case, x_0 is the unique minimizer of φ over C.

(iv) A special case of (iii) is when $\varphi\colon \mathcal{H} \to \mathbb{R}$ is convex, twice continuously Fréchet-differentiable, and that

$$(\forall (x, y) \in \mathcal{H}^2) \ \ \alpha\|y\|^2 \leq \left\langle y \,\middle|\, \nabla^2\varphi(x) y \right\rangle \leq \beta\|y\|^2. \tag{2.19}$$

This follows from [14, Theorem 4].

(v) Let $a \in \mathcal{H}$ and suppose that $Q\colon x \mapsto a$. Then x_0 is the projection of a onto C.

Remark 2.5 In Theorem 2.3, $\operatorname{Fix} T_\varepsilon$ may vary with ε. For instance, let $(C_\varepsilon)_{\varepsilon\in]0,1[}$ be closed convex subsets of $\mathcal{H}$ such that $C = \bigcap_{\varepsilon\in]0,1[} C_\varepsilon \neq \varnothing$ and such that the associated projectors $(T_\varepsilon)_{\varepsilon\in]0,1[}$ satisfy $(\forall x \in \mathcal{H})$ $T_\varepsilon x \rightharpoonup P_C x$ as $\varepsilon \to 0$. Furthermore, fix $a \in \mathcal{H}$ and set $Q\colon x \mapsto a$ and $S_\varepsilon \equiv \mathrm{Id}$. Then $(\forall \varepsilon \in]0,1[)$ $\operatorname{Fix} T_\varepsilon = C_\varepsilon$ and (2.2) $\Rightarrow x_\varepsilon = T_\varepsilon\big(x_\varepsilon + \varepsilon(a - x_\varepsilon)\big) = T_\varepsilon a$. Therefore, (2.3) holds trivially and $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is $\mathfrak{T}$-focused with respect to $(T_\varepsilon)_{\varepsilon\in]0,1[}$. Indeed, $x_\varepsilon \rightharpoonup x \Leftrightarrow T_\varepsilon a \rightharpoonup x$. However, since $T_\varepsilon a \rightharpoonup P_C a$, we obtain $x = P_C a \in C$.

Remark 2.6 Let $(B_\varepsilon)_{\varepsilon\in]0,1[}$ be a family of operators from $\mathcal{H}$ to $\mathcal{H}$ with domain $\mathcal{H}$ which uniquely define a curve $(x_\varepsilon)_{\varepsilon\in]0,1[}$ via the equations $(\forall\varepsilon\in\,]0,1[)\;\; x_\varepsilon = T_\varepsilon(\mathrm{Id}-\varepsilon B_\varepsilon)x_\varepsilon$. Set $(\forall\varepsilon\in\,]0,1[)\;\; y_\varepsilon = x_\varepsilon - \varepsilon B_\varepsilon x_\varepsilon$. Then

$$(\forall\varepsilon\in\,]0,1[)\;\; y_\varepsilon = (\mathrm{Id}-\varepsilon B_\varepsilon)T_\varepsilon y_\varepsilon. \tag{2.20}$$

Thus, if $x_\varepsilon \to x_0$ as $\varepsilon\to 0$ and $(B_\varepsilon x_\varepsilon)_{\varepsilon\in]0,1[}$ is bounded, we also have $y_\varepsilon \to x_0$ as $\varepsilon \to 0$. This simple observation yields the following alternative approximating curve result. Let us make the same assumptions as in Theorem 2.3 and let us set $(\forall\varepsilon\in\,]0,1[)\;\; B_\varepsilon = \mathrm{Id}-QS_\varepsilon$. Then (2.20) becomes

$$(\forall\varepsilon\in\,]0,1[)\;\; y_\varepsilon = \varepsilon QS_\varepsilon T_\varepsilon y_\varepsilon + (1-\varepsilon)T_\varepsilon y_\varepsilon. \tag{2.21}$$

In view of (2.7), the family $(B_\varepsilon x_\varepsilon)_{\varepsilon\in]0,1[}$ is bounded. Therefore, Theorem 2.3 yields $y_\varepsilon \to x_0 = P_C(Qx_0)$ as $\varepsilon\to 0$. In particular, if $a\in\mathcal{H}$, $Q\colon x\mapsto a$, $T_\varepsilon\equiv T$, and $S_\varepsilon\equiv\mathrm{Id}$, we recover the classical result [9, Theorem 2] alluded to in Section 1 (see also [10, Theorem 1] and [17, Theorem 1] for alternate proofs of this result).

Remark 2.7 (Infeasible case) Suppose that we make the same assumptions as in Theorem 2.3, except that $C=\varnothing$ and $D=\bigcap_{\varepsilon\in]0,1[}\mathrm{Fix}\,S_\varepsilon\neq\varnothing$ (e.g., $S_\varepsilon\equiv\mathrm{Id}$). Then $\|x_\varepsilon\|\to+\infty$ as $\varepsilon\to 0$. Indeed, otherwise there would exist a bounded sequence $(x_{\varepsilon_n})_{n\in\mathbb{N}}$, where $]0,1[\,\ni\varepsilon_n\downarrow 0$. Taking $x\in D$ in (2.6), we would obtain the boundedness of $(QS_{\varepsilon_n}x_{\varepsilon_n}-x_{\varepsilon_n})_{n\in\mathbb{N}}$ and it would follow from (2.11) that $T_{\varepsilon_n}x_{\varepsilon_n}-x_{\varepsilon_n}\to 0$. On the other hand, we could extract a subsequence $(x_{\varepsilon_{k_n}})_{n\in\mathbb{N}}$ such that $x_{\varepsilon_{k_n}}\rightharpoonup w$. However, the $\mathfrak{T}$-focused assumption would yield $w\in C$, which is absurd.

