Generalized Mann iterates for constructing fixed points in Hilbert spaces

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The mean iteration scheme originally proposed by Mann is extended to a broad class of relaxed, inexact fixed point algorithms in Hilbert spaces. Weak and strong convergence results are established under general conditions on the underlying averaging process and the type of operators involved. This analysis significantly widens the range of applications of mean iteration methods. Several examples are given.

1. INTRODUCTION

Let F be a firmly nonexpansive operator defined from a real Hilbert space $(\mathcal{H}, \|\cdot\|)$ into itself, i.e.,

$$(\forall (x,y) \in \mathcal{H}^2) \;\; \|Fx - Fy\|^2 \leq \|x-y\|^2 - \|(F - \mathrm{Id}\,)x - (F - \mathrm{Id}\,)y\|^2 \quad (1)$$

or, equivalently, $2F - \mathrm{Id}$ is nonexpansive [16, Thm. 12.1]. It follows from a classical result due to Opial [24, Thm. 3] that, for any initial point x_0, the sequence of successive approximations

$$(\forall n \in \mathbb{N}) \;\; x_{n+1} = Fx_n \quad (2)$$

converge weakly to a fixed point of F if such a point exists. The extension of this result to the relaxed iterations

$$(\forall n \in \mathbb{N}) \;\; x_{n+1} = x_n + \lambda_n\big(Fx_n - x_n\big), \;\; \text{where} \;\; 0 < \lambda_n < 2 \quad (3)$$

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under the condition $\sum_{n\geq 0}\lambda_n(2-\lambda_n) = +\infty$ follows from [17, Coro. 3]. Now define

$$\mathfrak{T} = \big\{T\colon \operatorname{dom} T = \mathcal{H} \to \mathcal{H} \mid (\forall (x,y) \in \mathcal{H} \times \operatorname{Fix} T)\ \langle y - Tx \mid x - Tx\rangle \leq 0\big\}, \tag{4}$$

where $\operatorname{Fix} T$ denotes the fixed point set of an operator T and $\langle\cdot \mid \cdot\rangle$ the scalar product of $\mathcal{H}$. This class of operators includes firmly nonexpansive operators, resolvents of maximal monotone operators, projection operators, subgradient projection operators, operators of the form $T = (\mathrm{Id} + R)/2$ where R is quasi-nonexpansive, as well as various combinations of those [2, 10]. The fact that $F \in \mathfrak{T}$ suggests that (3) could be generalized to

$$(\forall n \in \mathbb{N})\ \ x_{n+1} = x_n + \lambda_n\big(T_n x_n - x_n\big), \ \text{ where } \ 0 < \lambda_n < 2 \ \text{ and } \ T_n \in \mathfrak{T}. \tag{5}$$

This iterative procedure was investigated in [2] and further studied in [10]. It was shown that, under suitable conditions, the iterations (5) converge weakly to a point in $\bigcap_{n\geq 0} \operatorname{Fix} T_n$. These results provide a unifying framework for numerous fixed point algorithms, including in particular the serial scheme of [5] for finding a common fixed point of a family of firmly nonexpansive operators and its block-iterative generalizations [7, 19], the proximal point algorithms of [14, 32] for finding a zero of a maximal monotone operator, the fixed point scheme of [23] for functional equations, the projection methods of [8] for convex feasibility problems, the subgradient projection methods of [1, 9] for systems of convex inequalities, and operator splitting methods for variational inequalities [14, 21].

In the above algorithms, the update x_{n+1} involves only the current iterate x_n and the past iterates $(x_j)_{0\leq j\leq n-1}$ are not exploited. In [22], Mann proposed a simple modification of the basic scheme (2) in which the updating rule incorporates the past history of the process. More precisely, his scheme for finding a fixed point of an operator $T\colon \mathcal{H} \to \mathcal{H}$ is governed by the recursion

$$(\forall n \in \mathbb{N})\ \ x_{n+1} = T\,\overline{x}_n, \tag{6}$$

where $\overline{x}_n$ denotes a convex combination of the points $(x_j)_{0\leq j\leq n}$, say $\overline{x}_n = \sum_{j=0}^{n} \alpha_{n,j} x_j$. Further work on this type of iterative process for certain types of operators was carried out in [4, 6, 11, 17, 18, 26, 31].

Most existing convergence results for the Mann iterates (6) require explicitly, e.g., [11, 15, 17, 31], or implicitly, e.g., [4, 6, 18, 26], that the averaging matrix $A = [\alpha_{n,j}]$ be *segmenting*, i.e.,

$$(\forall n \in \mathbb{N})(\forall j \in \{0,\ldots,n\})\ \ \alpha_{n+1,j} = (1-\alpha_{n+1,n+1})\alpha_{n,j}. \tag{7}$$

This property implies that the points $(\overline{x}_n)_{n\geq 0}$ generated in (6) satisfy

$$
\begin{aligned}
(\forall n \in \mathbb{N})\ \ \overline{x}_{n+1} &= \alpha_{n+1,n+1}x_{n+1} + \sum_{j=0}^{n} \alpha_{n+1,j}x_j \\
&= \alpha_{n+1,n+1}x_{n+1} + (1-\alpha_{n+1,n+1})\sum_{j=0}^{n} \alpha_{n,j}x_j \\
&= \alpha_{n+1,n+1}T\overline{x}_n + (1-\alpha_{n+1,n+1})\overline{x}_n. \qquad (8)
\end{aligned}
$$

In other words, one is really just applying (3) with a specific relaxation strategy, namely,

$$(\forall n \in \mathbb{N})\ \ \lambda_n = \alpha_{n+1,n+1}. \qquad (9)$$

For that reason, (3) is commonly referred to as "Mann iterates" in the literature, although it merely corresponds to a special case of (6). Under (7), convergence results for (6) can be inferred from known results for (3). For instance, suppose that T is a quasi-nonexpansive operator such that $\operatorname{Fix} T \neq \varnothing$ and $T - \operatorname{Id}$ is demiclosed. Then any sequence $(\overline{x}_n)_{n\geq 0}$ conforming to (8) satisfies the following properties: $T\overline{x}_n - \overline{x}_n \to 0$ and $(\overline{x}_n)_{n\geq 0}$ converges weakly to a point in $\operatorname{Fix} T$ under either of the following conditions

(i) $\underline{\lim}\, \alpha_{n,n} > 0$ and $\overline{\lim}\, \alpha_{n,n} < 1$ [11, Thm. 8].

