Chapter 9
Monotone operator methods for Nash equilibria in non-potential games

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Abstract We observe that a significant class of Nash equilibrium problems in non-potential games can be associated with monotone inclusion problems. We propose splitting techniques to solve such problems and establish their convergence. Applications to generalized Nash equilibria, zero-sum games, and cyclic proximation problems are demonstrated.

Key words: monotone operator, Nash equilibrium, potential game, proximal algorithm, splitting method, zero-sum game.

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9.1 Problem statement

Consider a game with $m \geq 2$ players indexed by $i \in \{1, \dots, m\}$. The strategy x_i of the ith player lies in a real Hilbert space $\mathcal{H}_i$ and the problem is to find $x_1 \in \mathcal{H}_1, \dots, x_m \in \mathcal{H}_m$ such that

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$$(\forall i \in \{1,\ldots,m\})\quad x_i \in \underset{x\in\mathcal{H}_i}{\operatorname{Argmin}}\, \boldsymbol{f}(x_1,\ldots,x_{i-1},x,x_{i+1},\ldots,x_m) + \boldsymbol{g}_i(x_1,\ldots,x_{i-1},x,x_{i+1},\ldots,x_m), \tag{9.1}$$

where $(\boldsymbol{g}_i)_{1\leq i\leq m}$ represents the individual penalty of player i depending on the strategies of all players and $\boldsymbol{f}$ is a convex penalty which is common to all players and models the collective discomfort of the group. At this level of generality, no reliable method exists for solving (9.1) and some hypotheses are required. In this paper we focus on the following setting.

Problem 9.1.1 Let $m \geq 2$ be an integer and let $\boldsymbol{f}\colon \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m \to \left]-\infty,+\infty\right]$ be a proper lower semicontinuous convex function. For every $i \in \{1,\ldots,m\}$, let $\boldsymbol{g}_i\colon \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m \to \left]-\infty,+\infty\right]$ be such that, for every $x_1 \in \mathcal{H}_1,\ldots,x_m \in \mathcal{H}_m$, the function $x \mapsto \boldsymbol{g}_i(x_1,\ldots,x_{i-1},x,x_{i+1},\ldots,x_m)$ is convex and differentiable on $\mathcal{H}_i$, and denote by $\nabla_i\, \boldsymbol{g}_i(x_1,\ldots,x_m)$ its derivative at x_i. Moreover,

$$\big(\forall (x_1,\ldots,x_m) \in \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m\big)\big(\forall (y_1,\ldots,y_m) \in \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m\big)\quad \sum_{i=1}^m \langle \nabla_i\, \boldsymbol{g}_i(x_1,\ldots,x_m) - \nabla_i\, \boldsymbol{g}_i(y_1,\ldots,y_m) \mid x_i - y_i\rangle \geq 0. \tag{9.2}$$

The problem is to find $x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m$ such that

$$\begin{cases} x_1 \in \underset{x\in\mathcal{H}_1}{\operatorname{Argmin}}\;\; \boldsymbol{f}(x,x_2,\ldots,x_m) + \boldsymbol{g}_1(x,x_2,\ldots,x_m)\\ \quad\vdots\\ x_m \in \underset{x\in\mathcal{H}_m}{\operatorname{Argmin}}\;\; \boldsymbol{f}(x_1,\ldots,x_{m-1},x) + \boldsymbol{g}_m(x_1,\ldots,x_{m-1},x). \end{cases} \tag{9.3}$$

In the special case when, for every $i \in \{1,\ldots,m\}$, $\boldsymbol{g}_i = \boldsymbol{g}$ is convex, Problem 9.1.1 amounts to finding a Nash equilibrium of a potential game, i.e., a game in which the penalty of every player can be represented by a common potential $\boldsymbol{f}+\boldsymbol{g}$ [14]. Hence, Nash equilibria can be found by solving

$$\underset{x_1\in\mathcal{H}_1,\ldots,x_m\in\mathcal{H}_m}{\operatorname{minimize}}\;\; \boldsymbol{f}(x_1,\ldots,x_m) + \boldsymbol{g}(x_1,\ldots,x_m). \tag{9.4}$$

Thus, the problem reduces to the minimization of the sum of two convex functions on the Hilbert space $\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_m$ and various methods are available to tackle it under suitable assumptions (see for instance [5, Chapter 27]). On the other hand, in the particular case when $\boldsymbol{f}$ is separable, a review of methods for solving (9.3) is provided in [8]. In this paper we address the more challenging non-potential setting, in which the functions $(\boldsymbol{g}_i)_{1\leq i\leq m}$ need not be identical nor convex, but they must satisfy (9.2) and $\boldsymbol{f}$ need not be separable. Let us note that (9.2) actually implies, for every $i \in \{1, \dots, m\}$, the convexity of $\boldsymbol{g}_i$ with respect to its ith variable.

Our methodology consists of using monotone operator splitting techniques for solving an auxiliary monotone inclusion, the solutions of which are Nash equilibria of Problem 9.1.1. In Section 9.2 we review the notation and background material needed subsequently. In Section 9.3 we introduce the auxiliary monotone inclusion problem and provide conditions ensuring the existence of solutions to the auxiliary problem. We also propose two methods for solving Problem 9.1.1 and establish their convergence. Finally, in Section 9.4 the proposed methods are applied to the construction of generalized Nash equilibria, to zero-sum games, and to cyclic proximation problems.