We close this section with a special case of Theorem 2.3.

Corollary 2.8 *Let $T\colon \mathrm{dom}\,T=\mathcal{H}\to\mathcal{H}$ be a nonexpansive operator such that $\mathrm{Fix}\,T\neq\varnothing$ and let $Q\colon \mathrm{dom}\,Q=\mathcal{H}\to\mathcal{H}$ be a strict contraction. Then there exists a unique point $x_0\in\mathrm{Fix}\,T$ such that $x_0=P_{\mathrm{Fix}\,T}(Q\,x_0)$. Now let $(\lambda_\varepsilon)_{\varepsilon\in]0,1[}$ and $(\mu_\varepsilon)_{\varepsilon\in]0,1[}$ be families in $[0,1]$ such that $\inf_{\varepsilon\in]0,1[}\lambda_\varepsilon>0$ and set*

$$(\forall\varepsilon\in\,]0,1[)\;\; x_\varepsilon = \big(\mathrm{Id}+\lambda_\varepsilon(T-\mathrm{Id})\big)\big(x_\varepsilon+\varepsilon\big(Q\big(x_\varepsilon+\mu_\varepsilon(Tx_\varepsilon-x_\varepsilon)\big)-x_\varepsilon\big)\big). \tag{2.22}$$

Then $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is uniquely defined and $x_\varepsilon\to x_0$ as $\varepsilon\to 0$.

Proof. Set $C=\mathrm{Fix}\,T$ and, for every $\varepsilon\in\,]0,1[$, $T_\varepsilon=\mathrm{Id}+\lambda_\varepsilon(T-\mathrm{Id})$ and $S_\varepsilon=\mathrm{Id}+\mu_\varepsilon(T-\mathrm{Id})$. Then, for every $\varepsilon\in\,]0,1[$, T_ε and S_ε are nonexpansive, and $\mathrm{Fix}\,S_\varepsilon=C$ or $\mathcal{H}$, according as $0<\mu_\varepsilon\leq 1$ or $\mu_\varepsilon=0$. On the other hand, since $\inf_{\varepsilon\in]0,1[}\lambda_\varepsilon>0$, $\mathrm{Fix}\,T_\varepsilon\equiv C$. Altogether, $\varnothing\neq C=\bigcap_{\varepsilon\in]0,1[}\mathrm{Fix}\,T_\varepsilon\subset\bigcap_{\varepsilon\in]0,1[}\mathrm{Fix}\,S_\varepsilon$. Moreover, Example 2.2 shows that $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is $\mathfrak{T}$-focused with respect to $(T_\varepsilon)_{\varepsilon\in]0,1[}$, while (2.3) is readily verified. Thus, the result is a special case of Theorem 2.3. □

In particular, setting $Q\colon x\mapsto a$ and $\lambda_\varepsilon\equiv 1$ in Corollary 2.8, we recover the fact that the limit of the approximating curve (1.1) is the best approximation to a from $\mathrm{Fix}\,T$.

3 Monotone inclusion problems

Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued operator. The sets $\operatorname{dom} A = \{x \in \mathcal{H} \mid Ax \neq \varnothing\}$, $\operatorname{ran} A = \{u \in \mathcal{H} \mid (\exists\, x \in \mathcal{H})\ u \in Ax\}$, and $\operatorname{gr} A = \{(x,u) \in \mathcal{H}^2 \mid u \in Ax\}$ are the domain, the range, and the graph of A, respectively. The inverse A^{-1} of A is the set-valued operator with graph $\{(u,x) \in \mathcal{H}^2 \mid u \in Ax\}$, the resolvent of A is $J_A = (\operatorname{Id} + A)^{-1}$, and its Yosida approximation of index $\gamma \in \left]0,+\infty\right[$ is ${}^{\gamma}\!A = (\operatorname{Id} - J_{\gamma A})/\gamma$. Moreover, A is monotone if

$$(\forall (x,u) \in \operatorname{gr} A)(\forall (y,v) \in \operatorname{gr} A) \quad \langle x-y \mid u-v \rangle \geq 0, \tag{3.1}$$

and maximal monotone if, furthermore, $\operatorname{gr} A$ is not properly contained in the graph of any monotone operator $B\colon \mathcal{H} \to 2^{\mathcal{H}}$. If A is monotone and $\operatorname{dom} A \neq \varnothing$, the associated Fitzpatrick function [15] is the proper lower semicontinuous convex function $f_A\colon \mathcal{H} \times \mathcal{H} \to \left]-\infty,+\infty\right]$ defined by

$$(\forall (x,w) \in \mathcal{H} \times \mathcal{H}) \ \ f_A(x,w) = \langle x \mid w \rangle + \sup_{(y,v) \in \operatorname{gr} A} \langle x-y \mid v-w \rangle. \tag{3.2}$$