(ii) $\sum_{n\geq 0} \alpha_{n,n}(1-\alpha_{n,n}) = +\infty$ and T is nonexpansive [17, Coro. 3].

It therefore follows that the Mann sequence $(x_n)_{n\geq 0}$ in (6) converges weakly to a point in $\operatorname{Fix} T$ (whereas the standard successive approximations $x_{n+1} = Tx_n$ do not converge in general in this case: take $T = -\operatorname{Id}$ and $x_0 \neq 0$). Let us note that, under the segmenting condition (7), the value of $\alpha_{n,n}$ fixes those of $(\alpha_{n,j})_{0\leq j\leq n-1}$. This condition is therefore very restrictive.

The goal of this paper is to introduce and analyze a common algorithmic framework encompassing and extending the above iterative methods. The algorithm under consideration is the following inexact, Mann-like generalization of (5)

$$
\begin{aligned}
&(\forall n \in \mathbb{N})\ \ x_{n+1} = \overline{x}_n + \lambda_n\big(T_n\overline{x}_n + e_n - \overline{x}_n\big), \\
&\qquad\qquad \text{where}\ \ e_n \in \mathcal{H},\ 0 < \lambda_n < 2,\ \text{and}\ T_n \in \mathfrak{T}. \qquad (10)
\end{aligned}
$$

Here, e_n stands for the error made in the computation of $T_n\overline{x}_n$; incorporating such errors provides a more realistic model of the actual implementation of the algorithm. Throughout, the convex combinations in (10) are defined as

$$(\forall n \in \mathbb{N})\ \ \overline{x}_n = \sum_{j=0}^{n} \alpha_{n,j}x_j, \qquad (11)$$

where $(\alpha_{n,j})_{n,j\geq 0}$ are the entries of an infinite lower triangular row stochastic matrix A, i.e.,

$$(\forall n \in \mathbb{N}) \quad \begin{cases} (\forall j \in \mathbb{N}) \ \ \alpha_{n,j} \geq 0 \\ (\forall j \in \mathbb{N}) \ \ j > n \ \Rightarrow \ \alpha_{n,j} = 0 \\ \sum_{j=0}^{n} \alpha_{n,j} = 1, \end{cases} \tag{12}$$

which satisfies the regularity condition

$$(\forall j \in \mathbb{N}) \quad \lim_{n\to+\infty} \alpha_{n,j} = 0. \tag{13}$$

Our analysis will not rely on the segmenting condition (7) and will provide convergence results for the inexact, extended Mann iterations (10) for a wide range of averaging schemes.

Fig. 1 sheds some light on the geometrical structure of Algorithm (10). At iteration n, the points $(x_j)_{0\leq j\leq n}$ are available. A convex combination $\overline{x}_n$ of these points is formed and an operator $T_n \in \mathfrak{T}$ is selected, such that $\operatorname{Fix} T_n$ contains the solution set S of the underlying problem. If $\overline{x}_n \notin \operatorname{Fix} T_n$, then, by (4),

$$H_n = \big\{x \in \mathcal{H} \mid \langle x - T_n\overline{x}_n \mid \overline{x}_n - T_n\overline{x}_n\rangle \leq 0\big\} \tag{14}$$

is a closed affine half-space containing $\operatorname{Fix} T_n$ and onto which $T_n\overline{x}_n$ is the projection of $\overline{x}_n$. The update x_{n+1} is a point on the open segment between $\overline{x}_n$ and its approximate reflection, $2(T_n\overline{x}_n + e_n) - \overline{x}_n$, with respect to H_n. Thus, (10) offers much more flexibility in defining the update than (5) and, thereby, may be more advantageous in certain numerical applications. For instance, a problem that has been reported in some applications of (5) to convex feasibility is a tendency of its orbits to "zig-zag" [9, 29]. Acting on an average of past iterates rather than on the latest one alone as in (5) naturally centers the iterations and mitigates zig-zagging. Another numerical shortcoming of (5) that has been reported in operator splitting applications is the "spiralling" of the orbits around the solution set [12, Sec. 7.1], [13]. The averaging taking place in (10) has the inherent ability to avoid such undesirable convergence patterns.

The remainder of the paper is organized as follows. In Section 2, we introduce a special type of averaging matrix A which will be suitable for studying Algorithm (10). In Section 3, conditions for the weak and strong convergence of Algorithm (10) to a point in $\bigcap_{n\geq 0} \operatorname{Fix} T_n$ are established. Applications are discussed in Section 4.

2. CONCENTRATING AVERAGING MATRICES

Without further conditions on the averaging matrix A, Algorithm (10) may fail to converge. For instance, if we set $\alpha_{n,n-1} = 1$ for $n \geq 1$ then,

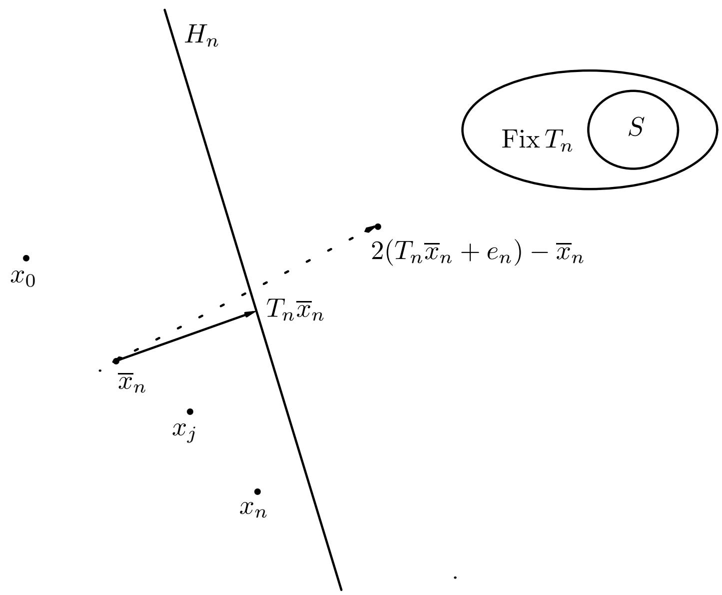

FIG. 1 An iteration of Algorithm (10); x_{n+1} lies on the dashed-line segment.