9.2 Notation and background

Throughout this paper, $\mathcal{H}$, $\mathcal{G}$, and $(\mathcal{H}_i)_{1 \leq i \leq m}$ are real Hilbert spaces. For convenience, their scalar products are all denoted by $\langle \cdot \mid \cdot \rangle$ and the associated norms by $\| \cdot \|$. Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued operator. The domain of A is

$$\operatorname{dom} A = \big\{ x \in \mathcal{H} \;\big|\; Ax \neq \varnothing \big\}, \tag{9.5}$$

the set of zeros of A is

$$\operatorname{zer} A = \big\{ x \in \mathcal{H} \;\big|\; 0 \in Ax \big\}, \tag{9.6}$$

the graph of A is

$$\operatorname{gra} A = \big\{ (x,u) \in \mathcal{H} \times \mathcal{H} \;\big|\; u \in Ax \big\}, \tag{9.7}$$

the range of A is

$$\operatorname{ran} A = \big\{ u \in \mathcal{H} \;\big|\; (\exists\, x \in \mathcal{H})\; u \in Ax \big\}, \tag{9.8}$$

the inverse of A is the set-valued operator

$$A^{-1}\colon \mathcal{H} \to 2^{\mathcal{H}}\colon u \mapsto \big\{ x \in \mathcal{H} \;\big|\; u \in Ax \big\}, \tag{9.9}$$

and the resolvent of A is

$$J_A = (\mathrm{Id} + A)^{-1}. \tag{9.10}$$

In addition, A is monotone if

$$(\forall (x,y) \in \mathcal{H} \times \mathcal{H})(\forall (u,v) \in Ax \times Ay) \quad \langle x - y \mid u - v \rangle \geq 0 \tag{9.11}$$

and it is maximally monotone if, furthermore, every monotone operator $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ such that $\operatorname{gra} A \subset \operatorname{gra} B$ coincides with A.

We denote by $\Gamma_0(\mathcal{H})$ the class of lower semicontinuous convex functions $\varphi\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ which are proper in the sense that $\operatorname{dom}\varphi = \big\{x \in \mathcal{H} \;\big|\; \varphi(x) < +\infty\big\} \neq \varnothing$. Let $\varphi \in \Gamma_0(\mathcal{H})$. The proximity operator of φ is

$$\operatorname{prox}_\varphi\colon \mathcal{H} \to \mathcal{H}\colon x \mapsto \underset{y\in\mathcal{H}}{\operatorname{argmin}}\; \varphi(y) + \frac{1}{2}\|x-y\|^2, \tag{9.12}$$

and the subdifferential of φ is the maximally monotone operator

$$\partial\varphi\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{u \in \mathcal{H} \;\big|\; (\forall y \in \mathcal{H})\;\; \langle y-x \mid u\rangle + \varphi(x) \leq \varphi(y)\big\}. \tag{9.13}$$

We have

$$\underset{x\in\mathcal{H}}{\operatorname{Argmin}}\; \varphi(x) = \operatorname{zer}\partial\varphi \quad \text{and} \quad \operatorname{prox}_\varphi = J_{\partial\varphi}. \tag{9.14}$$

Let $\beta \in \left]0,+\infty\right[$. An operator $T\colon \mathcal{H} \to \mathcal{H}$ is β-cocoercive (or βT is firmly nonexpansive) if

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \langle x-y \mid Tx-Ty\rangle \geq \beta\|Tx-Ty\|^2, \tag{9.15}$$

which implies that it is monotone and β^{-1}–Lipschitzian. Let C be a nonempty convex subset of $\mathcal{H}$. The indicator function of C is

$$\iota_C\colon \mathcal{H} \to \left]-\infty,+\infty\right]\colon x \mapsto \begin{cases} 0, & \text{if } x \in C;\\ +\infty, & \text{if } x \notin C \end{cases} \tag{9.16}$$

and $\partial\iota_C = N_C$ is the normal cone operator of C, i.e.,

$$N_C\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \begin{cases} \big\{u \in \mathcal{H} \;\big|\; (\forall y \in C)\;\; \langle y-x \mid u\rangle \leq 0\big\}, & \text{if } x \in C;\\ \varnothing, & \text{otherwise.} \end{cases} \tag{9.17}$$

If C is closed, for every $x \in \mathcal{H}$, there exists a unique point $P_C x \in C$ such that $\|x - P_C x\| = \inf_{y\in C}\|x-y\|$; $P_C x$ is called the projection of x onto C and we have $P_C = \operatorname{prox}_{\iota_C}$. In addition, the symbols $\rightharpoonup$ and $\to$ denote respectively weak and strong convergence. For a detailed account of the tools described above, see [5].

