Equilibrium Programming in Hilbert Spaces

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Abstract

Several methods for solving systems of equilibrium problems in Hilbert spaces – and for finding best approximations thereof – are presented and their convergence properties are established. The proposed methods include proximal-like block-iterative algorithms for general systems, as well as regularization and splitting algorithms for single equilibrium problems. The problem of constructing approximate equilibria in the case of inconsistent systems is also considered.

1 Introduction

Let $\mathcal{H}$ be a real Hilbert space, let K be a nonempty closed convex subset of $\mathcal{H}$, and let $(F_i)_{i\in I}$ be a countable family of bifunctions from K^2 to $\mathbb{R}$. We address the broad class of problems whose basic formulation reduces to solving the system of equilibrium problems

$$\text{find } \ x \in K \ \text{ such that } \ (\forall i \in I)(\forall y \in K) \ \ F_i(x, y) \geq 0. \tag{1.1}$$

In the case of a single equilibrium, i.e., I is a singleton, the formulation (1.1) was shown in [5, 24] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, Nash equilibria in noncooperative games, vector equilibrium problems, as well as certain fixed point problems (see also [15]). The above formulation extends this formalism to systems of such problems, covering in particular various forms of feasibility problems [2, 11]. We shall also address the problem of finding a best approximation to a point $a \in \mathcal{H}$ from the solutions to (1.1), namely

$$\text{project } a \text{ onto } S = \bigcap_{i\in I} S_i, \ \text{ where } \ (\forall i \in I) \ \ S_i = \big\{z \in K \mid (\forall y \in K) \ \ F_i(z, y) \geq 0\big\}. \tag{1.2}$$

Our main objective is to devise algorithms for solving (1.1) and (1.2) and to analyze their asymptotic behavior under the standing assumption that the bifunctions $(F_i)_{i\in I}$ all satisfy the following set of standard properties.

Condition 1.1 The bifunction $F\colon K^2 \to \mathbb{R}$ is such that:

(i) $(\forall x \in K)\;\; F(x,x) = 0$.

(ii) $(\forall (x,y) \in K^2)\;\; F(x,y) + F(y,x) \le 0$.

(iii) For every $x \in K$, $F(x,\cdot)\colon K \to \mathbb{R}$ is lower semicontinuous and convex.

(iv) $(\forall (x,y,z) \in K^3)\;\; \overline{\lim_{\varepsilon\to 0^+}} F\big((1-\varepsilon)x + \varepsilon z, y\big) \le F(x,y)$.

While some methods have been proposed to solve (1.1) in this context in the case of a single bifunction (see [22, 23] and, in the case of Euclidean spaces, [13, 17, 20]), we are not aware of any result for systems of equilibrium problems. In addition, there does not seem to be any iterative algorithm in the literature to solve (1.2), even in the case of a single bifunction. Our analysis will also bring to light new connections between equilibrium programming and standard optimization methods.

The remainder of the paper is organized as follows. In section 2, we define our notation and provide technical facts that will be used in subsequent sections. In section 3, we establish convergence results for parallel, proximal-like, block-iterative methods to solve (1.1) and (1.2). In the case of a single equilibrium problem, an alternative approach to (1.2) based on regularization ideas is presented in section 4. In section 5, we address problems with a single bifunction which can be split into two components, and devise forward-backward-like algorithms for solving them. In section 6, these results are applied to the problem of finding approximate solutions to (1.1) and (1.2) in the inconsistent case, i.e., when $S = \varnothing$.

2 Notation and preliminary results

2.1 Notation

Throughout the paper $\mathbb{N}$ denotes the set of nonnegative integers and $\mathcal{H}$ is a real Hilbert space with scalar product $\langle\cdot \mid \cdot\rangle$, norm $\|\cdot\|$, and distance d. K is a fixed nonempty closed convex subset of $\mathcal{H}$ and Id denotes the identity operator on $\mathcal{H}$. The expressions $x_n \rightharpoonup x$ and $x_n \to x$ denote respectively the weak and strong convergence to x of a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$, and $\mathfrak{W}(x_n)_{n\in\mathbb{N}}$ its set of weak cluster points. The class of all proper, lower semicontinuous, convex functions from $\mathcal{H}$ to $\left]-\infty,+\infty\right]$ is denoted by $\Gamma_0(\mathcal{H})$. Now let C be a subset of $\mathcal{H}$. Then $\operatorname{int} C$ is the interior of C, d_C its distance function, and ι_C its indicator function, which takes the value 0 on C and $+\infty$ on its complement. If C is nonempty, closed, and convex, then P_C denotes the projection operator onto C. Finally, $\operatorname{Fix} T = \{x \in \mathcal{H} \mid Tx = x\}$ denotes the set of fixed points of an operator $T\colon \mathcal{H} \to \mathcal{H}$.

2.2 Set-valued operators

Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued operator. The sets $\operatorname{dom} A = \{x \in \mathcal{H} \mid Ax \neq \varnothing\}$ and $\operatorname{gr} A = \{(x,u) \in \mathcal{H}^2 \mid u \in Ax\}$ are the domain and the graph of A, respectively. The inverse A^{-1} of A is the set-valued operator with graph $\{(u,x) \in \mathcal{H}^2 \mid u \in Ax\}$. The resolvent of A is $J_A = (\operatorname{Id} + A)^{-1}$ and its Yosida approximation of index $\gamma \in \left]0,+\infty\right[$ is

$$ {}^{\gamma}A = \frac{\operatorname{Id} - J_{\gamma A}}{\gamma}. \tag{2.1} $$

Moreover, A is monotone if

$$ (\forall (x,u) \in \operatorname{gr} A)(\forall (y,v) \in \operatorname{gr} A) \quad \langle x-y \mid u-v\rangle \geq 0, \tag{2.2} $$

and maximal monotone if, furthermore, $\operatorname{gr} A$ is not properly contained in the graph of any monotone operator $B\colon \mathcal{H} \to 2^{\mathcal{H}}$. If A is monotone, then J_A is single-valued on $\operatorname{dom} J_A$; in addition, if A is maximal monotone, then $\operatorname{dom} J_A = \mathcal{H}$ (see [1, section 3.5] for details).

Lemma 2.1 *Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator and let $\gamma \in \left]0,+\infty\right[$. Then*

$$ J_{({}^{\gamma}A)} = \operatorname{Id} + \frac{1}{\gamma+1}\left(J_{(\gamma+1)A} - \operatorname{Id}\right). \tag{2.3} $$

Proof. Let $(x,y) \in \mathcal{H}^2$. Then

$$ \begin{aligned} y = J_{({}^{\gamma}A)}x \quad &\Leftrightarrow \quad y = \left(\frac{\gamma+1}{\gamma}\operatorname{Id} - \frac{1}{\gamma}J_{\gamma A}\right)^{-1} x \\ &\Leftrightarrow \quad (\gamma+1)y - \gamma x = J_{\gamma A}y \\ &\Leftrightarrow \quad y \in (\gamma+1)y - \gamma x + \gamma A\big((\gamma+1)y - \gamma x\big) \\ &\Leftrightarrow \quad x - y \in A\big((\gamma+1)y - \gamma x\big) \\ &\Leftrightarrow \quad x - \big((\gamma+1)y - \gamma x\big) \in (\gamma+1)A\big((\gamma+1)y - \gamma x\big) \\ &\Leftrightarrow \quad (\gamma+1)y - \gamma x = J_{(\gamma+1)A}x \\ &\Leftrightarrow \quad y = x + \frac{1}{\gamma+1}\left(J_{(\gamma+1)A}x - x\right). \end{aligned} \tag{2.4} $$

□

The subdifferential of a proper function $f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ is the set-valued operator

$$ \partial f\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{u \in \mathcal{H} \mid (\forall y \in \mathcal{H}) \quad \langle y-x \mid u\rangle + f(x) \leq f(y)\big\}. \tag{2.5} $$

The normal cone operator of a nonempty closed convex set $C \subset \mathcal{H}$ is $N_C = \partial\iota_C$. If $f \in \Gamma_0(\mathcal{H})$, then ∂f is maximal monotone and

$$ \operatorname{prox}_f = J_{\partial f} \tag{2.6} $$

is Moreau's proximity operator [21]; moreover, the Moreau envelope of index $\gamma \in \left]0,+\infty\right[$ of f is the function ${}^{\gamma}f\colon x \mapsto \min_{y\in\mathcal{H}} f(y) + \frac{1}{2\gamma}\|x-y\|^2$.