with $\lambda_n \equiv 1$, $T_n \equiv \operatorname{Id}$, and $x_0 = 0$, (10) becomes

$$(\forall n \in \mathbb{N})\ \ x_{n+1} = \sum_{0 \leq j \leq n/2} e_{n-2j}. \tag{15}$$

In particular, if $e_0 \neq 0$ and $e_n = 0$ for $n \geq 1$, then $x_n = 0$ for n even and $x_n = e_0$ for n odd. It will turn out that the following property of the averaging matrix A prevents this kind of behavior. Henceforth, ℓ^1 (resp. ℓ^1_+) denotes the class of summable sequences in $\mathbb{R}$ (resp. $\mathbb{R}_+$). Moreover, given a sequence $(\xi_n)_{n\geq 0}$ in $\mathbb{R}$, $(\overline{\xi}_n)_{n\geq 0}$ denotes the sequence defined through the same averaging process as in (11).

Definition 2.1. *A is* concentrating *if every sequence $(\xi_n)_{n\geq 0}$ in $\mathbb{R}_+$ such that*

$$\big(\exists\, (\varepsilon_n)_{n\geq 0} \in \ell^1_+\big)(\forall n \in \mathbb{N})\ \ \xi_{n+1} \leq \overline{\xi}_n + \varepsilon_n, \tag{16}$$

converges.

The following facts will be useful in checking whether a matrix is concentrating.

Lemma 2.2. ([10, Lem. 3.1]) *Let $(\xi_n)_{n\geq 0}$, $(\beta_n)_{n\geq 0}$, and $(\varepsilon_n)_{n\geq 0}$ be sequences in $\mathbb{R}_+$ such that $(\varepsilon_n)_{n\geq 0} \in \ell^1$ and*

$$(\forall n \in \mathbb{N})\ \ \xi_{n+1} \leq \xi_n - \beta_n + \varepsilon_n. \tag{17}$$

Then $(\xi_n)_{n\geq 0}$ *converges and* $(\beta_n)_{n\geq 0} \in \ell^1$.

LEMMA 2.3. *Let* $(\xi_n)_{n\geq 0}$ *be a sequence in* $\mathbb{R}_+$ *that satisfies* (16) *and set, for every* $n \in \mathbb{N}$, $\check{\xi}_n = \max_{0\leq j\leq n} \xi_j$. *Then*

(i) $(\check{\xi}_n)_{n\geq 0}$ *converges.*

(ii) $(\xi_n)_{n\geq 0}$ *is bounded.*

(iii) $(\overline{\xi}_n)_{n\geq 0}$ *is bounded.*

Proof. (i): For every $n \in \mathbb{N}$, $\xi_{n+1} \leq \overline{\xi}_n + \varepsilon_n \leq \check{\xi}_n + \varepsilon_n$ and therefore $\check{\xi}_{n+1} \leq \check{\xi}_n + \varepsilon_n$. Hence, by Lemma 2.2, $(\check{\xi}_n)_{n\geq 0}$ converges. (ii)&(iii): For every $n \in \mathbb{N}$, $0 \leq \xi_n \leq \check{\xi}_n$ and $0 \leq \overline{\xi}_n \leq \check{\xi}_n$, where $(\check{\xi}_n)_{n\geq 0}$ is bounded by (i). ∎

Our first example is an immediate consequence of Lemma 2.2.

EXAMPLE 2.4. *If* $\alpha_{n,n} \equiv 1$, *then* A *is the identity matrix, which is concentrating. In this case* (10) *reverts to* (5) *and we recover the standard* $\mathfrak{T}$*-class methods of [2] and [10].*

The next example involves a relaxation of the segmenting condition (7).

EXAMPLE 2.5. *Set* $(\forall n \in \mathbb{N})\ \tau_n = \sum_{j=0}^{n} |\alpha_{n+1,j} - (1-\alpha_{n+1,n+1})\alpha_{n,j}|$. *Suppose that* $(\tau_n)_{n\geq 0} \in \ell^1$ *and that* $\underline{\lim}\, \alpha_{n,n} > 0$. *Then* A *is concentrating.*

Proof. Let $(\xi_n)_{n\geq 0}$ be a sequence in $\mathbb{R}_+$ satisfying (16). By Lemma 2.3(ii), $\gamma = \sup_{n\geq 0} \xi_n < +\infty$. We have

$$\begin{aligned}
(\forall n \in \mathbb{N})\ \ \overline{\xi}_{n+1} &= \alpha_{n+1,n+1}\xi_{n+1} + \sum_{j=0}^{n} \alpha_{n+1,j}\xi_j \\
&= \overline{\xi}_n + \sum_{j=0}^{n} \big(\alpha_{n+1,j} - (1-\alpha_{n+1,n+1})\alpha_{n,j}\big)\xi_j \\
&\quad - \alpha_{n+1,n+1}(\overline{\xi}_n - \xi_{n+1} + \varepsilon_n) + \alpha_{n+1,n+1}\varepsilon_n \\
&\leq \overline{\xi}_n - \alpha_{n+1,n+1}(\overline{\xi}_n - \xi_{n+1} + \varepsilon_n) + (\gamma\tau_n + \varepsilon_n)
\end{aligned}$$

where, by (16), $\overline{\xi}_n - \xi_{n+1} + \varepsilon_n \geq 0$. We thus get from Lemma 2.2 that $(\overline{\xi}_n)_{n\geq 0}$ converges and that $\big(\alpha_{n+1,n+1}(\overline{\xi}_n - \xi_{n+1} + \varepsilon_n)\big)_{n\geq 0} \in \ell^1$. Hence, since $\underline{\lim}\, \alpha_{n,n} > 0$, $\overline{\xi}_n - \xi_{n+1} + \varepsilon_n \to 0$ and $(\xi_n)_{n\geq 0}$ converges to the same limit as $(\overline{\xi}_n)_{n\geq 0}$. ∎

An example of an averaging matrix satisfying the above conditions can be constructed by choosing $\alpha_{n,n} = \alpha$ for $n \geq 1$, where $\alpha \in \,]0,1[$. Then (7) yields

$$(\forall n \in \mathbb{N})(\forall j \in \{0,\ldots,n\}) \;\; \alpha_{n,j} = \begin{cases} (1-\alpha)^n & \text{for } \; j=0 \\ \alpha(1-\alpha)^{n-j} & \text{for } \; 1 \le j \le n. \end{cases}$$

The next example offers an alternative to the approximate segmenting condition used in Example 2.5.