9.3 Model, algorithms, and convergence

We investigate an auxiliary monotone inclusion problem the solutions of which are Nash equilibria of Problem 9.1.1 and propose two splitting methods to solve it. Both involve the proximity operator $\operatorname{prox}_{\boldsymbol{f}}$, which can be computed explicitly in several instances [5, 7]. We henceforth denote by $\boldsymbol{\mathcal{H}}$ the direct sum of the Hilbert spaces $(\mathcal{H}_i)_{1\leq i\leq m}$, i.e., the product space $\mathcal{H}_1 \times \cdots \times \mathcal{H}_m$ equipped with the scalar product

$$\langle\langle \cdot \mid \cdot \rangle\rangle \colon \big((x_i)_{1\leq i\leq m}, (y_i)_{1\leq i\leq m}\big) \mapsto \sum_{i=1}^{m} \langle x_i \mid y_i \rangle. \tag{9.18}$$

We denote the associated norm by $|||\cdot|||$, a generic element of $\boldsymbol{\mathcal{H}}$ by $\boldsymbol{x} = (x_i)_{1\leq i\leq m}$, and the identity operator on $\boldsymbol{\mathcal{H}}$ by $\mathbf{Id}$.

9.3.1 A monotone inclusion model

With the notation and hypotheses of Problem 9.1.1, let us set

$$\boldsymbol{A} = \partial \boldsymbol{f} \quad \text{and} \quad \boldsymbol{B} \colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}} \colon \boldsymbol{x} \mapsto \big(\nabla_1 \boldsymbol{g}_1(\boldsymbol{x}), \ldots, \nabla_m \boldsymbol{g}_m(\boldsymbol{x})\big). \tag{9.19}$$

We consider the inclusion problem

$$\text{find} \quad \boldsymbol{x} \in \operatorname{zer}(\boldsymbol{A} + \boldsymbol{B}). \tag{9.20}$$

Since $\boldsymbol{f} \in \Gamma_0(\boldsymbol{\mathcal{H}})$, $\boldsymbol{A}$ is maximally monotone. On the other hand, it follows from (9.2) that $\boldsymbol{B}$ is monotone. The following result establishes a connection between the monotone inclusion problem (9.20) and Problem 9.1.1.

Proposition 9.3.1 *Using the notation and hypotheses of Problem 9.1.1, let $\boldsymbol{A}$ and $\boldsymbol{B}$ be as in (9.19). Then every point in* $\operatorname{zer}(\boldsymbol{A} + \boldsymbol{B})$ *is a solution to Problem 9.1.1.*

Proof. Suppose that $\operatorname{zer}(\boldsymbol{A} + \boldsymbol{B}) \neq \varnothing$ and let $(x_1, \ldots, x_m) \in \boldsymbol{\mathcal{H}}$. Then [5, Proposition 16.6] asserts that

$$\boldsymbol{A}(x_1, \ldots, x_m) \subset \partial\big(\boldsymbol{f}(\cdot, x_2, \ldots, x_m)\big)(x_1) \times \cdots \times \partial\big(\boldsymbol{f}(x_1, \ldots, x_{m-1}, \cdot)\big)(x_m). \tag{9.21}$$

Hence, since $\operatorname{dom} \boldsymbol{g}_1(\cdot, x_2, \ldots, x_m) = \mathcal{H}_1, \ \ldots, \ \operatorname{dom} \boldsymbol{g}_m(x_1, \ldots, x_{m-1}, \cdot) = \mathcal{H}_m$, we derive from (9.19), (9.14), and [5, Corollary 16.38(iii)] that

$$\begin{aligned}
&(x_1, \ldots, x_m) \in \operatorname{zer}(\boldsymbol{A} + \boldsymbol{B}) \\
&\qquad \Leftrightarrow \quad -\boldsymbol{B}(x_1, \ldots, x_m) \in \boldsymbol{A}(x_1, \ldots, x_m) \\
&\qquad \Rightarrow \quad \begin{cases} -\nabla_1 \boldsymbol{g}_1(x_1, \ldots, x_m) \in \partial\big(\boldsymbol{f}(\cdot, x_2, \ldots, x_m)\big)(x_1) \\ \qquad\qquad \vdots \\ -\nabla_m \boldsymbol{g}_m(x_1, \ldots, x_m) \in \partial\big(\boldsymbol{f}(x_1, \ldots, x_{m-1}, \cdot)\big)(x_m) \end{cases} \\
&\qquad \Leftrightarrow \quad (x_1, \ldots, x_m) \ \text{ solves Problem 9.1.1,}
\end{aligned} \tag{9.22}$$

which yields the result.

Proposition 9.3.1 asserts that we can solve Problem 9.1.1 by solving (9.20), provided that the latter has solutions. The following result provides instances in which this property is satisfied. First, we need the following definitions (see [5, Chapters 21–24]).

Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be monotone. Then A is 3^* monotone if $\operatorname{dom} A \times \operatorname{ran} A \subset \operatorname{dom} F_A$, where

$$F_A\colon \mathcal{H} \times \mathcal{H} \to \left]-\infty,+\infty\right] : (x,u) \mapsto \langle x \mid u\rangle - \inf_{(y,v)\in \operatorname{gra} A} \langle x-y \mid u-v\rangle. \tag{9.23}$$

On the other hand, A is uniformly monotone if there exists an increasing function $\phi\colon \left[0,+\infty\right[\to [0,+\infty]$ vanishing only at 0 such that

$$\big(\forall (x,y) \in \mathcal{H}\times\mathcal{H}\big)\big(\forall (u,v) \in Ax\times Ay\big) \quad \langle x-y \mid u-v\rangle \geq \phi(\|x-y\|). \tag{9.24}$$