Lemma 2.2 *Let $f \in \Gamma_0(\mathcal{H})$ and $\gamma \in \left]0,+\infty\right[$. Then*

$$\operatorname{prox}_{\left({}^{\gamma}f\right)} = \operatorname{Id} + \frac{1}{\gamma+1}\left(\operatorname{prox}_{(\gamma+1)f} - \operatorname{Id}\right). \tag{2.7}$$

Proof. Since [21, Proposition 7.d] implies that $\partial\left({}^{\gamma}f\right) = {}^{\gamma}(\partial f)$, it suffices to set $A = \partial f$ in Lemma 2.1. □

2.3 Nonlinear operators

Definition 2.3 *Let $T\colon \mathcal{H} \to \mathcal{H}$ be a single-valued operator with* $\operatorname{dom} T = \mathcal{H}$. *Then T*

(i) *belongs to the* class $\mathfrak{T}$ [3] *if* $(\forall (x,y) \in \mathcal{H} \times \operatorname{Fix} T)$ $\langle y - Tx \mid x - Tx\rangle \leq 0$;

(ii) *is* nonexpansive *if* $(\forall (x,y) \in \mathcal{H}^2)$ $\|Tx - Ty\| \leq \|x-y\|$;

(iii) *is* firmly nonexpansive *if* $(\forall (x,y) \in \mathcal{H}^2)$ $\|Tx - Ty\|^2 \leq \langle Tx - Ty \mid x - y\rangle$;

(iv) *is* α-averaged *for some* $\alpha \in \left]0,1\right[$ *if* $T = (1-\alpha)\operatorname{Id} + \alpha R$ *for some nonexpansive operator* $R\colon \operatorname{dom} R = \mathcal{H} \to \mathcal{H}$ [8]. *The class of α-averaged operators on $\mathcal{H}$ is denoted by $\mathcal{A}(\alpha)$.*

The following relationships exist between these types of operators (see, e.g., [3, Proposition 2.3] and [12, Lemma 2.1]):

$$\begin{array}{ccccc} T \in \mathfrak{T} & \Leftarrow & T \text{ is firmly nonexpansive} & \Leftrightarrow & \operatorname{Id} - T \text{ is firmly nonexpansive} \\ & & \Downarrow & & \Updownarrow \\ & & T \text{ is nonexpansive} & & T \in \mathcal{A}(\tfrac{1}{2}). \end{array} \tag{2.8}$$

Lemma 2.4 [11, Proposition 2.4] *Let $(T_i)_{i\in I}$ be a finite family of operators in $\mathfrak{T}$ such that $C = \bigcap_{i\in I} \operatorname{Fix} T_i \neq \varnothing$ and let $(\omega_i)_{i\in I}$ be real numbers in $]0,1]$ such that $\sum_{i\in I}\omega_i = 1$. Define*

$$(\forall x \in \mathcal{H}) \;\; L\big(x, (T_i)_{i\in I}, (\omega_i)_{i\in I}\big) = \begin{cases} \dfrac{\sum_{i\in I}\omega_i \|T_i x - x\|^2}{\|\sum_{i\in I}\omega_i T_i x - x\|^2}, & \text{if } x \notin C; \\ 1, & \text{otherwise}, \end{cases} \tag{2.9}$$

and

$$T\colon \mathcal{H} \to \mathcal{H}\colon x \mapsto x + \lambda(x)\left(\sum_{i\in I}\omega_i T_i x - x\right), \;\; \text{where } 0 < \lambda(x) \leq L\big(x, (T_i)_{i\in I}, (\omega_i)_{i\in I}\big). \tag{2.10}$$

Then:

(i) *For all $x \in \mathcal{H}$, $L\big(x, (T_i)_{i\in I}, (\omega_i)_{i\in I}\big)$ is a well defined number in $[1,+\infty[$.*

(ii) $\operatorname{Fix} T = C$.

(iii) $T \in \mathfrak{T}$.

2.4 Convergence of two $\mathfrak{T}$-class algorithms

Algorithm 2.5 Given $\varepsilon \in]0,1]$ and $x_0 \in \mathcal{H}$, a sequence $(x_n)_{n\in\mathbb{N}}$ is constructed inductively as follows: for every $n \in \mathbb{N}$, select $T_n \in \mathfrak{T}$ and set $x_{n+1} = x_n + (2-\varepsilon)(T_n x_n - x_n)$.

Theorem 2.6 *Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary orbit of Algorithm 2.5 and let C be a nonempty closed convex subset of $\mathcal{H}$ such that $C \subset \bigcap_{n\in\mathbb{N}} \operatorname{Fix} T_n$. Then:*

(i) *$(x_n)_{n\in\mathbb{N}}$ is bounded.*

(ii) *$\sum_{n\in\mathbb{N}} \|x_{n+1} - x_n\|^2 < +\infty$ and $\sum_{n\in\mathbb{N}} \|T_n x_n - x_n\|^2 < +\infty$.*

(iii) *$(x_n)_{n\in\mathbb{N}}$ converges weakly to a point in C if and only if $\mathfrak{W}(x_n)_{n\in\mathbb{N}} \subset C$.*

Proof. For $C = \bigcap_{n\in\mathbb{N}} \operatorname{Fix} T_n$, this is precisely [3, Theorem 2.9]. However, in view of [3, Proposition 2.1], the results remain true as stated above. □

The second algorithm, which goes back to [16] (see also [10]), concerns the best approximation of the point a in (1.2).

Algorithm 2.7

Step 0. Set $n = 0$ and $x_0 = a$.

Step 1. Select $T_n \in \mathfrak{T}$.

Step 2. Set $\pi_n = \langle x_0 - x_n \mid x_n - T_n x_n \rangle$, $\mu_n = \|x_0 - x_n\|^2$, $\nu_n = \|x_n - T_n x_n\|^2$, and $\rho_n = \mu_n \nu_n - \pi_n^2$.

Step 3. If $\rho_n = 0$ and $\pi_n < 0$, then stop; otherwise set

$$x_{n+1} = \begin{cases} T_n x_n, & \text{if } \rho_n = 0 \text{ and } \pi_n \geq 0; \\ x_0 + (1 + \pi_n/\nu_n)(T_n x_n - x_n), & \text{if } \rho_n > 0 \text{ and } \pi_n \nu_n \geq \rho_n; \\ x_n + \dfrac{\nu_n}{\rho_n}\big(\pi_n(x_0 - x_n) + \mu_n(T_n x_n - x_n)\big), & \text{if } \rho_n > 0 \text{ and } \pi_n \nu_n < \rho_n. \end{cases} \tag{2.11}$$

Step 4. Set $n = n + 1$ and go to Step 1.

As shown in [3, Proposition 3.4(v)], the above algorithm does generate an infinite sequence $(x_n)_{n\in\mathbb{N}}$ for any starting point $x_0 \in \mathcal{H}$ provided that $\bigcap_{n\in\mathbb{N}} \operatorname{Fix} T_n \neq \varnothing$.

Theorem 2.8 *Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary orbit of Algorithm 2.7 and let C be a nonempty closed convex subset of $\mathcal{H}$ such that $C \subset \bigcap_{n\in\mathbb{N}} \operatorname{Fix} T_n$. Then:*

(i) $(x_n)_{n\in\mathbb{N}}$ *is bounded.*

(ii) $\sum_{n\in\mathbb{N}} \|x_{n+1} - x_n\|^2 < +\infty$ *and* $\sum_{n\in\mathbb{N}} \|T_n x_n - x_n\|^2 < +\infty$.

(iii) $(x_n)_{n\in\mathbb{N}}$ *converges strongly to the projection of* a *onto* C *if and only if* $\mathfrak{W}(x_n)_{n\in\mathbb{N}} \subset C$.

Proof. For $C = \bigcap_{n\in\mathbb{N}} \operatorname{Fix} T_n$, this is precisely [3, Theorem 3.5(ii)&(v)&(iv)]. However, an inspection of the proofs of [3, section 3] shows that the assertions remain true as stated above. □

2.5 Convergence of compositions of averaged operators

Algorithm 2.9 Fix $x_0 \in \mathcal{H}$ and, for every $n \in \mathbb{N}$, set

$$x_{n+1} = x_n + \lambda_n\big(T_{1,n}\big(T_{2,n}x_n + e_{2,n}\big) + e_{1,n} - x_n\big), \tag{2.12}$$

where $T_{1,n} \in \mathcal{A}(\alpha_{1,n})$ and $T_{2,n} \in \mathcal{A}(\alpha_{2,n})$, with $(\alpha_{1,n}, \alpha_{2,n}) \in \,]0,1[^2$, $(e_{1,n}, e_{2,n}) \in \mathcal{H}^2$, and $\lambda_n \in \,]0,1]$.

In the above iteration, $e_{1,n}$ and $e_{2,n}$ model errors induced by the inexact evaluation of the operators $T_{1,n}$ and $T_{2,n}$, respectively.

Theorem 2.10 [12, Theorem 6.3] *Suppose that the following conditions are satisfied.*

(i) $G = \bigcap_{n\in\mathbb{N}} \operatorname{Fix}\big(T_{1,n}T_{2,n}\big) \neq \varnothing$.

(ii) $\underline{\lim}\, \lambda_n > 0$, $\overline{\lim}\, \alpha_{1,n} < 1$, *and* $\overline{\lim}\, \alpha_{2,n} < 1$.

(iii) *For every subsequence* $(x_{k_n})_{n\in\mathbb{N}}$ *of an orbit* $(x_n)_{n\in\mathbb{N}}$ *generated by Algorithm 2.9, we have*

$$\begin{cases} (\forall x \in G) \ \displaystyle\sum_{n\in\mathbb{N}} \|(\operatorname{Id} - T_{1,n})T_{2,n}x_n + (\operatorname{Id} - T_{2,n})x\|^2 < +\infty \\ (\forall x \in G) \ \displaystyle\sum_{n\in\mathbb{N}} \|(\operatorname{Id} - T_{2,n})x_n - (\operatorname{Id} - T_{2,n})x\|^2 < +\infty \\ \displaystyle\sum_{n\in\mathbb{N}} \|T_{1,n}T_{2,n}x_n - x_n\|^2 < +\infty \\ x_{k_n} \rightharpoonup z \end{cases} \quad \Rightarrow \quad z \in G. \tag{2.13}$$

(iv) $\sum_{n\in\mathbb{N}} \|e_{1,n}\| < +\infty$ *and* $\sum_{n\in\mathbb{N}} \|e_{2,n}\| < +\infty$.