EXAMPLE 2.6. *Set* $(\forall j \in \mathbb{N})$ $\tau_j = \max\{0, \sum_{n \ge j} \alpha_{n,j} - 1\}$ *and* $(\forall n \in \mathbb{N})$ $J_n = \{j \in \mathbb{N} \mid \alpha_{n,j} > 0\}$. *Suppose that* $\sum_{j \ge 0} \tau_j < +\infty$, *that*

$$(\forall n \in \mathbb{N}) \;\; J_{n+1} \subset J_n \cup \{n+1\}, \tag{18}$$

and that there exists $\underline{\alpha} \in \,]0,1[$ *such that*

$$(\forall n \in \mathbb{N})(\forall j \in J_n) \;\; \alpha_{n,j} \ge \underline{\alpha}. \tag{19}$$

Then A is concentrating.

Proof. Let $(\xi_n)_{n \ge 0}$ be a sequence in $\mathbb{R}_+$ satisfying (16). Then it follows from Lemma 2.3(ii)&(iii) that $\gamma = \sup_{n \ge 0} \xi_n < +\infty$ and $\gamma' = \sup_{n \ge 0} \overline{\xi}_n < +\infty$. Now define $(\forall n \in \mathbb{N})$ $\sigma_n = \big(\sum_{j=0}^n \alpha_{n,j} |\xi_j - \overline{\xi}_n|^2\big)^{1/2}$ and $\varepsilon'_n = 2\gamma' \varepsilon_n + \varepsilon_n^2$. Then $(\varepsilon'_n)_{n \ge 0} \in \ell^1$ and, by (16),

$$\begin{aligned} (\forall n \in \mathbb{N}) \;\; \sigma_n^2 \;\; &= \;\; \sum_{j=0}^n \alpha_{n,j} \xi_j^2 - \overline{\xi}_n^2 \\ &\le \;\; \sum_{j=0}^n \alpha_{n,j} \xi_j^2 - \xi_{n+1}^2 + 2\overline{\xi}_n \varepsilon_n + \varepsilon_n^2 \\ &\le \;\; \sum_{j=0}^n \alpha_{n,j} \xi_j^2 - \xi_{n+1}^2 + \varepsilon'_n. \end{aligned}$$

Whence,

$$\begin{aligned} (\forall N \in \mathbb{N}) \quad \sum_{n=0}^N \sigma_n^2 \;\; &\le \;\; \sum_{n=0}^N \sum_{j=0}^n \alpha_{n,j} \xi_j^2 - \sum_{n=0}^N \xi_{n+1}^2 + \sum_{n=0}^N \varepsilon'_n \\ &= \;\; \sum_{j=0}^N \sum_{n=j}^N \alpha_{n,j} \xi_j^2 - \sum_{j=1}^{N+1} \xi_j^2 + \sum_{n=0}^N \varepsilon'_n \\ &\le \;\; \xi_0^2 + \sum_{j=0}^N \tau_j \xi_j^2 + \sum_{n=0}^N \varepsilon'_n \\ &\le \;\; \gamma^2 \left(1 + \sum_{n=0}^N \tau_n\right) + \sum_{n=0}^N \varepsilon'_n, \end{aligned}$$

and we infer from the assumptions that $(\sigma_n^2)_{n\geq 0} \in \ell^1$.

It follows from (16) that, for every $n \geq 0$, $\xi_{n+1} \leq \widetilde{\xi}_n + \varepsilon_n$, where $\widetilde{\xi}_n = \max_{j\in J_n} \xi_j$. Consequently, by condition (18),

$$(\forall n \in \mathbb{N}) \;\; \widetilde{\xi}_{n+1} \leq \widetilde{\xi}_n + \varepsilon_n, \tag{20}$$

and $(\widetilde{\xi}_n)_{n\geq 0}$ converges by Lemma 2.2. On the other hand, (19) and Jensen's inequality yield

$$(\forall n \in \mathbb{N}) \;\; |\xi_n - \widetilde{\xi}_n| \leq |\xi_n - \overline{\xi}_n| + |\widetilde{\xi}_n - \overline{\xi}_n| \leq \frac{1}{\underline{\alpha}} \sum_{j=0}^{n} \alpha_{n,j} |\xi_j - \overline{\xi}_n| \leq \frac{\sigma_n}{\underline{\alpha}}.$$

Since $\sigma_n \to 0$, the convergence of $(\xi_n)_{n\geq 0}$ follows from that of $(\widetilde{\xi}_n)_{n\geq 0}$. ∎

As an example, take strictly positive numbers $(a_i)_{0\leq i\leq m}$ such that $\sum_{i=0}^m a_i = 1$ and define the averaging matrix A by

$$\begin{cases} (\forall n \in \{0,\ldots,m-1\})(\forall j \in \{0,\ldots,n\}) \;\; \alpha_{n,j} = \begin{cases} 0 & \text{if } \; 0 \leq j < n \\ 1 & \text{if } \; j = n, \end{cases} \\[2ex] (\forall n \geq m)(\forall j \in \{0,\ldots,n\}) \;\; \alpha_{n,j} = \begin{cases} 0 & \text{if } \; 0 \leq j < n-m \\ a_{n-j} & \text{if } \; n-m \leq j \leq n. \end{cases} \end{cases} \tag{21}$$

Then it is easily checked that the conditions of Example 2.6 are satisfied. More general stationary averaging processes can be obtained by exploiting a root condition from the theory of linear dynamical systems.

EXAMPLE 2.7. *Suppose there exist numbers $(a_i)_{0\leq i\leq m}$ in $\mathbb{R}_+$ such that* (21) *holds and the roots of the polynomial $z \mapsto z^{m+1} - \sum_{j=0}^m a_j z^{m-j}$ are all within the unit disc, with exactly one root on its boundary. Then A is concentrating.*

Proof. The claim follows from [27, Lemma 4]. ∎

The conditions of the previous example are frequently used in the numerical integration literature; several specific examples can be found, for instance, in [25].