A function $\varphi \in \Gamma_0(\mathcal{H})$ is uniformly convex if there exists an increasing function $\phi\colon \left[0,+\infty\right[\to [0,+\infty]$ vanishing only at 0 such that

$$\begin{aligned}&(\forall (x,y) \in \operatorname{dom}\varphi \times \operatorname{dom}\varphi)(\forall \alpha \in \left]0,1\right[)\\ &\quad \varphi(\alpha x + (1-\alpha)y) + \alpha(1-\alpha)\phi(\|x-y\|) \leq \alpha\varphi(x) + (1-\alpha)\varphi(y).\end{aligned} \tag{9.25}$$

The function ϕ in (9.24) and (9.25) is called the modulus of uniform monotonicity and of uniform convexity, respectively, and it is said to be supercoercive if $\lim_{t\to+\infty} \phi(t)/t = +\infty$.

Proposition 9.3.2 *With the notation and hypotheses of Problem 9.1.1, let $\boldsymbol{B}$ be as in (9.19). Suppose that $\boldsymbol{B}$ is maximally monotone and that one of the following holds.*

(i) $\lim_{|||\boldsymbol{x}|||\to+\infty} \inf |||\partial \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{B}\boldsymbol{x}||| = +\infty$.
(ii) $\partial\boldsymbol{f} + \boldsymbol{B}$ *is uniformly monotone with a supercoercive modulus.*
(iii) $(\operatorname{dom}\partial\boldsymbol{f}) \cap \operatorname{dom}\boldsymbol{B}$ *is bounded.*
(iv) $\boldsymbol{f} = \iota_{\boldsymbol{C}}$, *where $\boldsymbol{C}$ is a nonempty closed convex bounded subset of $\boldsymbol{\mathcal{H}}$.*
(v) $\boldsymbol{f}$ *is uniformly convex with a supercoercive modulus.*
(vi) $\boldsymbol{B}$ *is 3^* monotone, and $\partial\boldsymbol{f}$ or $\boldsymbol{B}$ is surjective.*
(vii) $\boldsymbol{B}$ *is uniformly monotone with a supercoercive modulus.*
(viii) $\boldsymbol{B}$ *is linear and bounded, there exists $\beta \in \left]0,+\infty\right[$ such that $\boldsymbol{B}$ is β–cocoercive, and $\partial\boldsymbol{f}$ or $\boldsymbol{B}$ is surjective.*

Then $\operatorname{zer}(\partial\boldsymbol{f} + \boldsymbol{B}) \neq \varnothing$. *In addition, if (ii), (v), or (vii) holds,* $\operatorname{zer}(\partial\boldsymbol{f} + \boldsymbol{B})$ *is a singleton.*

Proof. First note that, for every $(x_i)_{1\leq i\leq m} \in \boldsymbol{\mathcal{H}}$, $\operatorname{dom}\nabla_1 \boldsymbol{g}_1(\cdot, x_2,\ldots,x_m) = \mathcal{H}_1, \ldots, \operatorname{dom}\nabla_m \boldsymbol{g}_m(x_1,\ldots,x_{m-1},\cdot) = \mathcal{H}_m$. Hence, it follows from (9.19) that $\operatorname{dom}\boldsymbol{B} = \boldsymbol{\mathcal{H}}$ and, therefore, from [5, Corollary 24.4(i)] that $\partial\boldsymbol{f} + \boldsymbol{B}$ is maximally monotone. In addition, it follows from [5, Example 24.9] that $\partial\boldsymbol{f}$ is 3^* monotone.

(i): This follows from [5, Corollary 21.20].

(ii): This follows from [5, Corollary 23.37(i)].

(iii): Since $\operatorname{dom}(\partial \boldsymbol{f}+\boldsymbol{B})=(\operatorname{dom}\partial\boldsymbol{f})\cap\operatorname{dom}\boldsymbol{B}$, the result follows from [5, Proposition 23.36(iii)].

(iv)⇒(iii): $\boldsymbol{f}=\iota_{\boldsymbol{C}}\in\Gamma_0(\boldsymbol{\mathcal{H}})$ and $\operatorname{dom}\partial\boldsymbol{f}=\boldsymbol{C}$ is bounded.

(v)⇒(ii): It follows from (9.19) and [5, Example 22.3(iii)] that $\partial\boldsymbol{f}$ is uniformly monotone. Hence, $\partial\boldsymbol{f}+\boldsymbol{B}$ is uniformly monotone.

(vi): This follows from [5, Corollary 24.22(ii)].

(vii)⇒(ii): Clear.

(viii)⇒(vi): This follows from [5, Proposition 24.12].

Finally, the uniqueness of a zero of $\partial\boldsymbol{f}+\boldsymbol{B}$ in cases (ii), (v), and (vii) follows from the strict monotonicity of $\partial\boldsymbol{f}+\boldsymbol{B}$.

9.3.2 Forward-backward-forward algorithm

Our first method for solving Problem 9.1.1 is derived from an algorithm proposed in [6], which itself is a variant of a method proposed in [16].