Then every orbit of Algorithm 2.9 converges weakly to a point in G.

2.6 Resolvents of bifunctions

The following notion appears implicitly in [5].

Definition 2.11 *The* resolvent *of a bifunction $F\colon K^2 \to \mathbb{R}$ is the set-valued operator*

$$J_F\colon \mathcal{H} \to 2^K \colon x \mapsto \big\{z \in K \mid (\forall y \in K)\ \ F(z,y) + \langle z - x \mid y - z\rangle \geq 0\big\}. \tag{2.14}$$

Lemma 2.12 *Suppose that $F\colon K^2 \to \mathbb{R}$ satisfies Condition 1.1 and let*

$$S_F = \big\{x \in K \mid (\forall y \in K)\ \ F(x,y) \geq 0\big\}. \tag{2.15}$$

Then:

(i) $\operatorname{dom} J_F = \mathcal{H}$.

(ii) *J_F is single-valued and firmly nonexpansive.*

(iii) $\operatorname{Fix} J_F = S_F$.

(iv) *S_F is closed and convex.*

Proof. (i): [5, Corollary 1] asserts that for every $x \in \mathcal{H}$ there exists a point $z \in K$ such that

$$(\forall y \in K)\ \ F(z,y) + \langle z - x \mid y - z\rangle \geq 0. \tag{2.16}$$

(ii): This statement is implicitly given in [5, p. 135], we provide the details for completeness. Fix $(x,x') \in \mathcal{H}^2$ and let $z \in J_F x$, $z' \in J_F x'$. Then $F(z,z') \geq \langle x - z \mid z' - z\rangle$ and $F(z',z) \geq \langle x' - z' \mid z - z'\rangle$. Therefore, by Condition 1.1(ii), $0 \geq F(z,z') + F(z',z) \geq \langle (x - x') - (z - z') \mid z' - z\rangle$, hence

$$\big\langle x - x' \;\big|\; z - z'\big\rangle \geq \|z - z'\|^2. \tag{2.17}$$

In particular, for $x = x'$, we obtain $z = z'$, which implies that J_F is single-valued. In turn, we derive from (2.17) that J_F is firmly nonexpansive. (iii): Take $x \in K$. Then $x \in \operatorname{Fix} J_F \Leftrightarrow x = J_F x$ $\Leftrightarrow (\forall y \in K)\ F(x,y) + \langle x - x \mid y - x\rangle \geq 0 \Leftrightarrow (\forall y \in K)\ F(x,y) \geq 0 \Leftrightarrow x \in S_F$. (iv): Follows from (iii), (ii), and (2.8) since the fixed point set of a nonexpansive operator is closed and convex [14, Proposition 1.5.3]. ☐

Lemma 2.13 *Suppose that $F\colon K^2 \to \mathbb{R}$ satisfies Condition 1.1. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ and $(\gamma_n)_{n\in\mathbb{N}}$ a sequence in $]0,+\infty[$. Define*

$$(\forall n \in \mathbb{N})\ \ z_n = J_{\gamma_n F} x_n \quad \textit{and} \quad u_n = (x_n - z_n)/\gamma_n, \tag{2.18}$$

and suppose that

$$z_n \rightharpoonup x \quad \textit{and} \quad u_n \to u. \tag{2.19}$$

Then $x \in K$ and $(\forall y \in K)\ \ F(x,y) + \langle u \mid x - y\rangle \geq 0$.

Proof. It follows from Lemma 2.12(i)&(ii) that the sequence $(z_n)_{n\in\mathbb{N}}$ is well defined and from (2.14) that it lies in K, which is weakly closed. Therefore $x \in K$. On the other hand, it follows from Condition 1.1(iii) that $F(y,\cdot)$ is weak lower semicontinuous for every $y \in K$. Therefore, we derive from Condition 1.1(ii), (2.18), and (2.14) that

$$(\forall y \in K)\ \ F(y,x) \leq \underline{\lim}\, F(y,z_n) \leq \underline{\lim}\, -F(z_n,y) \leq \underline{\lim}\, \langle u_n \mid z_n - y\rangle = \langle u \mid x - y\rangle\,, \tag{2.20}$$

where the last equality follows from (2.19) and the sequential continuity of $\langle \cdot \mid \cdot \rangle$ on $\mathcal{H}^{\text{strong}} \times \mathcal{H}^{\text{weak}}$. Now fix $y \in K$ and define, for every $\varepsilon \in \,]0,1]$, $x_\varepsilon = (1-\varepsilon)x + \varepsilon y$. Then, for every $\varepsilon \in \,]0,1]$, $x_\varepsilon \in K$ by convexity of K and, in turn, Condition 1.1(i), Condition 1.1(iii), and (2.20) yield

$$\begin{aligned} 0 &= F(x_\varepsilon, x_\varepsilon) \\ &\leq (1-\varepsilon)F(x_\varepsilon, x) + \varepsilon F(x_\varepsilon, y) \\ &\leq (1-\varepsilon)\langle u \mid x - x_\varepsilon\rangle + \varepsilon F(x_\varepsilon, y) \\ &= \varepsilon(1-\varepsilon)\langle u \mid x - y\rangle + \varepsilon F(x_\varepsilon, y), \end{aligned} \tag{2.21}$$

whence $F(x_\varepsilon, y) \geq (1-\varepsilon)\langle u \mid y - x\rangle$. In view of Condition 1.1(iv), we conclude that $F(x,y) \geq \overline{\lim}_{\varepsilon\to 0^+} F(x_\varepsilon, y) \geq \langle u \mid y - x\rangle$. ☐

Next, we recall an important technical fact that will be required subsequently.

Lemma 2.14 [5, Lemma 1] *Let C_1 and C_2 be two nonempty convex subsets of $\mathcal{H}$ such that C_1 is weakly compact. Let the function $\Phi\colon C_1 \times C_2 \to \mathbb{R}$ be concave and upper semicontinuous in the first argument, and convex in the second argument. Assume furthermore that*

$$(\forall y \in C_2)\ \ \max_{u\in C_1} \Phi(u,y) \geq 0. \tag{2.22}$$

Then

$$(\exists\, u \in C_1)(\forall y \in C_2)\ \ \Phi(u,y) \geq 0. \tag{2.23}$$

We complete this section with concrete examples of resolvents of bifunctions.

Lemma 2.15 *Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator, let $f \in \Gamma_0(\mathcal{H})$, let $C \subset \mathcal{H}$ be a nonempty closed convex set, and let $\mu \in \,]0,+\infty[$.*

(i) *Suppose that $K \subset \operatorname{int}\operatorname{dom} A$ and set*

$$(\forall (x,y) \in K^2)\ \ F(x,y) = \max_{u\in Ax} \langle u \mid y - x\rangle\,. \tag{2.24}$$

Then F satisfies Condition 1.1 and $J_F = J_{A+N_K}$.

(ii) *Set*

$$K = \mathcal{H} \quad \text{and} \quad (\forall (x,y) \in \mathcal{H}^2)\ \ F(x,y) = \langle {}^{\mu}\!A x \mid y - x\rangle\,. \tag{2.25}$$

Then F satisfies Condition 1.1 and $J_F = \operatorname{Id} + \frac{1}{\mu+1}\big(J_{(\mu+1)A} - \operatorname{Id}\big)$.

(iii) *Set*

$$K = \mathcal{H} \quad \textit{and} \quad (\forall (x,y) \in \mathcal{H}^2) \;\; F(x,y) = \big\langle x - \operatorname{prox}_{\mu f} x \mid y - x \big\rangle / \mu. \tag{2.26}$$

Then F satisfies Condition 1.1 and $J_F = \operatorname{Id} + \frac{1}{\mu+1}(\operatorname{prox}_{(\mu+1)f} - \operatorname{Id})$.

(iv) *Set*

$$K = \mathcal{H} \quad \textit{and} \quad (\forall (x,y) \in \mathcal{H}^2) \;\; F(x,y) = \langle x - P_C x \mid y - x \rangle / \mu. \tag{2.27}$$

Then F satisfies Condition 1.1 and $J_F = \operatorname{Id} + \frac{1}{\mu+1}(P_C - \operatorname{Id})$.

(v) *Suppose that $K \subset \operatorname{dom} f$ and set*

$$(\forall (x,y) \in K^2) \;\; F(x,y) = f(y) - f(x). \tag{2.28}$$

Then F satisfies Condition 1.1 and $J_F = \operatorname{prox}_{f + \iota_K}$.