Definition 3.1 *Let $(A_\varepsilon)_{\varepsilon \in]0,1[}$ be a family of maximal monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$ and let $(x_\varepsilon)_{\varepsilon \in]0,1[}$ be a family in $\mathcal{H}$. Then $(x_\varepsilon)_{\varepsilon \in]0,1[}$ is $\mathcal{A}$-*focused *with respect to $(A_\varepsilon)_{\varepsilon \in]0,1[}$ if, for every $x \in \mathcal{H}$ and every sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ in $]0,1[$ such that $\varepsilon_n \downarrow 0$,*

$$\big[\, x_{\varepsilon_n} \rightharpoonup x \quad \text{and} \quad {}^{1}\!A_{\varepsilon_n} x_{\varepsilon_n} \to 0 \,\big] \quad \Rightarrow \quad (\forall \varepsilon \in]0,1[)\ \ 0 \in A_\varepsilon x. \tag{3.3}$$

Example 3.2 Let $(A_\varepsilon)_{\varepsilon \in]0,1[}$ be a family of maximal monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$ which graph-converges to some maximal monotone operator $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ such that $A^{-1}0 = \bigcap_{\varepsilon \in]0,1[} A_\varepsilon^{-1}0$, and take $(x_\varepsilon)_{\varepsilon \in]0,1[}$ in $\mathcal{H}$. Then $(x_\varepsilon)_{\varepsilon \in]0,1[}$ is $\mathcal{A}$-focused with respect to $(A_\varepsilon)_{\varepsilon \in]0,1[}$.

Proof. Suppose that $]0,1[\, \ni \varepsilon_n \downarrow 0$, $x_{\varepsilon_n} \rightharpoonup x$, and ${}^{1}\!A_{\varepsilon_n} x_{\varepsilon_n} \to 0$. Then $J_{A_{\varepsilon_n}} x_{\varepsilon_n} \rightharpoonup x$ and ${}^{1}\!A_{\varepsilon_n} x_{\varepsilon_n} \to 0$, while $(\forall n \in \mathbb{N})\ \big(J_{A_{\varepsilon_n}} x_{\varepsilon_n}, {}^{1}\!A_{\varepsilon_n} x_{\varepsilon_n}\big) \in \operatorname{gr} A_{\varepsilon_n}$. Therefore, [1, Proposition 3.59] yields $(x,0) \in \operatorname{gr} A$. □

We start with an application of Theorem 2.3.

Corollary 3.3 *Let $(A_\varepsilon)_{\varepsilon \in]0,1[}$ be a family of maximal monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$ such that $C = \bigcap_{\varepsilon \in]0,1[} A_\varepsilon^{-1}0 \neq \varnothing$ and let $Q\colon \operatorname{dom} Q = \mathcal{H} \to \mathcal{H}$ be a strict contraction. Then there exists a unique point $x_0 \in C$ such that $x_0 = P_C(Q\,x_0)$. Now take $(\rho_\varepsilon)_{\varepsilon \in]0,1[}$ and $(\nu_\varepsilon)_{\varepsilon \in]0,1[}$ in $[0,2]$ such that $\inf_{\varepsilon \in]0,1[} \rho_\varepsilon > 0$, and set*

$$(\forall \varepsilon \in]0,1[)\ \ x_\varepsilon = \big(\operatorname{Id} + \rho_\varepsilon (J_{A_\varepsilon} - \operatorname{Id})\big)\big(x_\varepsilon + \varepsilon\big(Q\big(x_\varepsilon + \nu_\varepsilon (J_{A_\varepsilon} x_\varepsilon - x_\varepsilon)\big) - x_\varepsilon\big)\big). \tag{3.4}$$

Then the family $(x_\varepsilon)_{\varepsilon \in]0,1[}$ is uniquely defined. In addition, if $(x_\varepsilon)_{\varepsilon \in]0,1[}$ is $\mathcal{A}$-focused with respect to $(A_\varepsilon)_{\varepsilon \in]0,1[}$, then $x_\varepsilon \to x_0$ as $\varepsilon \to 0$.

Proof. Set $(\forall \varepsilon \in]0,1[)$ $T_\varepsilon = \operatorname{Id} + \rho_\varepsilon (J_{A_\varepsilon} - \operatorname{Id})$ and $S_\varepsilon = \operatorname{Id} + \nu_\varepsilon (J_{A_\varepsilon} - \operatorname{Id})$. For every $\varepsilon \in]0,1[$, since A_ε is maximal monotone, $2J_{A_\varepsilon} - \operatorname{Id}$ is nonexpansive with domain $\mathcal{H}$ and fixed point set $A_\varepsilon^{-1}0$