Theorem 9.3.3 *In Problem 9.1.1, suppose that there exist $(z_1,\ldots,z_m)\in\boldsymbol{\mathcal{H}}$ such that*

$$-\big(\nabla_1\,\boldsymbol{g}_1(z_1,\ldots,z_m),\ldots,\nabla_m\,\boldsymbol{g}_m(z_1,\ldots,z_m)\big)\in\partial\boldsymbol{f}(z_1,\ldots,z_m) \tag{9.26}$$

and $\chi\in\,]0,+\infty[$ such that

$$(\forall(x_1,\ldots,x_m)\in\boldsymbol{\mathcal{H}})(\forall(y_1,\ldots,y_m)\in\boldsymbol{\mathcal{H}})\quad \sum_{i=1}^m\|\nabla_i\,\boldsymbol{g}_i(x_1,\ldots,x_m)-\nabla_i\,\boldsymbol{g}_i(y_1,\ldots,y_m)\|^2\leq\chi^2\sum_{i=1}^m\|x_i-y_i\|^2. \tag{9.27}$$

Let $\varepsilon\in\,]0,1/(\chi+1)[$ and let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon,(1-\varepsilon)/\chi]$. Moreover, for every $i\in\{1,\ldots,m\}$, let $x_{i,0}\in\mathcal{H}_i$, and let $(a_{i,n})_{n\in\mathbb{N}}$, $(b_{i,n})_{n\in\mathbb{N}}$, and $(c_{i,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{H}_i$. Now consider the following routine.

$$(\forall n\in\mathbb{N})\quad\left\lfloor\begin{array}{l}\text{For } i=1,\ldots,m\\ \lfloor\; y_{i,n}=x_{i,n}-\gamma_n(\nabla_i\,\boldsymbol{g}_i(x_{1,n},\ldots,x_{m,n})+a_{i,n})\\ (p_{1,n},\ldots,p_{m,n})=\operatorname{prox}_{\gamma_n\boldsymbol{f}}(y_{1,n},\ldots,y_{m,n})+(b_{1,n},\ldots,b_{m,n})\\ \text{For } i=1,\ldots,m\\ \left\lfloor\begin{array}{l} q_{i,n}=p_{i,n}-\gamma_n(\nabla_i\,\boldsymbol{g}_i(p_{1,n},\ldots,p_{m,n})+c_{i,n})\\ x_{i,n+1}=x_{i,n}-y_{i,n}+q_{i,n}.\end{array}\right.\end{array}\right. \tag{9.28}$$

Then there exists a solution $(\overline{x}_1,\ldots,\overline{x}_m)$ to Problem 9.1.1 such that, for every $i\in\{1,\ldots,m\}$, $x_{i,n}\rightharpoonup\overline{x}_i$ and $p_{i,n}\rightharpoonup\overline{x}_i$.

Proof. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be defined as (9.19). Then (9.26) yields $\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B}) \neq \varnothing$, and for every $\gamma \in \left]0,+\infty\right[$ (9.14) yields $J_{\gamma \boldsymbol{A}} = \operatorname{prox}_{\gamma \boldsymbol{f}}$. In addition, we deduce from (9.2) and (9.27) that $\boldsymbol{B}$ is monotone and χ–Lipschitzian. Now set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \boldsymbol{x}_n = (x_{1,n}, \ldots, x_{m,n}) \\ \boldsymbol{y}_n = (y_{1,n}, \ldots, y_{m,n}) \\ \boldsymbol{p}_n = (p_{1,n}, \ldots, p_{m,n}) \\ \boldsymbol{q}_n = (q_{1,n}, \ldots, q_{m,n}) \end{cases} \tag{9.29}$$

and

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \boldsymbol{a}_n = (a_{1,n}, \ldots, a_{m,n}) \\ \boldsymbol{b}_n = (b_{1,n}, \ldots, b_{m,n}) \\ \boldsymbol{c}_n = (c_{1,n}, \ldots, c_{m,n}). \end{cases} \tag{9.30}$$

Then (9.28) is equivalent to

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \boldsymbol{y}_n = \boldsymbol{x}_n - \gamma_n(\boldsymbol{B}\boldsymbol{x}_n + \boldsymbol{a}_n) \\ \boldsymbol{p}_n = J_{\gamma_n \boldsymbol{A}} \boldsymbol{y}_n + \boldsymbol{b}_n \\ \boldsymbol{q}_n = \boldsymbol{p}_n - \gamma_n(\boldsymbol{B}\boldsymbol{p}_n + \boldsymbol{c}_n) \\ \boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \boldsymbol{y}_n + \boldsymbol{q}_n. \end{array} \right. \tag{9.31}$$

Thus, the result follows from [6, Theorem 2.5(ii)] and Proposition 9.3.1.

Note that two (forward) gradient steps involving the individual penalties $(\boldsymbol{g}_i)_{1 \leq i \leq m}$ and one (backward) proximal step involving the common penalty $\boldsymbol{f}$ are required at each iteration of (9.28).

9.3.3 Forward-backward algorithm

Our second method for solving Problem 9.1.1 is somewhat simpler than (9.28) but requires stronger hypotheses on $(\boldsymbol{g}_i)_{1 \leq i \leq m}$. This method is an application of the forward-backward splitting algorithm (see [3, 9] and the references therein for background).