3. CONVERGENCE ANALYSIS

In this section we study the convergence of the generalized Mann iteration scheme (10). Henceforth, $\mathfrak{W}(y_n)_{n\geq 0}$ and $\mathfrak{S}(y_n)_{n\geq 0}$ denote respectively the sets of weak and strong cluster points of a sequence $(y_n)_{n\geq 0}$ in $\mathcal{H}$, whereas $\rightharpoonup$ and $\to$ denote respectively weak and strong convergence.

In the case of Algorithm (3), a key property of the operator F to establish weak convergence to a point in $\operatorname{Fix} F$ is the demiclosedness of $F - \mathrm{Id}$

at 0, i.e., whenever $y_n \rightharpoonup y$ and $Fy_n - y_n \to 0$, then $y = Fy$ [24]. The following extended notion of demiclosedness will prove pertinent to establish the weak convergence of (10).

CONDITION 3.1. *For every bounded sequence* $(y_n)_{n\geq 0}$ *in* $\mathcal{H}$,

$$T_n y_n - y_n \to 0 \;\;\Rightarrow\;\; \mathfrak{W}(y_n)_{n\geq 0} \subset \bigcap_{n\geq 0} \operatorname{Fix} T_n. \tag{22}$$

Likewise, to study the strong convergence of (3), a central property is the demicompactness of F at 0, i.e., every bounded sequence $(y_n)_{n\geq 0}$ clusters strongly whenever $Fy_n - y_n \to 0$ [28]. For our purposes, a suitable extension of this property will be

CONDITION 3.2. *For every bounded sequence* $(y_n)_{n\geq 0}$ *in* $\mathcal{H}$,

$$T_n y_n - y_n \to 0 \;\;\Rightarrow\;\; \mathfrak{S}(y_n)_{n\geq 0} \neq \varnothing. \tag{23}$$

The following two lemmas will also be required.

LEMMA 3.3. [10, Prop. 2.3(ii)] *Let* $T \in \mathfrak{T}$ *and* $\lambda \in [0,2]$. *Then*

$$(\forall y \in \mathcal{H})(\forall x \in \operatorname{Fix} T) \;\; \|y + \lambda(Ty - y) - x\|^2 \leq \|y - x\|^2 - \lambda(2-\lambda)\|Ty - y\|^2.$$

LEMMA 3.4. [20, Thm. 3.5.4] *Let* $(\xi_n)_{n\geq 0}$ *be a sequence in* $\mathbb{R}$. *Then* $\xi_n \to \xi \Rightarrow \overline{\xi}_n \to \xi$.

Our main convergence result can now be stated.

THEOREM 3.5. *Let* $(x_n)_{n\geq 0}$ *be an arbitrary sequence generated by* (10). *Suppose that* A *is concentrating, that* $(T_n)_{n\geq 0}$ *satisfies Condition 3.1 with* $S = \bigcap_{n\geq 0} \operatorname{Fix} T_n \neq \varnothing$, *that* $(\lambda_n)_{n\geq 0}$ *lies in* $[\delta, 2-\delta]$ *for some* $\delta \in \,]0,1[$, *and that* $(\|e_n\|)_{n\geq 0} \in \ell^1$. *Then:*

(i) $(x_n)_{n\geq 0}$ *converges weakly to a point in* S.

(ii) *If* $(T_n)_{n\geq 0}$ *satisfies Condition 3.2,* $(x_n)_{n\geq 0}$ *converges strongly to a point in* S.

Proof. Take a point $x \in S$. In view of (10), Lemma 3.3, and the convexity of $\|\cdot\|$,

$$\begin{aligned}(\forall n \in \mathbb{N}) \;\; \|x_{n+1} - x\| &\leq \|\overline{x}_n + \lambda_n\big(T_n\overline{x}_n - \overline{x}_n\big) - x\| + \lambda_n\|e_n\| \\ &\leq \|\overline{x}_n - x\| + 2\|e_n\| \\ &\leq \sum_{j=0}^{n} \alpha_{n,j}\|x_j - x\| + 2\|e_n\|. \end{aligned} \tag{24}$$

Therefore, since A is concentrating and $(\|e_n\|)_{n\geq 0} \in \ell^1$, $(\|x_n - x\|)_{n\geq 0}$ converges to some number $\ell(x)$. It then follows from Lemma 3.4 and (24) that

$$\|\overline{x}_n - x\| \to \ell(x). \tag{25}$$

Hence $\gamma = 4\sup_{n\geq 0} \|\overline{x}_n - x\| < +\infty$ and the sequence $(\varepsilon_n)_{n\geq 0}$ defined by

$$(\forall n \in \mathbb{N}) \;\; \varepsilon_n = \gamma\|e_n\| + 4\|e_n\|^2 \tag{26}$$

lies in ℓ^1. Invoking Lemma 3.3, the convexity of $\|\cdot\|^2$, and the restrictions on $(\lambda_n)_{n\geq 0}$, we obtain

$$\begin{aligned}(\forall n \in \mathbb{N}) \;\; \|x_{n+1} - x\|^2 \;\; &\leq \;\; \big(\|\overline{x}_n + \lambda_n\big(T_n\overline{x}_n - \overline{x}_n\big) - x\| + \lambda_n\|e_n\|\big)^2 \\ &\leq \;\; \|\overline{x}_n - x\|^2 - \lambda_n(2-\lambda_n)\|T_n\overline{x}_n - \overline{x}_n\|^2 \\ &\quad\;\; + 2\lambda_n\|\overline{x}_n - x\|\cdot\|e_n\| + \lambda_n^2\|e_n\|^2 \\ &\leq \;\; \sum_{j=0}^{n}\alpha_{n,j}\|x_j - x\|^2 - \delta^2\|T_n\overline{x}_n - \overline{x}_n\|^2 + \varepsilon_n.\end{aligned}$$

Consequently,

$$(\forall n \in \mathbb{N}) \;\; \|T_n\overline{x}_n - \overline{x}_n\|^2 \leq \\ \delta^{-2}\left(\sum_{j=0}^{n}\alpha_{n,j}\|x_j - x\|^2 - \|x_{n+1} - x\|^2 + \varepsilon_n\right). \tag{27}$$