[16, Section 1.11]; consequently, T_ε and S_ε are nonexpansive, $\operatorname{Fix} T_\varepsilon = A_\varepsilon^{-1}0$ (since $\rho_\varepsilon > 0$), and $\operatorname{Fix} S_\varepsilon = A_\varepsilon^{-1}0$ or $\mathcal{H}$, according as $0 < \nu_\varepsilon \leq 2$ or $\nu_\varepsilon = 0$. Consequently, $\varnothing \neq C = \bigcap_{\varepsilon\in]0,1[} \operatorname{Fix} T_\varepsilon \subset \bigcap_{\varepsilon\in]0,1[} \operatorname{Fix} S_\varepsilon$. Now take $]0,1[\, \ni \varepsilon_n \downarrow 0$. Since $\inf_{\varepsilon\in]0,1[} \rho_\varepsilon > 0$, $x_{\varepsilon_n} - T_{\varepsilon_n} x_{\varepsilon_n} \to 0 \Rightarrow {}^1\!A_{\varepsilon_n} x_{\varepsilon_n} \to 0$, and it follows from (3.3) that $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is $\mathfrak{T}$-focused with respect to $(T_\varepsilon)_{\varepsilon\in]0,1[}$. Finally, suppose that $x_{\varepsilon_n} \to x \in C$. Then $x_{\varepsilon_n} - T_{\varepsilon_n} x_{\varepsilon_n} \to 0 \Rightarrow 2\|{}^1\!A_{\varepsilon_n} x_{\varepsilon_n}\| \to 0 \Rightarrow \nu_{\varepsilon_n}\|{}^1\!A_{\varepsilon_n} x_{\varepsilon_n}\| \to 0 \Rightarrow \|x_{\varepsilon_n} - S_{\varepsilon_n} x_{\varepsilon_n}\| \to 0 \Rightarrow S_{\varepsilon_n} x_{\varepsilon_n} \to x$. Hence, (2.3) holds. Altogether, since (3.4) is a special case of (2.2), the claims follow from Theorem 2.3. ☐

Corollary 3.4 *Let $(A_\varepsilon)_{\varepsilon\in]0,1[}$ be a family of maximal monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$ such that $C = \bigcap_{\varepsilon\in]0,1[} A_\varepsilon^{-1}0 \neq \varnothing$ and let $B = \operatorname{Id} - Q$, where $Q\colon \operatorname{dom} Q = \mathcal{H} \to \mathcal{H}$ is a strict contraction. Then there exists a unique point $x_0 \in C$ such that $x_0 = P_C(x_0 - Bx_0)$. Now let*

$$(\forall \varepsilon \in \,]0,1[)\;\; 0 \in A_\varepsilon x_\varepsilon + \varepsilon B x_\varepsilon. \tag{3.5}$$

Then the family $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is uniquely defined. In addition, if $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is $\mathcal{A}$-focused with respect to $(A_\varepsilon)_{\varepsilon\in]0,1[}$, then $x_\varepsilon \to x_0$ as $\varepsilon \to 0$.

Proof. Setting $\rho_\varepsilon \equiv 1$ and $\nu_\varepsilon \equiv 0$ in (3.4), we obtain (3.5). We can then apply Corollary 3.3. ☐

Remark 3.5 We can rewrite (3.5) as $(\forall \varepsilon \in \,]0,1[)\;\; x_\varepsilon = J_{A_\varepsilon/\varepsilon}\big(x_\varepsilon - Bx_\varepsilon\big)$. In particular, for $A_\varepsilon \equiv A$ and $B = \operatorname{Id}$, we obtain

$$(\forall \varepsilon \in \,]0,1[)\;\; x_\varepsilon = J_{A/\varepsilon}0. \tag{3.6}$$

(i) In this case, Corollary 3.4 coincides with [11, Lemma 1], i.e., $J_{A/\varepsilon}0 \to P_{A^{-1}0}0$ as $\varepsilon \to 0$ (here (3.3) follows from the fact that, by maximal monotonicity of A, $\operatorname{gr} A$ is sequentially weakly-strongly closed in $\mathcal{H} \times \mathcal{H}$). This result can be traced back to [20] (see also [23, Theorem 1 and Remark 2] for a Banach space version, and [9, Theorem 1] for a related result; moreover, Remark 2.7 corresponds to [22, Theorem 2], i.e., $\|x_\varepsilon\| \to +\infty$ if $0 \notin \operatorname{ran} A$).

(ii) Let $U\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator, let $x \in \mathcal{H}$, and let $A = U^{-1} - x$. Then (3.6) becomes $(\forall \varepsilon \in \,]0,1[)\;\; x_\varepsilon = {}^\varepsilon U x$. Therefore, (i) asserts that

   (a) if $x \in \operatorname{dom} U$, that is $0 \in \operatorname{ran} A$, then $x_\varepsilon \to P_{A^{-1}0}0 = P_{Ux}0$ as $\varepsilon \to 0$;

   (b) if $x \notin \operatorname{dom} U$, that is $0 \notin \operatorname{ran} A$, then $\|x_\varepsilon\| \to +\infty$ as $\varepsilon \to 0$.

   This classical result can be found in [7, Proposition 2.6(iii)&(iv)].

In Corollary 3.4, the approximating curve (3.5) converges strongly to the solution x_0 to the variational inequality

$$0 \in N_{\left(\bigcap_{\varepsilon\in]0,1[} A_\varepsilon^{-1}0\right)} x_0 + Bx_0, \tag{3.7}$$

where B is a special type of single-valued strongly monotone operator (see Example 2.4 for specific examples). In Theorem 3.10 below, we extend this result to a more general type of set-valued strictly monotone operator B. First, we require the following facts, starting with a generalization of the notion of strong monotonicity.