Theorem 9.3.4 *In Problem 9.1.1, suppose that there exist* $(z_1, \ldots, z_m) \in \boldsymbol{\mathcal{H}}$ *such that*

$$-\big(\nabla_1 \boldsymbol{g}_1(z_1, \ldots, z_m), \ldots, \nabla_m \boldsymbol{g}_m(z_1, \ldots, z_m)\big) \in \partial \boldsymbol{f}(z_1, \ldots, z_m) \tag{9.32}$$

and $\chi \in \left]0,+\infty\right[$ *such that*

$$(\forall(x_1,\ldots,x_m)\in\boldsymbol{\mathcal{H}})(\forall(y_1,\ldots,y_m)\in\boldsymbol{\mathcal{H}})$$
$$\sum_{i=1}^m \langle\nabla_i\,\boldsymbol{g}_i(x_1,\ldots,x_m)-\nabla_i\,\boldsymbol{g}_i(y_1,\ldots,y_m)\mid x_i-y_i\rangle$$
$$\geq\frac{1}{\chi}\sum_{i=1}^m\|\nabla_i\,\boldsymbol{g}_i(x_1,\ldots,x_m)-\nabla_i\,\boldsymbol{g}_i(y_1,\ldots,y_m)\|^2. \quad (9.33)$$

Let $\varepsilon\in\,]0,2/(\chi+1)[$ and let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon,(2-\varepsilon)/\chi]$. Moreover, for every $i\in\{1,\ldots,m\}$, let $x_{i,0}\in\mathcal{H}_i$, and let $(a_{i,n})_{n\in\mathbb{N}}$ and $(b_{i,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{H}_i$. Now consider the following routine.

$$(\forall n\in\mathbb{N})\quad\left\lfloor\begin{array}{l}\textit{For } i=1,\ldots,m\\ \lfloor\; y_{i,n}=x_{i,n}-\gamma_n(\nabla_i\,\boldsymbol{g}_i(x_{1,n},\ldots,x_{m,n})+a_{i,n})\\ (x_{1,n+1},\ldots,x_{m,n+1})=\operatorname{prox}_{\gamma_n\boldsymbol{f}}(y_{1,n},\ldots,y_{m,n})+(b_{1,n},\ldots,b_{m,n}).\end{array}\right. \quad (9.34)$$

Then there exists a solution $(\overline{x}_1,\ldots,\overline{x}_m)$ to Problem 9.1.1 such that, for every $i\in\{1,\ldots,m\}$, $x_{i,n}\rightharpoonup\overline{x}_i$ and $\nabla_i\,\boldsymbol{g}_i(x_{1,n},\ldots,x_{m,n})\to\nabla_i\,\boldsymbol{g}_i(\overline{x}_1,\ldots,\overline{x}_m)$.

Proof. If we define $\boldsymbol{A}$ and $\boldsymbol{B}$ as in (9.19), (9.32) is equivalent to $\operatorname{zer}(\boldsymbol{A}+\boldsymbol{B})\neq\varnothing$, and it follows from (9.33) that $\boldsymbol{B}$ is χ^{-1}–cocoercive. Moreover, (9.34) can be recast as

$$(\forall n\in\mathbb{N})\quad\left\lfloor\begin{array}{l}\boldsymbol{y}_n=\boldsymbol{x}_n-\gamma_n(\boldsymbol{B}\boldsymbol{x}_n+\boldsymbol{a}_n)\\ \boldsymbol{x}_{n+1}=J_{\gamma_n\boldsymbol{A}}\boldsymbol{y}_n+\boldsymbol{b}_n.\end{array}\right. \quad (9.35)$$

The result hence follows from Proposition 9.3.1 and [3, Theorem 2.8(i)&(ii)].

As illustrated in the following example, Theorem 9.3.4 imposes more restrictions on $(\boldsymbol{g}_i)_{1\leq i\leq m}$. However, unlike the forward-backward-forward algorithm used in Section 9.3.2, it employs only one forward step at each iteration. In addition, this method allows for larger gradient steps since the sequence $(\gamma_n)_{n\in\mathbb{N}}$ lies in $]0,2/\chi[$, as opposed to $]0,1/\chi[$ in Theorem 9.3.3.

Example 9.3.5 In Problem 9.1.1, set $m=2$, let $L\colon\mathcal{H}_1\to\mathcal{H}_2$ be linear and bounded, and set

$$\begin{cases}\boldsymbol{g}_1\colon(x_1,x_2)\mapsto\langle Lx_1\mid x_2\rangle\\ \boldsymbol{g}_2\colon(x_1,x_2)\mapsto-\langle Lx_1\mid x_2\rangle.\end{cases} \quad (9.36)$$

It is readily checked that all the assumptions of Problem 9.1.1 are satisfied, as well as (9.27) with $\chi=\|L\|$. However, (9.33) does not hold since

$$(\forall(x_1,x_2)\in\mathcal{H}_1\oplus\mathcal{H}_2)(\forall(y_1,y_2)\in\mathcal{H}_1\oplus\mathcal{H}_2)$$
$$\langle\nabla_1\,\boldsymbol{g}_1(x_1,x_2)-\nabla_1\,\boldsymbol{g}_1(y_1,y_2)\mid x_1-y_1\rangle$$
$$+\langle\nabla_2\,\boldsymbol{g}_2(x_1,x_2)-\nabla_2\,\boldsymbol{g}_2(y_1,y_2)\mid x_2-y_2\rangle=0. \quad (9.37)$$

9.4 Applications

The previous results can be used to solve a wide variety of instances of Problem 9.1.1. We discuss several examples.