However, since $(\|x_n - x\|^2)_{n\geq 0}$ converges, Lemma 3.4 asserts that

$$\sum_{j=0}^{n}\alpha_{n,j}\|x_j - x\|^2 - \|x_{n+1} - x\|^2 \to 0.$$

It therefore follows from (27) that

$$T_n\overline{x}_n - \overline{x}_n \to 0. \tag{28}$$

Moreover, since

$$(\forall n \in \mathbb{N}) \;\; \|x_{n+1} - \overline{x}_n\| = \lambda_n\|T_n\overline{x}_n + e_n - \overline{x}_n\| \leq 2\big(\|T_n\overline{x}_n - \overline{x}_n\| + \|e_n\|\big),$$

(28) yields

$$x_{n+1} - \overline{x}_n \to 0. \tag{29}$$

(i): Take two points x and x' in $\mathfrak{W}(\overline{x}_n)_{n\geq 0} \cap S$. From (25), the sequences $(\|\overline{x}_n\|^2 - 2\langle \overline{x}_n \mid x\rangle)_{n\geq 0}$ and $(\|\overline{x}_n\|^2 - 2\langle \overline{x}_n \mid x'\rangle)_{n\geq 0}$ converge and therefore so does $(\langle \overline{x}_n \mid x - x'\rangle)_{n\geq 0}$. Consequently, it must hold that $\langle x \mid x - x'\rangle = \langle x' \mid x - x'\rangle$, i.e., $x = x'$. Thus, the bounded sequence $(\overline{x}_n)_{n\geq 0}$ has at most one weak cluster point in S. Since (22) and (28) imply that $\mathfrak{W}(\overline{x}_n)_{n\geq 0} \subset S$, we deduce that $(\overline{x}_n)_{n\geq 0}$ converges weakly to a point $x \in S$. In view of (29), $x_n \rightharpoonup x$.

(ii): It follows from (28) and (23) that $\mathfrak{S}(\overline{x}_n)_{n\geq 0} \neq \varnothing$. However, by (i), there exists a point $x \in S$ such that $\overline{x}_n \rightharpoonup x$. Whence, $\mathfrak{S}(\overline{x}_n)_{n\geq 0} = \{x\} \subset S$ and therefore $\ell(x) = 0$ in (25). We conclude $x_n \to x$. ∎

As an immediate by-product of this theorem, we obtain convergence results for the alternative averaging scheme

$$(\forall n \in \mathbb{N})\ \ x_{n+1} = \sum_{j=0}^{n} \alpha_{n,j}\big(x_j + \lambda_j(T_j x_j + e_j - x_j)\big),$$
$$\text{where}\ \ e_n \in \mathcal{H},\ 0 < \lambda_n < 2,\ \text{and}\ T_n \in \mathfrak{T}, \tag{30}$$

special cases of which have been investigated, for instance, in [3] and [30]. If the T_ns are resolvents of a maximal monotone operator, then (30) can be shown to correspond to a linear multi-step method described in [27].

Corollary 3.6. *Let $(x_n)_{n\geq 0}$ be an arbitrary sequence generated by (30). Suppose that A is concentrating, that $(T_n)_{n\geq 0}$ satisfies Condition 3.1 with $S = \bigcap_{n\geq 0} \operatorname{Fix} T_n \neq \varnothing$, that $(\lambda_n)_{n\geq 0}$ lies in $[\delta, 2-\delta]$ for some $\delta \in]0,1[$, and that $(\|e_n\|)_{n\geq 0} \in \ell^1$. Then:*

(i) *$(x_n)_{n\geq 0}$ converges weakly to a point in S.*

(ii) *If $(T_n)_{n\geq 0}$ satisfies Condition 3.2, $(x_n)_{n\geq 0}$ converges strongly to a point in S.*

Proof. Define $(\forall j \in \mathbb{N})\ y_j = x_j + \lambda_j(T_j x_j + e_j - x_j)$. Then, by (30), for every $n \in \mathbb{N}$, $x_{n+1} = \overline{y}_n$, whence $y_{n+1} = \overline{y}_n + \lambda_{n+1}(T_{n+1}\overline{y}_n + e_{n+1} - \overline{y}_n)$. (i): By Theorem 3.5(i), $y_n \rightharpoonup x \in S$, i.e., $(\forall z \in \mathcal{H})\ \langle y_n \mid z\rangle \to \langle x \mid z\rangle$. In turn, Lemma 3.4 yields $(\forall z \in \mathcal{H})\ \langle \overline{y}_n \mid z\rangle \to \langle x \mid z\rangle$, i.e., $x_n \rightharpoonup x$. (ii): By Theorem 3.5(ii), $y_n \to x \in S$ and Lemma 3.4 yields $\sum_{j=0}^{n} \alpha_{n,j}\|y_j - x\| \to 0$. Since $(\forall n \in \mathbb{N})\ \|x_{n+1} - x\| = \|\overline{y}_n - x\| \leq \sum_{j=0}^{n} \alpha_{n,j}\|y_j - x\|$, we conclude $x_n \to x$. ∎

4. APPLICATIONS

Algorithm (5) covers essentially all Fejér-monotone methods [2, Prop. 2.7] and perturbed versions thereof [10]. Theorem 3.5 provides convergence results for the Mann-like extension of these methods described by (10). To demonstrate the wide range of applicability of these results, a few examples are detailed below.

4.1. Mean iterations for common fixed points

Our first application concerns the problem of finding a common fixed point of a finite family of operators $(R_i)_{i\in I}$ such that

$$(\forall i \in I)\ \ R_i \in \mathfrak{T} \ \text{ and } \ R_i - \mathrm{Id} \text{ is demiclosed at } 0. \tag{31}$$

For every $n \in \mathbb{N}$, let $(\omega_{i,n})_{i\in I}$ be weights in $]0,1]$ such that $\sum_{i\in I}\omega_{i,n} = 1$. It follows from [10, Eq. (18)] that

$$\begin{aligned} x \in \mathrm{Fix} \sum_{i\in I}\omega_{i,n}R_i \ \Leftrightarrow\ \Big\|\sum_{i\in I}\omega_{i,n}R_i x - x\Big\| = 0 \ \Leftrightarrow\ \sum_{i\in I}\omega_{i,n}\|R_i x - x\|^2 = 0 \\ \Leftrightarrow\ x \in \bigcap_{i\in I}\mathrm{Fix}\, R_i. \end{aligned}$$

Hence, the function

$$L_n \colon \mathcal{H} \to [1,+\infty[\colon x \mapsto \begin{cases} \dfrac{\sum_{i\in I}\omega_{i,n}\|R_i x - x\|^2}{\|\sum_{i\in I}\omega_{i,n}R_i x - x\|^2} & \text{if } x \notin \bigcap_{i\in I}\mathrm{Fix}\, R_i \\ 1 & \text{otherwise} \end{cases}$$

is well defined.