Definition 3.6 *Let* $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ *be a set-valued operator with* $\operatorname{dom} B \neq \varnothing$ *and let* $c\colon\ [0,+\infty[\ \to [0,+\infty[$ *be a nondecreasing function that vanishes only at* 0 *and such that* $\lim_{t\to+\infty} c(t)/t = +\infty$. *Then* B *is* c-uniformly monotone *if*

$$(\forall (x,u) \in \operatorname{gr} B)(\forall (y,v) \in \operatorname{gr} B) \quad \langle x-y \mid u-v\rangle \geq c(\|x-y\|). \tag{3.8}$$

If $c\colon t \mapsto \alpha t^2$ *for some* $\alpha \in\ ]0,+\infty[$, *then* B *is* α-strongly monotone.

Lemma 3.7 *Let* $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ *be a c-uniformly monotone operator. Then* $(\operatorname{dom} B) \times \mathcal{H} \subset \operatorname{dom} f_B$.

Proof. Fix $(x,u) \in \operatorname{gr} B$ and $w \in \mathcal{H}$, and set $\gamma = \|u-w\|$ and $\psi\colon\ [0,+\infty[\ \to \mathbb{R}\colon t \mapsto \gamma t - c(t)$. Since $\lim_{t\to+\infty} c(t)/t = +\infty$, we can find $\tau \in [0,+\infty[$ such that $\psi(t) < 0 = \psi(0)$ whenever $t > \tau$. Thus, $\sup_{t\in[0,+\infty[} \psi(t) = \sup_{t\in[0,\tau]} \psi(t) \leq \gamma\tau < +\infty$. Therefore, (3.2), (3.8), and Cauchy-Schwarz yield

$$f_B(x,w) - \langle x \mid w\rangle = \sup_{(y,v)\in\operatorname{gr} B} \langle x-y \mid v-u\rangle + \langle x-y \mid u-w\rangle \leq \sup_{y\in\operatorname{dom} B} \psi(\|x-y\|) < +\infty. \tag{3.9}$$

In other words, $(x,w) \in \operatorname{dom} f_B$. ☐

Lemma 3.8 *Let* $A, B\colon \mathcal{H} \to 2^{\mathcal{H}}$ *be maximal monotone operators such that* $A+B$ *is maximal monotone and* B *is c-uniformly monotone. Suppose that, in addition,* $(\operatorname{dom} A) \times (\operatorname{ran} A) \subset \operatorname{dom} f_A$ *or* $\operatorname{dom} A \subset \operatorname{dom} B$. *Then:*

(i) $\operatorname{ran}(A+B) = \mathcal{H}$.

(ii) *The inclusion* $0 \in Ax + Bx$ *admits a unique solution.*

Proof. (i): Fix $(y,v) \in \operatorname{gr} B$. Then (3.8) yields

$$\begin{aligned}(\forall (x,u) \in \operatorname{gr} B) \quad \|x-y\| \cdot \|u\| &\geq \langle x-y \mid u\rangle \\ &= \langle x-y \mid u-v\rangle + \langle x-y \mid v\rangle \\ &\geq c(\|x-y\|) - \|x-y\| \cdot \|v\|. \end{aligned} \tag{3.10}$$

Accordingly, since $\lim_{t\to+\infty} c(t)/t = +\infty$, we have $\lim_{\substack{x\in\operatorname{dom} B\\ \|x\|\to+\infty}} \inf_{u\in Bx} \|u\| = +\infty$ whenever $\operatorname{dom} B$ is unbounded. It then follows from [28, Corollary 32.35] that $\operatorname{ran} B = \mathcal{H}$ and, in turn, from Lemma 3.7 that $(\operatorname{dom} B) \times (\operatorname{ran} B) = (\operatorname{dom} B) \times \mathcal{H} \subset \operatorname{dom} f_B$. We then deduce from the Brézis-Haraux theorem [8, Théorèmes 3 and 4] that $\operatorname{int}\operatorname{ran}(A+B) = \operatorname{int}(\operatorname{ran} A + \operatorname{ran} B) = \mathcal{H}$.

(ii): Since A is monotone and B is strictly monotone, $A+B$ is strictly monotone. Hence, the inclusion $0 \in Ax + Bx$ has at most one solution. Existence follows from (i). ☐

Remark 3.9 Fitzpatrick functions have recently been shown to be remarkably useful in establishing concise proofs of various key results in monotone operator theory (see [6, 24] and the references therein). In the same vein, S. Simons (personal communication, April 7, 2005) has produced a new proof of the Brézis-Haraux theorem in Banach spaces.

Theorem 3.10 *Let $(A_\varepsilon)_{\varepsilon\in]0,1[}$ be a family of maximal monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$ such that $C = \bigcap_{\varepsilon\in]0,1[} A_\varepsilon^{-1}0 \neq \varnothing$ and let $B\colon \operatorname{dom} B = \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator which is c-uniformly monotone. Then there exists a unique point $x_0 \in \mathcal{H}$ such that $0 \in N_C x_0 + Bx_0$. Now let*

$$(\forall \varepsilon \in]0,1[)\ \ 0 \in A_\varepsilon x_\varepsilon + \varepsilon B x_\varepsilon. \tag{3.11}$$

Then the family $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is uniquely defined. In addition, if B maps every bounded subset into a bounded subset and if $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is $\mathcal{A}$-focused with respect to $(A_\varepsilon)_{\varepsilon\in]0,1[}$, then $x_\varepsilon \to x_0$ as $\varepsilon \to 0$.