Proof. (i): By [25, Theorem 2.28], A is locally bounded on $\operatorname{int} \operatorname{dom} A$. Therefore, it follows from [1, Proposition 3.5.6.1] that the sets $(Ax)_{x \in K}$ are weakly compact and that, for every $(x,y) \in K^2$, the weakly continuous function $\langle \cdot \mid y - x \rangle$ achieves its maximum over Ax. Hence, F is well defined. In addition, F satisfies Condition 1.1: Indeed, item (i) there is obvious, item (ii) follows at once from the monotonicity of A, and item (iii) from the fact that $F(x,\cdot)$ is the supremum of the family of lower semicontinuous convex functions $(\langle u \mid \cdot - x \rangle)_{u \in Ax}$. Finally, to establish item (iv) in Condition 1.1, let us observe that our assumptions imply that $A|_{\operatorname{int} \operatorname{dom} A}$ is a weak-usco operator [25, section 7]. Hence, it follows from [4, Théorème VI.3.2] that, for every $y \in K$, $F(\cdot, y)$ is upper semicontinuous and, therefore, that Condition 1.1(iv) holds. Now take x and z in $\mathcal{H}$. Then Lemma 2.12(ii) and (2.14) yield

$$\begin{aligned}
z = J_F x \quad &\Leftrightarrow \quad z \in K \quad \text{and} \quad (\forall y \in K) \;\; \max_{u \in Az} \langle y - z \mid u + z - x \rangle \geq 0 \\
&\Leftrightarrow \quad z \in K \quad \text{and} \quad (\exists\, u \in Az)(\forall y \in K) \;\; \langle y - z \mid u + z - x \rangle \geq 0 \qquad (2.29) \\
&\Leftrightarrow \quad (\exists\, u \in Az) \;\; x - z - u \in N_K z \\
&\Leftrightarrow \quad x \in z + Az + N_K z \\
&\Leftrightarrow \quad z = (\operatorname{Id} + A + N_K)^{-1} x \\
&\Leftrightarrow \quad z = J_{A + N_K} x, \qquad (2.30)
\end{aligned}$$

where we have used Lemma 2.14 (with $C_1 = Az$, $C_2 = K$, and $\Phi(u,y) = \langle y - z \mid u + z - x \rangle$) to get (2.29). (ii): ${}^{\mu}A$ is a single-valued maximal monotone operator with domain $\mathcal{H}$ [1, Theorem 3.5.9]. Using this operator in (2.24) yields (2.25), and it follows from (i) that $J_F = J_{({}^{\mu}A)}$, which proves the assertion via Lemma 2.1. (iii): Set $A = \partial f$ in (ii) and use (2.1) and (2.6). (iv): Set $f = \iota_C$ in (iii). (v): It is easily verified that F satisfies Condition 1.1. Now take x and z in $\mathcal{H}$. Then it follows from (2.14), (2.28), and (2.5) that

$$\begin{aligned}
z = J_F x \quad &\Leftrightarrow \quad z \in K \quad \text{and} \quad (\forall y \in K) \;\; \langle y - z \mid x - z \rangle + f(z) \leq f(y) \\
&\Leftrightarrow \quad (\forall y \in \mathcal{H}) \;\; \langle y - z \mid x - z \rangle + f(z) + \iota_K(z) \leq f(y) + \iota_K(y) \\
&\Leftrightarrow \quad x - z \in \partial(f + \iota_K)(z) \\
&\Leftrightarrow \quad z = \operatorname{prox}_{f + \iota_K} x. \qquad (2.31)
\end{aligned}$$

□

Remark 2.16 In all of the above examples, the function $F(\cdot, y)$ is actually upper semicontinuous for every $y \in K$.

3 Block-iterative algorithms

The main objective of this section is to apply Theorem 2.6 and Theorem 2.8 with a suitable choice of the sequence $(T_n)_{n\in\mathbb{N}}$ to solve (1.1) and (1.2). It is recalled that $(F_i)_{i\in I}$ is a countable (finite or countably infinite) family of bifunctions from K^2 to $\mathbb{R}$ which all satisfy Condition 1.1. We shall use the following construction, in which the operator L is defined by (2.9).

Procedure 3.1 Fix $\delta \in \,]0,1[$. At every iteration $n \in \mathbb{N}$, x_n is available and T_n is constructed according to the following steps.

① $\varnothing \neq I_n \subset I$, I_n finite.

② $(\forall i \in I_n)\ \ \gamma_{i,n} \in \,]0,+\infty[$.

③ $(\forall i \in I_n)$ $\omega_{i,n} \in [0,1]$, $\sum_{i\in I_n} \omega_{i,n} = 1$, and

$$(\exists\, j \in I_n) \ \begin{cases} \|J_{\gamma_{j,n}F_j}x_n - x_n\| = \max\limits_{i\in I_n} \|J_{\gamma_{i,n}F_i}x_n - x_n\| \\ \omega_{j,n} \geq \delta. \end{cases}$$

④ $(\forall x \in \mathcal{H})$ $\lambda_n(x) \in \big[\delta, L\big(x, (J_{\gamma_{i,n}F_i})_{i\in I_n^+}, (\omega_{i,n})_{i\in I_n^+}\big)\big]$, where $I_n^+ = \big\{i \in I_n \mid \omega_{i,n} > 0\big\}$.

⑤ $T_n\colon \mathcal{H} \to \mathcal{H}\colon x \mapsto x + \lambda_n(x)\left(\sum_{i\in I_n^+} \omega_{i,n} J_{\gamma_{i,n}F_i}x - x\right)$.

Condition 3.2

(i) The set S in (1.2) is nonempty.

(ii) $(T_n)_{n\in\mathbb{N}}$ is constructed as in Procedure 3.1.

(iii) There exist strictly positive integers $(M_i)_{i\in I}$ such that

$$(\forall (i,n) \in I \times \mathbb{N})\ \ i \in \bigcup_{k=n}^{n+M_i-1} I_k. \tag{3.1}$$

(iv) For every $i \in I$ and every strictly increasing sequence $(p_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $i \in \bigcap_{n\in\mathbb{N}} I_{p_n}$, $\inf_{n\in\mathbb{N}} \gamma_{i,p_n} > 0$.

Theorem 3.3 *Suppose that Condition 3.2 is satisfied. Then:*

(i) *Every orbit of Algorithm 2.5 converges weakly to a solution to (1.1).*

(ii) *Every orbit of Algorithm 2.7 converges strongly to the unique solution to (1.2).*

Proof. We are going to show that the two assertions follow from Theorem 2.6 and Theorem 2.8 with $C = S$. We first observe that Lemma 2.12(iv) guarantees that the sets $(S_i)_{i\in I}$ in (1.2) are closed and convex. Accordingly, it follows from Condition 3.2(i) that S is nonempty, closed, and convex. Problem (1.2) therefore possesses a unique solution. Moreover, for all $n \in \mathbb{N}$, the weights $(\omega_{i,n})_{i\in I_n^+}$ are real numbers in $]0,1]$ such that $\sum_{i\in I_n^+} \omega_{i,n} = 1$. On the other hand, it follows from Lemma 2.12(i)&(ii) and (2.8) that

$$(\forall n \in \mathbb{N})(\forall i \in I_n) \;\; J_{\gamma_{i,n}F_i} \in \mathfrak{T}. \tag{3.2}$$

Therefore, Lemma 2.4(iii) yields

$$(\forall n \in \mathbb{N}) \;\; T_n \in \mathfrak{T}. \tag{3.3}$$

Furthermore, we derive from (1.2), Lemma 2.12(iii), and Lemma 2.4(ii) that

$$(\forall n \in \mathbb{N}) \;\; S = \bigcap_{i\in I} S_i \subset \bigcap_{i\in I_n^+} \operatorname{Fix} J_{\gamma_{i,n}F_i} = \operatorname{Fix} T_n. \tag{3.4}$$

Therefore $S \subset \bigcap_{n\in\mathbb{N}} \operatorname{Fix} T_n$. Now let i be an index in I, let $(x_n)_{n\in\mathbb{N}}$ be a sequence generated by either algorithm, and let $x \in \mathfrak{W}(x_n)_{n\in\mathbb{N}}$, say $x_{k_n} \rightharpoonup x$. Then, in view of Theorem 2.6(iii) and Theorem 2.8(iii), it is enough to show that $x \in S_i$, i.e.,

$$x \in K \quad \text{and} \quad (\forall y \in K) \;\; F_i(x,y) \geq 0. \tag{3.5}$$

Theorem 2.6(ii) and Theorem 2.8(ii) assert that $\sum_{n\in\mathbb{N}} \|x_{n+1} - x_n\|^2 < +\infty$ and $\sum_{n\in\mathbb{N}} \|T_n x_n - x_n\|^2 < +\infty$. Now fix $z \in S$ and set $\beta = \sup_{n\in\mathbb{N}} \|x_n - z\|$. Then $\beta < +\infty$ by Theorem 2.6(i) and Theorem 2.8(i). On the other hand, we derive from (3.2), (3.4), and Definition 2.3(i) that

$$(\forall n \in \mathbb{N})(\forall j \in I_n) \;\; \|J_{\gamma_{j,n}F_j} x_n - x_n\|^2 \leq \left\langle z - x_n \;\middle|\; J_{\gamma_{j,n}F_j} x_n - x_n \right\rangle. \tag{3.6}$$

Thus, it follows from Procedure 3.1③, (3.6), the Cauchy-Schwarz inequality, and Procedure 3.1④&⑤ that

$$
\begin{aligned}
(\forall n \in \mathbb{N})\ \ \delta \max_{j \in I_n} \|J_{\gamma_{j,n} F_j} x_n - x_n\|^2 &\leq \sum_{j \in I_n} \omega_{j,n} \left\| J_{\gamma_{j,n} F_j} x_n - x_n \right\|^2 \\
&\leq \left\langle z - x_n \,\middle|\, \sum_{j \in I_n} \omega_{j,n} J_{\gamma_{j,n} F_j} x_n - x_n \right\rangle \\
&\leq \beta \left\| \sum_{j \in I_n^+} \omega_{j,n} J_{\gamma_{j,n} F_j} x_n - x_n \right\| \\
&\leq \frac{\beta \lambda_n(x_n)}{\delta} \left\| \sum_{j \in I_n^+} \omega_{j,n} J_{\gamma_{j,n} F_j} x_n - x_n \right\| \\
&= \beta \|T_n x_n - x_n\| / \delta.
\end{aligned} \tag{3.7}
$$