We consider the extrapolated parallel algorithm

$$(\forall n \in \mathbb{N})\ \ x_{n+1} = \overline{x}_n + \lambda_n\left(L_n(\overline{x}_n)\left(\sum_{i\in I}\omega_{i,n}R_i\overline{x}_n - \overline{x}_n\right) + e_n\right), \quad \text{where } e_n \in \mathcal{H} \text{ and } 0 < \lambda_n < 2. \tag{32}$$

In the standard case when A is the identity matrix, this type of extrapolated algorithm has been investigated at various levels of generality in [7, 9, 10, 19, 29]. It has been observed to enjoy fast convergence due to the large relaxation values attainable through the extrapolation functions $(L_n)_{n\geq 0}$ but, in some cases, to be subject to zig-zagging, which weakens its performance [9, 29]. As discussed in the Introduction, the averaging process that takes place in (32) can effectively reduce this phenomenon.

Corollary 4.1. *Let $(x_n)_{n\geq 0}$ be an arbitrary sequence generated by (32). Suppose that A is concentrating, that $\bigcap_{i\in I}\mathrm{Fix}\, R_i \neq \varnothing$, that $(\lambda_n)_{n\geq 0}$ lies in $[\delta, 2-\delta]$ for some $\delta \in]0,1[$, that $\zeta = \inf_{n\geq 0}\min_{i\in I}\omega_{i,n} > 0$, and that $(\|e_n\|)_{n\geq 0} \in \ell^1$. Then:*

(i) *$(x_n)_{n\geq 0}$ converges weakly to a point in $\bigcap_{i\in I}\mathrm{Fix}\, R_i$.*

(ii) *If one of the operators in $(R_i)_{i\in I}$ is demicompact at 0, $(x_n)_{n\geq 0}$ converges strongly to a point in $\bigcap_{i\in I}\mathrm{Fix}\, R_i$.*

Proof. For every $n \in \mathbb{N}$, the operator $T_n = \mathrm{Id} + L_n\big(\sum_{i\in I} \omega_{i,n} R_i - \mathrm{Id}\big)$ lies in $\mathfrak{T}$ and $\operatorname{Fix} T_n = \bigcap_{i\in I} \operatorname{Fix} R_i$ [10, Prop. 2.4]. Hence, with $(T_n)_{n\geq 0}$ thus defined, Algorithm (32) is immediately seen to be a particular realization of (10). Therefore, to prove (i), it suffices by Theorem 3.5 to check that Condition 3.1 is satisfied. To this end, take a bounded sequence $(y_n)_{n\geq 0}$ such that $T_n y_n - y_n \to 0$ and $y \in \mathfrak{W}(y_n)_{n\geq 0}$. Then we must show $y \in \bigcap_{i\in I} \operatorname{Fix} R_i$.

Take $z \in \bigcap_{i\in I} \operatorname{Fix} R_i$ and set $\beta = \sup_{n\geq 0} \|y_n - z\|$. Then

$$
\begin{aligned}
(\forall n \in \mathbb{N})\ \ \|T_n y_n - y_n\| \ &\geq\ \Big\|\sum_{i\in I} \omega_{i,n} R_i y_n - y_n\Big\| && (33)\\
&\geq\ \beta^{-1} \sum_{i\in I} \omega_{i,n} \|R_i y_n - y_n\|^2 && (34)\\
&\geq\ \beta^{-1} \zeta \max_{i\in I} \|R_i y_n - y_n\|^2, && (35)
\end{aligned}
$$

where (33) follows from the inequality $L_n(y_n) \geq 1$ and (34) from [10, Eq. (17)]. Consequently,

$$\max_{i\in I} \|R_i y_n - y_n\| \to 0 \tag{36}$$

and, since the operators $(R_i - \mathrm{Id})_{i\in I}$ are demiclosed at 0, we obtain $y \in \bigcap_{i\in I} \operatorname{Fix} R_i$. Assertion (i) is thus proven.

To prove (ii) it suffices to check that Condition 3.2 is satisfied, i.e., that $\mathfrak{S}(y_n)_{n\geq 0} \neq \varnothing$. Suppose that, for some $j \in I$, R_j is demicompact at 0. Then, by (36), $R_j y_n - y_n \to 0$ and, in turn, $\mathfrak{S}(y_n)_{n\geq 0} \neq \varnothing$. ∎

To illustrate this result, let us highlight specific applications.

EXAMPLE 4.2 (firmly nonexpansive operators). *$(R_i)_{i\in I}$ is a finite family of firmly nonexpansive operators from $\mathcal{H}$ to $\mathcal{H}$ with domain $\mathcal{H}$. Then, for each $i \in I$, $R_i \in \mathfrak{T}$ [2, Prop. 2.3] and $R_i - \mathrm{Id}$ is demiclosed [5, Lem. 4]. Corollary 4.1 therefore applies. In particular if, for every $i \in I$, R_i is the projector relative to a closed convex set S_i, then (32) provides a new projection algorithm to find a point in $\bigcap_{i\in I} S_i$ that reduces to Pierra's method [29] when A is the identity matrix, $e_n \equiv 0$, $\omega_{i,n} \equiv \omega_i$, and the range of the relaxation parameters $(\lambda_n)_{n\geq 0}$ is limited to $[\delta, 1]$.*

REMARK 4.3. *In [15], an elliptic Cauchy problem was shown to be equivalent to a fixed point problem for a nonexpansive affine operator T in a Hilbert space. This problem was solved with the Mann iterative process (6) under the segmenting condition (7). If we let $R = (\mathrm{Id} + T)/2$, then R is a firmly nonexpansive operator with $\operatorname{Fix} R = \operatorname{Fix} T$ and Example 4.2 (with the single operator R) provides new variants of the algorithm of [15] beyond the segmenting condition.*