Proof. By maximal monotonicity, the sets $(A_\varepsilon^{-1}0)_{\varepsilon\in]0,1[}$ are closed and convex, and so is therefore C. Accordingly, N_C is maximal monotone and, since $\operatorname{dom} B = \mathcal{H}$, Lemma 3.8(ii) guarantees that x_0 is uniquely defined. Likewise, it follows from (3.11) and Lemma 3.8(ii) that $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is uniquely defined.

To show the last assertion, we first derive from (3.11) that there exists a family $(b_\varepsilon)_{\varepsilon\in]0,1[}$ such that

$$(\forall \varepsilon \in]0,1[) \quad b_\varepsilon \in Bx_\varepsilon \quad \text{and} \quad -\varepsilon b_\varepsilon \in A_\varepsilon x_\varepsilon. \tag{3.12}$$

Now fix $x \in C$ and $u \in Bx$. Then $(\forall \varepsilon \in]0,1[)\ 0 \in A_\varepsilon x$. Hence, in view of (3.12), the monotonicity of the operators $(A_\varepsilon)_{\varepsilon\in]0,1[}$ yields

$$(\forall \varepsilon \in]0,1[)\ \ \langle x - x_\varepsilon \mid b_\varepsilon \rangle \geq 0, \tag{3.13}$$

while the c-uniform monotonicity of B yields

$$(\forall \varepsilon \in]0,1[)\ \ \langle x - x_\varepsilon \mid u - b_\varepsilon \rangle \geq c(\|x - x_\varepsilon\|). \tag{3.14}$$

Adding (3.13) and (3.14) we obtain

$$(\forall \varepsilon \in]0,1[)\ \ \langle x - x_\varepsilon \mid u \rangle \geq c(\|x - x_\varepsilon\|), \tag{3.15}$$

and therefore

$$(\forall \varepsilon \in]0,1[)\ \ \|x - x_\varepsilon\| \cdot \|u\| \geq c(\|x - x_\varepsilon\|). \tag{3.16}$$

Consequently, since $\lim_{t\to+\infty} c(t)/t = +\infty$, $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is bounded. In turn, it follows from the boundedness of B on bounded sets that

$$\beta = \sup_{\varepsilon\in]0,1[} \|b_\varepsilon\| < +\infty. \tag{3.17}$$

Now, observe that the monotonicity of the operators $(A_\varepsilon)_{\varepsilon\in]0,1[}$ and (3.12) yield

$$(\forall \varepsilon \in]0,1[)(\forall y \in C)\ \ \langle x_\varepsilon - y \mid b_\varepsilon \rangle \leq 0. \tag{3.18}$$

Likewise,

$$\begin{cases} -\varepsilon b_\varepsilon \in A_\varepsilon x_\varepsilon \\ {}^1\!A_\varepsilon x_\varepsilon \in A_\varepsilon J_{A_\varepsilon} x_\varepsilon \end{cases} \quad \Rightarrow \quad \left\langle {}^1\!A_\varepsilon x_\varepsilon \;\middle|\; {}^1\!A_\varepsilon x_\varepsilon + \varepsilon b_\varepsilon \right\rangle \leq 0 \quad \Rightarrow \quad \|{}^1\!A_\varepsilon x_\varepsilon\| \leq \varepsilon \|b_\varepsilon\| \leq \varepsilon\beta, \tag{3.19}$$

where the last implication follows from Cauchy-Schwarz and (3.17). We have thus shown that

$$\lim_{\varepsilon\to 0} \|\,{}^{1}\!A_\varepsilon x_\varepsilon\| = 0. \tag{3.20}$$

Now let $(\varepsilon_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in $]0,1[$ such that $\varepsilon_n \downarrow 0$. Then it remains to show that $x_{\varepsilon_n} \to x_0$. To this end, take a weak cluster point of $(x_{\varepsilon_n})_{n\in\mathbb{N}}$, say $x_{\varepsilon_{k_n}} \rightharpoonup w$. Then it follows from (3.20) and (3.3) that $w \in C$. In turn, (3.15) implies that $x_{\varepsilon_{k_n}} \to w$. Moreover, in view of (3.17), passing to a further subsequence if necessary, we assume that $(b_{\varepsilon_{k_n}})_{n\in\mathbb{N}}$ converges weakly, say $b_{\varepsilon_{k_n}} \rightharpoonup v$. Since B is maximal monotone, its graph is sequentially strongly-weakly closed in $\mathcal{H}\times\mathcal{H}$ and therefore $v \in Bw$. Altogether, $x_{\varepsilon_{k_n}} \to w$, $b_{\varepsilon_{k_n}} \rightharpoonup v$, and hence (3.18) yields

$$(\forall y \in C)\quad \big\langle x_{\varepsilon_{k_n}} - y \;\big|\; b_{\varepsilon_{k_n}} \big\rangle \to \langle w - y \mid v\rangle \leq \varlimsup_{n\to+\infty} \langle x_{\varepsilon_n} - y \mid b_{\varepsilon_n}\rangle \leq 0. \tag{3.21}$$

Consequently, $\sup_{y\in C} \langle w - y \mid v\rangle \leq 0$ and, therefore, $-v \in N_C w$. Recalling that $v \in Bw$, we obtain $0 \in N_C w + Bw$. However, since the inclusion $0 \in N_C x_0 + Bx_0$ admits a unique solution, $w = x_0$ is the unique weak cluster point of $(x_{\varepsilon_n})_{n\in\mathbb{N}}$ and therefore $x_{\varepsilon_n} \rightharpoonup x_0$. Invoking (3.15), we conclude that $x_{\varepsilon_n} \to x_0$. ☐