Therefore, since $T_n x_n - x_n \to 0$, we obtain

$$\max_{j \in I_n} \|J_{\gamma_{j,n} F_j} x_n - x_n\| \to 0. \tag{3.8}$$

After passing to a subsequence of $(x_{k_n})_{n \in \mathbb{N}}$ if necessary, we assume that, for every $n \in \mathbb{N}$, $k_{n+1} \geq k_n + M_i$. Then Condition 3.2(iii) asserts that there exists a sequence $(p_n)_{n \in \mathbb{N}}$ in $\mathbb{N}$ such that

$$(\forall n \in \mathbb{N})\ \ k_n \leq p_n \leq k_n + M_i - 1 < k_{n+1} \leq p_{n+1} \quad \text{and} \quad i \in I_{p_n}. \tag{3.9}$$

However, $x_{p_n} - x_{k_n} \to 0$ since $\sum_{n \in \mathbb{N}} \|x_{n+1} - x_n\|^2 < +\infty$ and

$$(\forall n \in \mathbb{N})\ \ \|x_{p_n} - x_{k_n}\| \leq \sum_{l=k_n}^{k_n+M_i-2} \|x_{l+1} - x_l\| \leq \sqrt{M_i - 1} \sqrt{\sum_{l \geq k_n} \|x_{l+1} - x_l\|^2}. \tag{3.10}$$

Hence, $x_{p_n} \rightharpoonup x$. On the other hand, we derive from (3.8) and (3.9) that

$$z_{p_n} - x_{p_n} \to 0, \quad \text{where} \quad (\forall n \in \mathbb{N})\ \ z_{p_n} = J_{\gamma_{i,p_n} F_i} x_{p_n}. \tag{3.11}$$

In turn, we obtain

$$z_{p_n} \rightharpoonup x. \tag{3.12}$$

Now set, for every $n \in \mathbb{N}$, $u_{p_n} = (x_{p_n} - z_{p_n})/\gamma_{i,p_n}$. Then (3.11) and Condition 3.2(iv) imply that

$$u_{p_n} \to 0. \tag{3.13}$$

Altogether, it follows from (3.12), (3.13), and Lemma 2.13 that (3.5) is satisfied. ☐

Remark 3.4

- By considering the special cases described in Lemma 2.15, one can recover from Theorem 3.3 various convergence results for block-iterative methods involving resolvents of maximal monotone operators, proximity operators, or projection operators (see [3] and [11] and the references therein). For instance, suppose that $I = \{1, \ldots, m\}$ and that $(S_i)_{i\in I}$ is a family of closed convex sets in $\mathcal{H}$ with associated projection operators $(P_i)_{i\in I}$. Now fix $\varepsilon \in \,]0,1[$, define $\mathrm{i}\colon \mathbb{N} \to I\colon n \mapsto (n \,\mathrm{modulo}\, m) + 1$, and set

$$K = \mathcal{H} \;\text{ and }\; (\forall i \in I)(\forall (x,y) \in \mathcal{H}^2)\;\; F_i(x,y) = \langle x - P_i x \mid y - x\rangle /(1-\varepsilon). \tag{3.14}$$

  Then it follows from Lemma 2.15(iv) that, for every $i \in I$, F_i satisfies Condition 1.1 and $J_{F_i} = \mathrm{Id} + (P_i - \mathrm{Id})/(2-\varepsilon)$. Now set

$$(\forall n \in \mathbb{N})\;\; \lambda_n \equiv 1,\; I_n = \{\mathrm{i}(n)\},\; \text{and } \gamma_{\mathrm{i}(n),n} = 1. \tag{3.15}$$

  Then Theorem 3.3(i) states that, if $S = \bigcap_{i\in I} S_i \neq \varnothing$, the sequence produced by the cyclic projections method

$$x_0 \in \mathcal{H} \quad \text{and} \quad (\forall n \in \mathbb{N})\;\; x_{n+1} = P_{\mathrm{i}(n)} x_n \tag{3.16}$$

  converges weakly to a point in S. This classical result is due to Bregman [6, Theorem 1].

- It follows from the analysis of [11] that the conclusion of Theorem 3.3(i) remains true if certain errors are present in evaluation of the resolvents in Procedure 3.1.

In the case when the family $(F_i)_{i\in I}$ consists of a single bifunction F, Problem (1.1) reduces to

$$\text{find } x \in K \quad \text{such that} \quad (\forall y \in K)\;\; F(x,y) \geq 0. \tag{3.17}$$

Moreover, we have $L \equiv 1$ and, for $\lambda_n \equiv 1/(2-\varepsilon)$ in Procedure 3.1, the iteration described by Algorithm 2.5 assumes the form

$$x_0 \in K \quad \text{and} \quad (\forall n \in \mathbb{N})\;\; x_{n+1} = J_{\gamma_n F} x_n, \;\; \text{where } \; \gamma_n \in \,]0,+\infty[\,. \tag{3.18}$$

Theorem 3.3(i) states that every sequence $(x_n)_{n\in\mathbb{N}}$ so constructed converges weakly to a solution to (3.17) provided that $\inf_{n\in\mathbb{N}} \gamma_n > 0$ (see also [13, 20, 22, 23] for related results). This statement can be refined as follows.

Theorem 3.5 *Suppose that $F\colon K^2 \to \mathbb{R}$ satisfies Condition 1.1 and that the set S of solutions to (3.17) is nonempty. Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence generated by (3.18), where $\sum_{n\in\mathbb{N}} \gamma_n^2 = +\infty$. Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point in S.*

Proof. Let $n \in \mathbb{N}$. It follows from (3.18) and (2.14) that

$$\begin{cases} 0 \leq \gamma_n F(x_{n+1}, x_{n+2}) + \langle x_{n+1} - x_n \mid x_{n+2} - x_{n+1}\rangle \\ 0 \leq \gamma_{n+1} F(x_{n+2}, x_{n+1}) + \langle x_{n+2} - x_{n+1} \mid x_{n+1} - x_{n+2}\rangle\,. \end{cases} \tag{3.19}$$

Now set $z_n = J_{\gamma_n F} x_n$ and $u_n = (x_n - z_n)/\gamma_n$. Then (3.19) yields

$$\begin{cases} \langle u_n \mid x_{n+2} - x_{n+1} \rangle \leq F(x_{n+1}, x_{n+2}) \\ \langle u_{n+1} \mid x_{n+1} - x_{n+2} \rangle \leq F(x_{n+2}, x_{n+1}) \end{cases} \tag{3.20}$$

and it therefore follows from Condition 1.1(ii) that

$$\langle u_n - u_{n+1} \mid x_{n+2} - x_{n+1} \rangle \leq F(x_{n+1}, x_{n+2}) + F(x_{n+2}, x_{n+1}) \leq 0. \tag{3.21}$$

Consequently, we have $\langle u_{n+1} - u_n \mid u_{n+1} \rangle \leq 0$ and, by Cauchy-Schwarz, $\|u_{n+1}\| \leq \|u_n\|$. Therefore

$$(\|u_n\|)_{n\in\mathbb{N}} \text{ converges.} \tag{3.22}$$

Let us now apply Theorem 2.6 with $(T_n)_{n\in\mathbb{N}} = (J_{\gamma_n F})_{n\in\mathbb{N}}$. We first obtain from Theorem 2.6(ii) that $\sum_{n\in\mathbb{N}} \gamma_n^2 \|u_n\|^2 = \sum_{n\in\mathbb{N}} \|z_n - x_n\|^2 < +\infty$. Since $\sum_{n\in\mathbb{N}} \gamma_n^2 = +\infty$, it follows that $\underline{\lim}\, \|u_n\| = 0$ and, consequently, (3.22) yields $u_n \to 0$. Now suppose that $x_{k_n} \rightharpoonup x$. Then, in view of Theorem 2.6(iii), it remains to show that $x \in S$. As seen above,

$$u_{k_n} \to 0. \tag{3.23}$$

On the other hand, since $z_n - x_n \to 0$, we have

$$z_{k_n} \rightharpoonup x. \tag{3.24}$$

Combining (3.23), (3.24), and Lemma 2.13, we conclude that x solves (3.17). ☐

Remark 3.6 Consider the setting of Lemma 2.15(i) with $K = \mathcal{H}$. Then, for every $n \in \mathbb{N}$, $J_{\gamma_n F} = (\mathrm{Id} + \gamma_n A)^{-1}$ reduces to the usual resolvent of $\gamma_n A$ and Theorem 3.5 therefore corresponds to [7, Proposition 8] (see also [8, Theorem 2.6(a)]).

4 A regularization method

In this section, we suppose that the family $(F_i)_{i\in I}$ consists of a single bifunction F. Then the problem (1.2) of finding the best approximation to the point a becomes

$$\text{project } a \text{ onto } S = \big\{ z \in K \mid (\forall y \in K) \;\; F(z, y) \geq 0 \big\}. \tag{4.1}$$

We now describe an alternative to Theorem 3.3(ii) to solve this problem.

Algorithm 4.1

Step 0. Set $n = 0$ and $x_0 = a$.

Step 1. Let $\alpha_n \in \left]0,1\right[$, $\gamma_n \in \left]0,+\infty\right[$, and $e_n \in \mathcal{H}$.

Step 2. Set $x_{n+1} = \alpha_n a + (1-\alpha_n)\big(J_{\gamma_n F} x_n + e_n\big)$.

Step 3. Set $n = n+1$ and go to Step 1.