EXAMPLE 4.4 (demicontractions). *For every $i \in I$,*

$$R_i = \frac{1-k_i}{2} T_i + \frac{1+k_i}{2} \mathrm{Id}\,, \tag{37}$$

where $T_i \colon \operatorname{dom} T_i = \mathcal{H} \to \mathcal{H}$ is demicontractive with constant $k_i \in [0,1[$, that is [18],

$$(\forall x \in \mathcal{H})(\forall y \in \operatorname{Fix} T_i) \;\; \|T_i x - y\|^2 \leq \|x-y\|^2 + k_i \|T_i x - x\|^2, \tag{38}$$

and $T_i - \mathrm{Id}$ is demiclosed at 0. Upon inserting (37) into (32), one obtains an algorithm to find a common fixed point of $(T_i)_{i\in I}$ whose convergence properties are given in Corollary 4.1. To see this, it suffices to show that, for every $i \in I$, (a) $\operatorname{Fix} R_i = \operatorname{Fix} T_i$, (b) $R_i - \mathrm{Id}$ is demiclosed at 0, and (c) $R_i \in \mathfrak{T}$. Properties (a) and (b) are immediate from (37). To check (c), fix $x \in \mathcal{H}$ and $y \in \operatorname{Fix} R_i$. Then we must show $\|R_i x - x\|^2 \leq \langle y - x \mid R_i x - x\rangle$. By (38), we have

$$\begin{aligned} \|T_i x - x\|^2 &= \|T_i x - y\|^2 + 2\langle y - x \mid T_i x - x\rangle - \|y-x\|^2 \\ &\leq k_i \|T_i x - x\|^2 + 2\langle y - x \mid T_i x - x\rangle\,. \end{aligned} \tag{39}$$

Hence,

$$\begin{aligned} \|R_i x - x\|^2 &= \big((1-k_i)/2\big)^2 \|T_i x - x\|^2 \\ &\leq (1-k_i)\langle y - x \mid T_i x - x\rangle /2 \\ &= \langle y - x \mid R_i x - x\rangle\,. \end{aligned} \tag{40}$$

EXAMPLE 4.5 (systems of convex inequalities). *Given a finite family $(f_i)_{i\in I}$ of continuous convex functions from $\mathcal{H}$ to $\mathbb{R}$ with nonempty level sets $\big(f_i^{-1}(]-\infty,0])\big)_{i\in I}$, we want to find a point $x \in \mathcal{H}$ such that*

$$(\forall i \in I) \;\; f_i(x) \leq 0. \tag{41}$$

Define

$$(\forall i \in I) \;\; R_i : x \mapsto \begin{cases} x - \dfrac{f_i(x)}{\|g_i(x)\|^2} g_i(x) & \text{if } f_i(x) > 0 \\ x & \text{if } f_i(x) \leq 0, \end{cases} \tag{42}$$

where g_i is a selection of the subdifferential ∂f_i of f_i. Then the operators $(R_i)_{i\in I}$ lie in $\mathfrak{T}$ [2, Prop. 2.3] and solving (41) is equivalent to finding one of their common fixed points. Moreover if, for every $i \in I$, ∂f_i maps bounded sets into bounded sets, then the operators $(R_i - \mathrm{Id})_{i\in I}$ are demiclosed at 0 (use the same arguments as in the proof of [2, Coro. 6.10]) and Corollary 4.1 can be invoked to solve (41). Here, R_i is demicompact at 0 if $f_i^{-1}(]-\infty,\eta])$ is boundedly compact (its intersection with any closed ball is compact) for some $\eta \in]0,+\infty[$.

4.2. Mean proximal iterations

We consider the standard problem of finding a zero of a set-valued maximal monotone operator $M : \mathcal{H} \to 2^{\mathcal{H}}$, i.e., a point in the set $M^{-1}0$. To solve this problem, we propose the mean proximal algorithm

$$(\forall n \in \mathbb{N})\ \ x_{n+1} = \overline{x}_n + \lambda_n\big((\mathrm{Id} + \gamma_n M)^{-1}\overline{x}_n + e_n - \overline{x}_n\big),\\ \text{where}\ \ e_n \in \mathcal{H},\ 0 < \lambda_n < 2,\ \text{and}\ 0 < \gamma_n < +\infty. \quad (43)$$

COROLLARY 4.6. *Let* $(x_n)_{n\geq 0}$ *be an arbitrary sequence generated by* (43). *Suppose that* A *is concentrating, that* $0 \in \operatorname{ran} M$, *that* $\inf_{n\geq 0} \gamma_n > 0$, *that* $(\lambda_n)_{n\geq 0}$ *lies in* $[\delta, 2-\delta]$ *for some* $\delta \in \,]0,1[$, *and that* $(\|e_n\|)_{n\geq 0} \in \ell^1$. *Then:*

(i) $(x_n)_{n\geq 0}$ *converges weakly to a point in* $M^{-1}0$.

(ii) *If* $\operatorname{dom} M$ *is boundedly compact,* $(x_n)_{n\geq 0}$ *converges strongly to a point in* $M^{-1}0$.

Proof. For every $n \in \mathbb{N}$, set $T_n = (\mathrm{Id} + \gamma_n M)^{-1}$. Then the operators $(T_n)_{n\geq 0}$ lie in $\mathfrak{T}$ and, for every $n \in \mathbb{N}$, $\operatorname{Fix} T_n = M^{-1}0$ [2, Prop. 2.3]. Therefore, to prove (i), it suffices by Theorem 3.5 to check that Condition 3.1 is satisfied. This can be done by following the same arguments as in the proof of [2, Coro. 6.1]. Finally, the fact that the bounded compactness of $\operatorname{dom} M$ in (ii) implies Condition 3.2 can be proved by proceeding as in the proof of [10, Thm. 6.9]. ∎

In particular, if A is the identity matrix, (43) relapses to the usual relaxed proximal point algorithm. In this case, Corollary 4.6(i) can be found in [14, Thm. 3], which itself contains Rockafellar's classical result [32, Thm. 1] for $\lambda_n \equiv 1$.

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