Remark 3.11 (Infeasible case) Suppose that we make the same assumptions as in Theorem 3.10, except that $C = \varnothing$. Then $\|x_\varepsilon\| \to +\infty$ as $\varepsilon \to 0$. Indeed, otherwise there would exist a bounded sequence $(x_{\varepsilon_n})_{n\in\mathbb{N}}$, where $]0,1[\, \ni \varepsilon_n \downarrow 0$. Hence, the sequence $(b_{\varepsilon_n})_{n\in\mathbb{N}}$ given by (3.12) would also be bounded and, as in (3.19), we would get $\|\,{}^{1}\!A_{\varepsilon_n} x_{\varepsilon_n}\| \leq \varepsilon_n \|b_{\varepsilon_n}\| \to 0$. Furthermore, we could extract a subsequence $(x_{\varepsilon_{k_n}})_{n\in\mathbb{N}}$ such that $x_{\varepsilon_{k_n}} \rightharpoonup w$, and (3.3) would force $w \in C = \varnothing$.

Remark 3.12

(i) As seen in Example 3.2, the $\mathcal{A}$-focused condition holds in Theorem 3.10 when $(A_\varepsilon)_{\varepsilon\in]0,1[}$ graph-converges to a maximal monotone operator $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ such that $A^{-1}0 = \bigcap_{\varepsilon\in]0,1[} A_\varepsilon^{-1}0$. In such instances, $0 \in N_{A^{-1}0}x_0 + Bx_0$.

(ii) Suppose that B is single-valued in Theorem 3.10. Then $x_\varepsilon \to x_0 = P_C(x_0 - Bx_0)$ as $\varepsilon \to 0$. However, as shown in (3.17), the family $(Bx_\varepsilon)_{\varepsilon\in]0,1[}$ is bounded. On the other hand, (3.11) can be rewritten as

$$(\forall \varepsilon \in\,]0,1[)\;\; x_\varepsilon = J_{A_\varepsilon}\big(\mathrm{Id} - \varepsilon B\big)x_\varepsilon. \tag{3.22}$$

Consequently, it follows from the observation made in Remark 2.6 that, under the same assumptions as in Theorem 3.10, the approximating curve defined by $(\forall \varepsilon \in\,]0,1[)\;\; y_\varepsilon = x_\varepsilon - \varepsilon Bx_\varepsilon$, i.e., by

$$(\forall \varepsilon \in\,]0,1[)\;\; y_\varepsilon = J_{A_\varepsilon} y_\varepsilon - \varepsilon B J_{A_\varepsilon} y_\varepsilon, \tag{3.23}$$

converges strongly to x_0 as $\varepsilon \to 0$.

(iii) Consider the special case when $A_\varepsilon \equiv A$. Then (3.11) reduces to

$$(\forall \varepsilon \in\,]0,1[)\;\; 0 \in Ax_\varepsilon + \varepsilon Bx_\varepsilon. \tag{3.24}$$

In this context, a result related to Theorem 3.10 – though based on different assumptions – is [28, Theorem 32.K]. If we further specialize by imposing that B be strongly monotone, then Theorem 3.10 and Remark 3.11 reduce to [18, Proposition 2.1]. Finally, when A is the subdifferential of a proper lower semicontinuous convex function $f\colon \mathcal{H} \to \,]-\infty, +\infty]$ and B the subdifferential of a uniformly convex function $g\colon \mathcal{H} \to \mathbb{R}$, x_ε in (3.24) is the minimizer of $f + \varepsilon g$, and we obtain the Tikhonov regularization setting (see [2, Section 5] for related results and [26] for classical work).

4 Further nonexpansive fixed point results

In this section, we derive from the results of Section 3 additional approximating curves for fixed point problems.

As seen in Example 2.4(i)&(ii), Theorem 2.3 asserts that if $B\colon \operatorname{dom} B = \mathcal{H} \to \mathcal{H}$ is strongly monotone and possesses additional properties then, for some suitable $\gamma \in \,]0, +\infty[$, the limit x_0 of the approximating curve

$$(\forall \varepsilon \in \,]0,1[)\ \ x_\varepsilon = T_\varepsilon\big(x_\varepsilon + \varepsilon((\operatorname{Id} - \gamma B)S_\varepsilon x_\varepsilon - x_\varepsilon)\big), \tag{4.1}$$

as $\varepsilon \to 0$, solves the variational inequality (2.17). We now investigate an alternative approximating curve, which allows for a more general type of operator B.

Corollary 4.1 *Let $(T_\varepsilon)_{\varepsilon\in]0,1[}$ be a family of nonexpansive operators from $\mathcal{H}$ to $\mathcal{H}$ with domain $\mathcal{H}$ such that $C = \bigcap_{\varepsilon\in]0,1[} \operatorname{Fix} T_\varepsilon \neq \varnothing$ and let $B\colon \operatorname{dom} B = \mathcal{H} \to \mathcal{H}$ be a maximal monotone operator which is c-uniformly monotone. Then there exists a unique point $x_0 \in C$ such that $x_0 = P_C(x_0 - Bx_0)$. Now set*

$$(\forall \varepsilon \in \,]0,1[)\ \ x_\varepsilon = T_\varepsilon\big(x_\varepsilon - \varepsilon B x_\varepsilon\big) - \varepsilon B x_\varepsilon. \tag{4.2}$$

Then $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is uniquely defined. In addition, if B maps every bounded subset into a bounded subset and $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is $\mathcal{T}$-focused with respect to $(T_\varepsilon)_{\varepsilon\in]0,1[}$, then $x_\varepsilon \to x_0$ as $\varepsilon \to 0$.