We shall study this iterative scheme in the context described below.

Condition 4.2

(i) The set S in (4.1) is nonempty.

(ii) $\alpha_n \to 0$ and $\sum_{n\in\mathbb{N}} \alpha_n = +\infty$.

(iii) $\gamma_n \to +\infty$.

(iv) $\sum_{n\in\mathbb{N}} \|e_n\| < +\infty$.

Theorem 4.3 *Suppose that $F\colon K^2 \to \mathbb{R}$ satisfies Condition 1.1 and that Condition 4.2 is satisfied. Then every orbit of Algorithm 4.1 converges strongly to the unique solution to (4.1).*

Proof. In view of Lemma 2.12(iv) and Condition 4.2(i), Problem (4.1) does possess a unique solution. Now let $(x_n)_{n\in\mathbb{N}}$ be an orbit of Algorithm 4.1 and set

$$(\forall n \in \mathbb{N})\ \ z_n = J_{\gamma_n F} x_n \quad \text{and} \quad u_n = (x_n - z_n)/\gamma_n. \tag{4.2}$$

It follows from Lemma 2.12(i)–(iii), and (2.8) that

$$(\forall x \in S)(\forall n \in \mathbb{N})\ \ \|z_n - x\| = \|J_{\gamma_n F} x_n - J_{\gamma_n F} x\| \leq \|x_n - x\|. \tag{4.3}$$

Therefore

$$\begin{aligned}(\forall x \in S)(\forall n \in \mathbb{N})\ \ \|x_{n+1} - x\| &\leq \alpha_n \|a - x\| + (1-\alpha_n)\big(\|z_n - x\| + \|e_n\|\big) \\ &\leq \alpha_n \|a - x\| + (1-\alpha_n)\|x_n - x\| + \|e_n\|. \end{aligned} \tag{4.4}$$

Since $x_0 = a$, we obtain by induction

$$(\forall x \in S)(\forall n \in \mathbb{N})\ \ \|x_{n+1} - x\| \leq \|a - x\| + \sum_{k=0}^{n} \|e_k\|. \tag{4.5}$$

Hence, it follows from Condition 4.2(iv) that $(x_n)_{n\in\mathbb{N}}$ is bounded and, in turn, from (4.3) that $(z_n)_{n\in\mathbb{N}}$ is bounded. Consequently, $(x_n - z_n)_{n\in\mathbb{N}}$ is bounded and we derive from (4.2) and Condition 4.2(iii) that

$$u_n \to 0. \tag{4.6}$$

Now let $(z_{k_n})_{n\in\mathbb{N}}$ be a subsequence of $(z_n)_{n\in\mathbb{N}}$ such that

$$\langle a - P_S a \mid z_{k_n} - P_S a \rangle \to \overline{\lim}\, \langle a - P_S a \mid z_n - P_S a \rangle\,, \tag{4.7}$$

and such that $(z_{k_n})_{n\in\mathbb{N}}$ converges weakly to some point $z \in \mathcal{H}$. Then it follows from (4.6) and Lemma 2.13 that $z \in S$. We therefore deduce from Lemma 2.12(iv) and the standard characterization of projections onto convex sets that

$$\langle a - P_S a \mid z_{k_n} - P_S a\rangle \to \langle a - P_S a \mid z - P_S a\rangle \leq 0. \tag{4.8}$$

Therefore, (4.7) and Condition 4.2(iv) imply that

$$\overline{\lim}\,\langle a - P_S a \mid z_n + e_n - P_S a\rangle \leq 0. \tag{4.9}$$

Next, we observe that

$$\begin{aligned}(\forall n \in \mathbb{N})\ \ \|x_{n+1} - P_S a\|^2 &= \|\alpha_n(a - P_S a) + (1-\alpha_n)\big(z_n + e_n - P_S a\big)\|^2\\ &= \alpha_n^2\|a - P_S a\|^2 + (1-\alpha_n)^2\|z_n + e_n - P_S a\|^2\\ &\quad + 2\alpha_n(1-\alpha_n)\,\langle a - P_S a \mid z_n + e_n - P_S a\rangle\\ &\leq \alpha_n^2\|a - P_S a\|^2 + (1-\alpha_n)\big(\|z_n - P_S a\| + \|e_n\|\big)^2\\ &\quad + 2\alpha_n(1-\alpha_n)\,\langle a - P_S a \mid z_n + e_n - P_S a\rangle\,. \end{aligned} \tag{4.10}$$

Now let us fix $\varepsilon \in\ ]0,+\infty[$. We infer from Condition 4.2(ii)&(iv) and (4.9) the existence of an index $q \in \mathbb{N}$ such that, for every $n \geq q$,

$$\alpha_n\|a - P_S a\|^2 \leq \varepsilon, \quad \sum_{k\geq q}\|e_k\| \leq \varepsilon, \quad \text{and} \quad \langle a - P_S a \mid z_n + e_n - P_S a\rangle \leq \varepsilon. \tag{4.11}$$

Thus, it follows from (4.10), (4.11), and (4.3) that, for every $n \geq q$,

$$\begin{aligned}\|x_{n+1} - P_S a\|^2 &\leq 3\alpha_n\varepsilon + (1-\alpha_n)\big(\|z_n - P_S a\|^2 + \beta\|e_n\|\big)\\ &\leq 3\alpha_n\varepsilon + (1-\alpha_n)\|x_n - P_S a\|^2 + \beta\|e_n\|,\end{aligned} \tag{4.12}$$

where $\beta = \sup_{n\in\mathbb{N}}\big(\|e_n\| + 2\|z_n - P_S a\|\big) < +\infty$. Hence, we obtain by induction that, for every $n \geq q$,

$$\begin{aligned}\|x_{n+1} - P_S a\|^2 \ \leq\ & 3\Big(1 - \prod_{k=q}^{n}(1-\alpha_k)\Big)\varepsilon\\ & + \Big(\prod_{k=q}^{n}(1-\alpha_k)\Big)\|x_q - P_S a\|^2 + \beta\sum_{k=q}^{n}\|e_k\|.\end{aligned} \tag{4.13}$$

However, it follows from Condition 4.2(ii) that $\prod_{k=q}^{n}(1-\alpha_k) \to 0$ [19, Theorem 3.7.7]. Therefore, (4.13), Condition 4.2(iv), and (4.11) yield

$$\overline{\lim}\,\|x_n - P_S a\|^2 \leq 3\varepsilon + \beta\sum_{k\geq q}\|e_k\| \leq \varepsilon(3+\beta), \tag{4.14}$$

and hence we conclude that $\|x_n - P_S a\|^2 \to 0$. ☐

Remark 4.4 Consider the setting of Lemma 2.15(i) with $K = \mathcal{H}$. Then, for every $n \in \mathbb{N}$, $J_{\gamma_n F} = (\mathrm{Id} + \gamma_n A)^{-1}$ and Theorem 4.3 therefore corresponds to [18, Theorem 1].

5 Splitting

In this section, we return to the single equilibrium problem (3.17) in instances when the bifunction F can be broken up into the sum of two terms, say

$$F\colon (x,y) \mapsto F_0(x,y) + \langle Bx \mid y-x \rangle\,, \text{ where } F_0\colon K^2 \to \mathbb{R} \text{ and } B\colon \mathcal{H} \to \mathcal{H}. \tag{5.1}$$

In this scenario, (3.17) becomes

$$\text{find } x \in K \text{ such that } (\forall y \in K)\ \ F_0(x,y) + \langle Bx \mid y-x \rangle \geq 0. \tag{5.2}$$

Our objective is to devise a splitting algorithm in which the bifunction F_0 and the operator B are employed in separate steps at each iteration. It is assumed throughout this section that

$$F_0\colon K^2 \to \mathbb{R} \text{ satisfies Condition 1.1} \tag{5.3}$$

and that

$$\beta B \text{ is firmly nonexpansive on } \operatorname{dom} B = \mathcal{H}, \text{ for some } \beta \in \left]0,+\infty\right[\,. \tag{5.4}$$

Moreover, we denote by G the set of solutions to (5.2), i.e.,

$$G = \big\{x \in K \mid (\forall y \in K)\ \ F_0(x,y) + \langle Bx \mid y-x \rangle \geq 0\big\}. \tag{5.5}$$

Remark 5.1 The bifunction F defined in (5.1) satisfies Condition 1.1. Indeed, by (5.4), B is continuous and monotone on $\mathcal{H}$, hence maximal monotone [1, Proposition 3.5.7]. Thus, it follows from Lemma 2.15(i) that the bifunction $(x,y) \mapsto \langle Bx \mid y-x \rangle$ satisfies Condition 1.1. In view of (5.3), so does the sum F in (5.1).

Proposition 5.2 *Let $\gamma \in \left]0,2\beta\right[$ and set $S_{F_0,B} = \big\{x \in K \mid Bx = 0 \text{ and } (\forall y \in K)\ \ F_0(x,y) \geq 0\big\}$. Then:*

(i) $G = \operatorname{Fix} J_{\gamma F_0}\big(\operatorname{Id} - \gamma B\big)$.