9.4.1 Saddle functions and zero-sum games

We consider an instance of Problem 9.1.1 with $m = 2$ players whose individual penalties $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are saddle functions.

Example 9.4.1 Let $\chi \in \left]0, +\infty\right[$, let $\boldsymbol{f} \in \Gamma_0(\mathcal{H}_1 \oplus \mathcal{H}_2)$, and let $\boldsymbol{\mathcal{L}} \colon \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathbb{R}$ be a differentiable function with a χ–Lipschitzian gradient such that, for every $x_1 \in \mathcal{H}_1$, $\boldsymbol{\mathcal{L}}(x_1, \cdot)$ is concave and, for every $x_2 \in \mathcal{H}_2$, $\boldsymbol{\mathcal{L}}(\cdot, x_2)$ is convex. The problem is to find $x_1 \in \mathcal{H}_1$ and $x_2 \in \mathcal{H}_2$ such that

$$\begin{cases} x_1 \in \underset{x \in \mathcal{H}_1}{\operatorname{Argmin}} \;\; \boldsymbol{f}(x, x_2) + \boldsymbol{\mathcal{L}}(x, x_2) \\ x_2 \in \underset{x \in \mathcal{H}_2}{\operatorname{Argmin}} \;\; \boldsymbol{f}(x_1, x) - \boldsymbol{\mathcal{L}}(x_1, x). \end{cases} \tag{9.38}$$

Proposition 9.4.2 *In Example 9.4.1, suppose that there exists* $(z_1, z_2) \in \mathcal{H}_1 \oplus \mathcal{H}_2$ *such that*

$$\big(-\nabla_1 \boldsymbol{\mathcal{L}}(z_1, z_2), \nabla_2 \boldsymbol{\mathcal{L}}(z_1, z_2)\big) \in \partial \boldsymbol{f}(z_1, z_2). \tag{9.39}$$

Let $\varepsilon \in \left]0, 1/(\chi+1)\right[$ *and let* $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence in* $[\varepsilon, (1-\varepsilon)/\chi]$. *Moreover, let* $(x_{1,0}, x_{2,0}) \in \mathcal{H}_1 \oplus \mathcal{H}_2$, *let* $(a_{1,n})_{n\in\mathbb{N}}$, $(b_{1,n})_{n\in\mathbb{N}}$, *and* $(c_{1,n})_{n\in\mathbb{N}}$ *be absolutely summable sequences in* $\mathcal{H}_1$, *and let* $(a_{2,n})_{n\in\mathbb{N}}$, $(b_{2,n})_{n\in\mathbb{N}}$, *and* $(c_{2,n})_{n\in\mathbb{N}}$ *be absolutely summable sequences in* $\mathcal{H}_2$. *Now consider the following routine.*

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_{1,n} = x_{1,n} - \gamma_n(\nabla_1 \boldsymbol{\mathcal{L}}(x_{1,n}, x_{2,n}) + a_{1,n}) \\ y_{2,n} = x_{2,n} + \gamma_n(\nabla_2 \boldsymbol{\mathcal{L}}(x_{1,n}, x_{2,n}) + a_{2,n}) \\ (p_{1,n}, p_{2,n}) = \operatorname{prox}_{\gamma_n \boldsymbol{f}}(y_{1,n}, y_{2,n}) + (b_{1,n}, b_{2,n}) \\ q_{1,n} = p_{1,n} - \gamma_n(\nabla_1 \boldsymbol{\mathcal{L}}(p_{1,n}, p_{2,n}) + c_{1,n}) \\ q_{2,n} = p_{2,n} + \gamma_n(\nabla_2 \boldsymbol{\mathcal{L}}(p_{1,n}, p_{2,n}) + c_{2,n}) \\ x_{1,n+1} = x_{1,n} - y_{1,n} + q_{1,n} \\ x_{2,n+1} = x_{2,n} - y_{2,n} + q_{2,n}. \end{array} \right. \tag{9.40}$$

Then there exists a solution $(\overline{x}_1, \overline{x}_1)$ *to Example 9.4.1 such that* $x_{1,n} \rightharpoonup \overline{x}_1$, $p_{1,n} \rightharpoonup \overline{x}_1$, $x_{2,n} \rightharpoonup \overline{x}_2$, *and* $p_{2,n} \rightharpoonup \overline{x}_2$.

Proof. Example 9.4.1 corresponds to the particular instance of Problem 9.1.1 in which $m = 2$, $\boldsymbol{g}_1 = \boldsymbol{\mathcal{L}}$, and $\boldsymbol{g}_2 = -\boldsymbol{\mathcal{L}}$. Indeed, it follows from [15, Theorem 1] that the operator

$$(x_1,x_2) \mapsto \big(\nabla_1 \boldsymbol{\mathcal{L}}(x_1,x_2), -\nabla_2 \boldsymbol{\mathcal{L}}(x_1,x_2)\big) \tag{9.41}$$

is monotone in $\mathcal{H}_1 \oplus \mathcal{H}_2$ and hence (9.2) holds. In addition, (9.39) implies (9.26) and, since $\nabla \boldsymbol{\mathcal{L}}$ is χ–Lipschitzian, (9.27) holds. Altogether, since (9.28) reduces to (9.40), the result follows from Theorem 9.3.3.