Proof. Set $(\forall \varepsilon \in \,]0,1[)$ $F_\varepsilon = (T_\varepsilon + \operatorname{Id})/2$ and $A_\varepsilon = F_\varepsilon^{-1} - \operatorname{Id}$. Since $(F_\varepsilon)_{\varepsilon\in]0,1[}$ is a family of firmly nonexpansive operators with domain $\mathcal{H}$, $(A_\varepsilon)_{\varepsilon\in]0,1[}$ is a family of maximal monotone operators [16, Section 1.11]. Moreover, it follows from Definitions 2.1 and 3.1 that $(x_\varepsilon)_{\varepsilon\in]0,1[}$ is $\mathcal{A}$-focused with respect to $(A_\varepsilon)_{\varepsilon\in]0,1[}$. Finally, since (4.2) is equivalent to (3.22), which is itself equivalent to (3.11), the results follow from Theorem 3.10. ☐

We conclude with two results on the approximation of a particular fixed point of a nonexpansive operator.

Corollary 4.2 *Let $T\colon \operatorname{dom} T = \mathcal{H} \to \mathcal{H}$ be a nonexpansive operator such that $\operatorname{Fix} T \neq \varnothing$, let $(\lambda_\varepsilon)_{\varepsilon\in]0,1[}$ be a family in $]0,1]$ such that $\inf_{\varepsilon\in]0,1[} \lambda_\varepsilon > 0$, and let $B\colon \operatorname{dom} B = \mathcal{H} \to \mathcal{H}$ be a maximal* *monotone operator which is c-uniformly monotone. Then there exists a unique point $x_0 \in \operatorname{Fix} T$ such that $x_0 = P_{\operatorname{Fix} T}(x_0 - Bx_0)$. Now set*

$$(\forall \varepsilon \in \,]0,1[) \;\; x_\varepsilon = T\big(x_\varepsilon - Bx_\varepsilon\big) + \varepsilon \frac{\lambda_\varepsilon - 2}{\lambda_\varepsilon} Bx_\varepsilon. \tag{4.3}$$

Then $(x_\varepsilon)_{\varepsilon \in]0,1[}$ is uniquely defined. In addition, if B maps every bounded subset into a bounded subset, then $x_\varepsilon \to x_0$ as $\varepsilon \to 0$.

Proof. Set $(\forall \varepsilon \in \,]0,1[)$ $T_\varepsilon = \operatorname{Id} + \lambda_\varepsilon(T - \operatorname{Id})$ in Corollary 4.1 and use Example 2.2. ☐

Corollary 4.3 *Let $T\colon \operatorname{dom} T = \mathcal{H} \to \mathcal{H}$ be a nonexpansive operator such that $\operatorname{Fix} T \neq \varnothing$ and let $B\colon \operatorname{dom} B = \mathcal{H} \to \mathcal{H}$ be a maximal monotone operator which is c-uniformly monotone. Then there exists a unique point $x_0 \in \operatorname{Fix} T$ such that $x_0 = P_{\operatorname{Fix} T}(x_0 - Bx_0)$. Now set*

$$(\forall \varepsilon \in \,]0,1[) \;\; x_\varepsilon = Tx_\varepsilon - \varepsilon Bx_\varepsilon. \tag{4.4}$$

Then $(x_\varepsilon)_{\varepsilon \in]0,1[}$ is uniquely defined. In addition, if B maps every bounded subset into a bounded subset, then $x_\varepsilon \to x_0$ as $\varepsilon \to 0$.

Proof. It suffices to set $A_\varepsilon \equiv \operatorname{Id} - T$ in Theorem 3.10. To check (3.3), take $]0,1[\, \ni \varepsilon_n \downarrow 0$, $x_{\varepsilon_n} \rightharpoonup x$, and ${}^1A_{\varepsilon_n} x_{\varepsilon_n} \to 0$. Letting $(\forall n \in \mathbb{N})$ $p_n = J_{A_{\varepsilon_n}} x_{\varepsilon_n}$, we obtain $p_n \rightharpoonup x$ and $p_n - Tp_n = A_{\varepsilon n} p_n = x_{\varepsilon_n} - p_n \to 0$. Then the demiclosed principle [10, Lemma 2] yields $x \in \operatorname{Fix} T \equiv A_\varepsilon^{-1} 0$. ☐

In particular, if $B = \operatorname{Id} - Q$, where $Q\colon \operatorname{dom} Q = \mathcal{H} \to \mathcal{H}$ is a strict contraction, then (4.4) reduces to

$$(\forall \varepsilon \in \,]0,1[) \;\; x_\varepsilon = \frac{\varepsilon}{\varepsilon + 1} Q x_\varepsilon + \frac{1}{\varepsilon + 1} T x_\varepsilon, \tag{4.5}$$

and Corollary 4.3 reduces to [21, Theorem 2.1].

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