(ii) $J_{\gamma F_0}\big(\operatorname{Id} - \gamma B\big)$ *is nonexpansive.*

(iii) G *is closed and convex.*

(iv) *Suppose that $S_{F_0,B} \neq \varnothing$. Then $G = S_{F_0,B}$.*

Proof. Note that, by (5.3) and Lemma 2.12(i), $\operatorname{dom} J_{\gamma F_0} = \mathcal{H}$. (i): Let $x \in \mathcal{H}$. Then, in view of (2.14) and Lemma 2.12(ii),

$$\begin{aligned} x \in G \quad &\Leftrightarrow \quad x \in K \text{ and } (\forall y \in K)\ \ F_0(x,y) + \langle Bx \mid y-x \rangle \geq 0 \\ &\Leftrightarrow \quad x \in K \text{ and } (\forall y \in K)\ \ \gamma F_0(x,y) + \langle x - (x - \gamma Bx) \mid y-x \rangle \geq 0 \\ &\Leftrightarrow \quad x = J_{\gamma F_0}\big(x - \gamma Bx\big). \end{aligned} \tag{5.6}$$

(ii): It follows from (5.4) and (2.8) that $\beta B \in \mathcal{A}(\frac{1}{2})$ and hence from [12, Lemma 2.3] that $\mathrm{Id} - \gamma B \in \mathcal{A}(\frac{\gamma}{2\beta})$. On the other hand, (5.3), Lemma 2.12(ii) and (2.8) yield $J_{\gamma F_0} \in \mathcal{A}(\frac{1}{2})$. Hence $J_{\gamma F_0}$ and $\mathrm{Id} - \gamma B$ are nonexpansive and so is their composition. (iii): The assertion follows from (ii), (i), and [14, Proposition 1.5.3]. Alternatively, this claim follows from Remark 5.1 and Lemma 2.12(iv). (iv): Set $T_1 = J_{\gamma F_0}$ and $T_2 = \mathrm{Id} - \gamma B$. Then, on the one hand, (i) yields $\mathrm{Fix}\, T_1 T_2 = G$ and, on the other hand, Lemma 2.12(iii) yields $\mathrm{Fix}\, T_1 \cap \mathrm{Fix}\, T_2 = S_{F_0,B}$. Now, as seen above, the operators T_1 and T_2 are averaged. Therefore it follows from [8, Lemma 2.1] that $\mathrm{Fix}\, T_1 \cap \mathrm{Fix}\, T_2 \neq \varnothing \Rightarrow \mathrm{Fix}\, T_1 \cap \mathrm{Fix}\, T_2 = \mathrm{Fix}\, T_1 T_2$, which completes the proof. □

We now describe a splitting algorithm and analyze its convergence.

Algorithm 5.3

Step 0. Fix $x_0 \in \mathcal{H}$ and set $n = 0$.

Step 1. Take $\lambda_n \in \left]0, 1\right]$, $\gamma_n \in \left]0, 2\beta\right[$, $a_n \in \mathcal{H}$, and $b_n \in \mathcal{H}$.

Step 2. Set $x_{n+1} = x_n + \lambda_n \Big(J_{\gamma_n F_0}\big(x_n - \gamma_n(Bx_n + b_n)\big) + a_n - x_n \Big)$.

Step 3. Set $n = n + 1$ and go to Step 1.

Theorem 5.4 *Suppose that the following conditions are satisfied:*

(i) $G \neq \varnothing$.

(ii) $\underline{\lim}\, \lambda_n > 0$ *and* $0 < \underline{\lim}\, \gamma_n \leq \overline{\lim}\, \gamma_n < 2\beta$.

(iii) $\sum_{n\in\mathbb{N}} \|a_n\| < +\infty$ *and* $\sum_{n\in\mathbb{N}} \|b_n\| < +\infty$.

Then every orbit of Algorithm 5.3 converges weakly to a point in G.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary orbit of Algorithm 5.3. Set

$$(\forall n \in \mathbb{N}) \;\; T_{1,n} = J_{\gamma_n F_0} \quad \text{and} \quad T_{2,n} = \mathrm{Id} - \gamma_n B. \tag{5.7}$$

Then the update rule at Step 2 of Algorithm 5.3 assumes the form given in (2.12) and, moreover, it follows from (5.3), Lemma 2.12(i)&(ii) and (2.8) that $(T_{1,n})_{n\in\mathbb{N}}$ lies in $\mathcal{A}(\frac{1}{2})$. Likewise, it follows from (5.4) and (2.8) that $\beta B \in \mathcal{A}(\frac{1}{2})$ and hence from [12, Lemma 2.3] that $(\forall n \in \mathbb{N})\; T_{2,n} \in \mathcal{A}(\frac{\gamma_n}{2\beta})$. Thus, Algorithm 5.3 is a special case of Algorithm 2.9 with $\alpha_{1,n} = 1/2$, $\alpha_{2,n} = \gamma_n/(2\beta)$, $e_{1,n} = a_n$, and $e_{2,n} = -\gamma_n b_n$. We shall show that all the conditions of Theorem 2.10 are satisfied. First, Proposition 5.2(i) yields

$$G = \bigcap_{n\in\mathbb{N}} \mathrm{Fix}\,\big(T_{1,n} T_{2,n}\big). \tag{5.8}$$

Hence, item (i) in Theorem 2.10 is implied by (i) above. Moreover, items (ii) and (iv) in Theorem 2.10 are implied by (ii) and (iii) above. Let us now verify item (iii) in Theorem 2.10. To this end, let us fix a suborbit $(x_{k_n})_{n\in\mathbb{N}}$ of Algorithm 5.3, $x \in G$, and set

$$(\forall n \in \mathbb{N})\;\; z_n = J_{\gamma_n F_0}(x_n - \gamma_n B x_n) \;\text{ and }\; u_n = \frac{x_n - z_n}{\gamma_n} - Bx_n. \tag{5.9}$$

In view of (5.7) and (ii), (2.13) holds if

$$\begin{cases} u_n \to -Bx \\ Bx_n \to Bx \\ z_n - x_n \to 0 \\ x_{k_n} \rightharpoonup z \end{cases} \quad \Rightarrow \quad z \in G. \tag{5.10}$$

Since B is continuous and monotone on $\mathcal{H}$, it is maximal monotone [1, Proposition 3.5.7]. Therefore $\operatorname{gr} B$ is sequentially weakly-strongly closed in $\mathcal{H}^2$ [1, Proposition 3.5.6.2], and the conditions $x_{k_n} \rightharpoonup z$ and $Bx_{k_n} \to Bx$ imply $Bx = Bz$. Consequently, the condition $z_n - x_n \to 0$ yields

$$z_{k_n} \rightharpoonup z \quad \text{and} \quad u_{k_n} \to -Bz. \tag{5.11}$$

It therefore follows from (5.9) and Lemma 2.13 that

$$z \in K \quad \text{and} \quad (\forall y \in K)\;\; F_0(z,y) + \langle Bz \mid y - z\rangle \geq 0, \tag{5.12}$$

i.e., $z \in G$. Therefore, the conclusion follows from Theorem 2.10. ☐

Remark 5.5 We point out some connections between Theorem 5.4 and existing results.

- In the special case when $\gamma_n \equiv \gamma$, $\lambda_n \equiv 1$, $a_n \equiv 0$, and $b_n \equiv 0$, Theorem 5.4 follows from [23, Theorem 1].
- As in Lemma 2.15(i), let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator such that $K \subset \operatorname{int}\operatorname{dom} A$ and set $F_0\colon K^2 \to \mathbb{R}\colon (x,y) \mapsto \max_{u\in Ax}\langle u \mid y - x\rangle$. Then

$$\begin{aligned} x \in \mathcal{H} \text{ solves (5.2)} \;\; &\Leftrightarrow \;\; x \in K \text{ and } (\forall y \in K)\;\; \max_{u\in Ax}\langle u + Bx \mid y - x\rangle \geq 0 \\ &\Leftrightarrow \;\; x \in K \text{ and } (\exists\, u \in Ax)(\forall y \in K)\;\; \langle u + Bx \mid y - x\rangle \geq 0 \quad (5.13) \\ &\Leftrightarrow \;\; (\exists\, u \in Ax)\;\; -u - Bx \in N_K x \\ &\Leftrightarrow \;\; 0 \in Ax + Bx + N_K x, \quad (5.14) \end{aligned}$$

  where we have used Lemma 2.14 to get (5.13). Thus, (5.2) coincides with the problem of finding a zero of $A + B + N_K$. In particular, if $K = \mathcal{H}$, then $J_{\gamma_n F_0} = J_{\gamma_n A}$ by Lemma 2.15(i) and therefore Algorithm 5.3 produces the forward-backward iteration

$$x_{n+1} = x_n + \lambda_n\Big(J_{\gamma_n A}\big(x_n - \gamma_n(Bx_n + b_n)\big) + a_n - x_n\Big), \tag{5.15}$$

  for finding a zero of $A + B$. In this context, for $\lambda_n \equiv 1$, $a_n \equiv 0$, and $b_n \equiv 0$, Theorem 5.4 corresponds to [26, Proposition 1(c)].

We now turn to the problem of finding the best approximation to a point $a \in \mathcal{H}$ from G.

Theorem 5.6 *Let $\varepsilon \in]0,1[$ and $\gamma \in]0,2\beta[$, and let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary orbit of Algorithm 2.7 generated with*

$$(\forall n \in \mathbb{N}) \;\; T_n = \operatorname{Id} + \lambda_n\big(J_{\gamma F_0}(\operatorname{Id} - \gamma B) - \operatorname{Id}\big), \;\; \textit{where } \; \varepsilon \le \lambda_n \le 1/2. \tag{5.16}$$

Then:

(i) *If $G \neq \varnothing$, $(x_n)_{n\in\mathbb{N}}$ converges strongly to $P_G a$.*

(ii) *If $G = \varnothing$, $\|x_n\| \to +\infty$.*

Proof. In view of Proposition 5.2(i)&(ii), the assertions follow from [3, Corollary 6.6(ii)]. ☐

6 Inconsistent problems

We now consider problems (1.1) and (1.2) with finitely many sets, say $I = \{0, \ldots, m\}$. In this section we aim at solving these problems when the assumption $S \neq \varnothing$ is relaxed. We shall use the following notion, which was introduced in [23].

Definition 6.1 *The* Yosida approximation *of index $\rho \in]0,+\infty[$ of a bifunction $F\colon K^2 \to \mathbb{R}$ is the set-valued operator ${}^{\rho}F\colon \mathcal{H}^2 \to 2^{\mathbb{R}}\colon (x,y) \mapsto \langle x - J_{\rho F}x \mid y - x\rangle/\rho$.*

In the present context, (1.1) becomes

$$\text{find } \; x \in K \; \text{ such that } \; (\forall i \in \{0,\ldots,m\})(\forall y \in K) \;\; F_i(x,y) \ge 0. \tag{6.1}$$

We recall that its set of solutions is denoted by S, while each F_i satisfies Condition 1.1. Now, let us fix $(\rho_i)_{1\le i\le m}$ in $]0,+\infty[$. When (6.1) has no solution, it can be approximated by the problem

$$\text{find } \; x \in K \; \text{ such that } \; (\forall y \in K) \;\; F_0(x,y) + \sum_{i=1}^{m} {}^{\rho_i}F_i(x,y) \ge 0, \tag{6.2}$$

in which the bifunctions $(F_i)_{1\le i\le m}$ have been replaced by their Yosida approximations, which are single-valued operators by Lemma 2.12(ii). This approximation is justified by item (iii) below. We henceforth denote by G the set of solutions to (6.2).

Proposition 6.2 *Let $\gamma \in \big]0, 2/(\sum_{i=1}^m \rho_i^{-1})\big[$. Then:*

(i) $G = \operatorname{Fix} J_{\gamma F_0}\big(\operatorname{Id} + \gamma \sum_{i=1}^m (J_{\rho_i F_i} - \operatorname{Id})/\rho_i\big)$.

(ii) *G is closed and convex.*

(iii) *Suppose that $S \neq \varnothing$. Then $G = S$.*

Proof. Problem (6.2) is a special case of Problem (5.2)–(5.4) with

$$B = \sum_{i=1}^{m} \frac{\operatorname{Id} - J_{\rho_i F_i}}{\rho_i} \quad \text{and} \quad \beta = \frac{1}{\sum_{i=1}^{m} 1/\rho_i}. \tag{6.3}$$

Indeed, set $(\forall i \in \{1,\ldots,m\})\ \omega_i = \beta/\rho_i$. Then $\sum_{i=1}^{m} \omega_i = 1$. Moreover, it follows from Lemma 2.12(ii) and (2.8) that the operators $(\operatorname{Id} - J_{\rho_i F_i})_{1\leq i\leq m}$ are firmly nonexpansive. Therefore, their convex combination $\beta B = \sum_{i=1}^{m} \omega_i(\operatorname{Id} - J_{\rho_i F_i})$ is also firmly nonexpansive. Hence (5.3) and (5.4) hold and we can apply Proposition 5.2(i)&(iii) to obtain at once (i) and (ii). To show (iii), we first observe that Lemma 2.12(iii) asserts that

$$S = S_{F_0} \cap \bigcap_{i=1}^{m} \operatorname{Fix} J_{\rho_i F_i}, \ \text{ where } \ S_{F_0} = \big\{x \in K \mid (\forall y \in K)\ \ F_0(x,y) \geq 0\big\}. \tag{6.4}$$

Now suppose that $S \neq \varnothing$. Then $\bigcap_{i=1}^{m} \operatorname{Fix} J_{\rho_i F_i} \neq \varnothing$ and it therefore follows from (2.8) and Lemma 2.4(ii) with $\lambda \equiv 1$ that $\operatorname{Fix} \sum_{i=1}^{m} \omega_i J_{\rho_i F_i} = \bigcap_{i=1}^{m} \operatorname{Fix} J_{\rho_i F_i}$. Consequently, it results from (6.3) that

$$(\forall x \in \mathcal{H})\ \ Bx = 0 \quad \Leftrightarrow \quad x \in \operatorname{Fix} \sum_{i=1}^{m} \omega_i J_{\rho_i F_i} = \bigcap_{i=1}^{m} \operatorname{Fix} J_{\rho_i F_i}, \tag{6.5}$$

and we deduce from (6.4) that S coincides with the set $S_{F_0,B}$ introduced in Proposition 5.2. Hence $S_{F_0,B} \neq \varnothing$ and Proposition 5.2(iv) yields $G = S_{F_0,B} = S$. ☐

Specializing Algorithm 5.3 and Theorem 5.4 to (6.3), we obtain the following result.

Algorithm 6.3

Step 0. Fix $x_0 \in \mathcal{H}$ and set $n = 0$.

Step 1. Take $\lambda_n \in \left]0,1\right]$, $\gamma_n \in \big]0, 2/(\sum_{i=1}^{m} \rho_i^{-1})\big[$, $a_n \in \mathcal{H}$, and $(b_{i,n})_{1\leq i\leq m} \in \mathcal{H}^m$.

Step 2. Set $x_{n+1} = x_n + \lambda_n \Big(J_{\gamma_n F_0}\big(x_n + \gamma_n \sum_{i=1}^{m} (J_{\rho_i F_i} x_n + b_{i,n} - x_n)/\rho_i\big) + a_n - x_n \Big)$.

Step 3. Set $n = n + 1$ and go to Step 1.

Corollary 6.4 *Suppose that the following conditions are satisfied.*

(i) $G \neq \varnothing$.

(ii) $\underline{\lim}\, \lambda_n > 0$ *and* $0 < \underline{\lim}\, \gamma_n \leq \overline{\lim}\, \gamma_n < 2/(\sum_{i=1}^{m} \rho_i^{-1})$.

(iii) $\sum_{n\in\mathbb{N}} \|a_n\| < +\infty$ *and* $\max_{1\leq i\leq m} \sum_{n\in\mathbb{N}} \|b_{i,n}\| < +\infty$.

Then every orbit of Algorithm 6.3 converges weakly to a point in G.

Likewise, a direct reformulation of Theorem 5.6 for the scenario (6.3) yields a sequence that converges strongly to the best approximation to a point $a \in \mathcal{H}$ from the solutions to (6.2).

Remark 6.5 Let $(S_i)_{1\leq i\leq m}$ be a family of closed convex sets in $\mathcal{H}$ with associated projection operators $(P_i)_{1\leq i\leq m}$, and let $(\rho_i)_{1\leq i\leq m}$ be numbers in $[1, +\infty[$ such that $\sum_{i=1}^m 1/\rho_i = 1$. Now set

$$K = \mathcal{H},\ F_0 = 0,\ \text{ and } (\forall i \in \{1,\ldots,m\})(\forall (x,y) \in \mathcal{H}^2)\ \ F_i(x,y) = \big(d^2_{S_i}(y) - d^2_{S_i}(x)\big)/2. \tag{6.6}$$

Then (6.1) corresponds to the basic convex feasibility problem of finding a point in $S = \bigcap_{i=1}^m S_i$ and the bifunctions $(F_i)_{1\leq i\leq m}$ satisfy Condition 1.1 by Lemma 2.15(v). Now let $(\forall i \in \{1,\ldots,m\})$ $f_i = \rho_i d^2_{S_i}/2 = {}^{1/\rho_i}\iota_{S_i}$. Then it follows from Lemma 2.15(v) and Lemma 2.2 that

$$(\forall i \in \{1,\ldots,m\})\ \ J_{\rho_i F_i} = \operatorname{prox}_{f_i} = \operatorname{Id} + \frac{\rho_i}{\rho_i + 1}(P_i - \operatorname{Id}). \tag{6.7}$$

Thus, we deduce from Proposition 6.2(i) that the set of solutions to (6.2) is $G = \operatorname{Fix} \sum_{i=1}^m J_{\rho_i F_i}/\rho_i$. Now set $(\forall i \in \{1,\ldots,m\})$ $\omega_i = 1/(\rho_i + 1)$. Then $x \in G \Leftrightarrow \sum_{i=1}^m \omega_i(x - P_i x) = 0 \Leftrightarrow \nabla \sum_{i=1}^m \omega_i d^2_{S_i}(x) = 0$. Hence, $G = \operatorname{Argmin} \sum_{i=1}^m \omega_i d^2_{S_i}$. On the other hand, Corollary 6.4 with $\lambda_n \equiv 1$, $a_n \equiv 0$, and $b_{i,n} \equiv 0$ asserts that, if $G \neq \varnothing$ and if $(\gamma_n)_{n\in\mathbb{N}} \in\ ]0,2[^{\mathbb{N}}$ satisfies $0 < \underline{\lim}\, \gamma_n \leq \overline{\lim}\, \gamma_n < 2$, every sequence $(x_n)_{n\in\mathbb{N}}$ generated by the iteration

$$x_0 \in \mathcal{H} \quad \text{and} \quad (\forall n \in \mathbb{N})\ \ x_{n+1} = x_n + \gamma_n \sum_{i=1}^m \omega_i (P_i x_n - x_n) \tag{6.8}$$

converges weakly to a point in G. This framework was considered in [9] to deal with inconsistent signal feasibility problems.

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