Next, we examine an application of Proposition 9.4.2 to 2-player finite zero-sum games.

Example 9.4.3 We consider a 2-player finite zero-sum game (for complements and background on finite games, see [17]). Let S_1 be the finite set of pure strategies of player 1, with cardinality N_1, and let

$$C_1 = \left\{ (\xi_j)_{1\leq j\leq N_1} \in [0,1]^{N_1} \;\middle|\; \sum_{j=1}^{N_1} \xi_j = 1 \right\} \tag{9.42}$$

be his set of mixed strategies (S_2, N_2, and C_2 are defined likewise). Moreover, let L be an $N_1 \times N_2$ real cost matrix such that

$$(\exists\, z_1 \in C_1)(\exists\, z_2 \in C_2) \quad -Lz_2 \in N_{C_1} z_1 \quad \text{and} \quad L^\top z_1 \in N_{C_2} z_2. \tag{9.43}$$

The problem is to

$$\text{find} \quad x_1 \in \mathbb{R}^{N_1} \quad \text{and} \quad x_2 \in \mathbb{R}^{N_2} \quad \text{such that} \quad \begin{cases} x_1 \in \underset{x\in C_1}{\operatorname{Argmin}} \;\; x^\top L x_2 \\ x_2 \in \underset{x\in C_2}{\operatorname{Argmax}} \;\; x_1^\top L x. \end{cases} \tag{9.44}$$

Since the penalty function of player 1 is $(x_1,x_2) \mapsto x_1^\top L x_2$ and the penalty function of player 2 is $(x_1,x_2) \mapsto -x_1^\top L x_2$, (9.44) is a zero-sum game. It corresponds to the particular instance of Example 9.4.1 in which $\mathcal{H}_1 = \mathbb{R}^{N_1}$, $\mathcal{H}_2 = \mathbb{R}^{N_2}$, $\boldsymbol{f} \colon (x_1,x_2) \mapsto \iota_{C_1}(x_1) + \iota_{C_2}(x_2)$, and $\boldsymbol{\mathcal{L}} \colon (x_1,x_2) \mapsto x_1^\top L x_2$. Indeed, since C_1 and C_2 are nonempty closed convex sets, $\boldsymbol{f} \in \Gamma_0(\mathcal{H}_1 \oplus \mathcal{H}_2)$. Moreover, $x_1 \mapsto \boldsymbol{\mathcal{L}}(x_1,x_2)$ and $x_2 \mapsto -\boldsymbol{\mathcal{L}}(x_1,x_2)$ are convex, and $\nabla \boldsymbol{\mathcal{L}} \colon (x_1,x_2) \mapsto (Lx_2, L^\top x_1)$ is linear and bounded, with $\|\nabla \boldsymbol{\mathcal{L}}\| = \|L\|$. In addition, for every $\gamma \in \left]0,+\infty\right[$, $\operatorname{prox}_{\gamma \boldsymbol{f}} = (P_{C_1}, P_{C_2})$ [5, Proposition 23.30]. Hence, (9.40) reduces to (we set the error terms to zero for simplicity)

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_{1,n} = x_{1,n} - \gamma_n L x_{2,n} \\ y_{2,n} = x_{2,n} + \gamma_n L^\top x_{1,n} \\ p_{1,n} = P_{C_1} y_{1,n} \\ p_{2,n} = P_{C_2} y_{2,n} \\ q_{1,n} = p_{1,n} - \gamma_n L p_{2,n} \\ q_{2,n} = p_{2,n} + \gamma_n L^\top p_{1,n} \\ x_{1,n+1} = x_{1,n} - y_{1,n} + q_{1,n} \\ x_{2,n+1} = x_{2,n} - y_{2,n} + q_{2,n}, \end{array} \right. \tag{9.45}$$

where $(\gamma_n)_{n\in\mathbb{N}}$ is a sequence in $[\varepsilon, (1-\varepsilon)/\|L\|]$ for some arbitrary $\varepsilon \in \left]0, 1/(\|L\|+1)\right[$. Since $\partial \boldsymbol{f} \colon (x_1, x_2) \mapsto N_{C_1}x_1 \times N_{C_2}x_2$, (9.43) yields (9.39). Altogether, Proposition 9.4.2 asserts that the sequence $(x_{1,n}, x_{2,n})_{n\in\mathbb{N}}$ generated by (9.45) converges to $(\overline{x}_1, \overline{x}_2) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, such that $(\overline{x}_1, \overline{x}_2)$ is a solution to (9.44).

9.4.2 Generalized Nash equilibria

We consider the particular case of Problem 9.1.1 in which $\boldsymbol{f}$ is the indicator function of a closed convex subset of $\boldsymbol{\mathcal{H}} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$.

Example 9.4.4 Let $\boldsymbol{C} \subset \boldsymbol{\mathcal{H}}$ be a nonempty closed convex set and, for every $i \in \{1, \ldots, m\}$, let $\boldsymbol{g}_i \colon \boldsymbol{\mathcal{H}} \to \left]-\infty, +\infty\right]$ be a function which is differentiable with respect to its ith variable. Suppose that

$$\big(\forall (x_1, \ldots, x_m) \in \boldsymbol{\mathcal{H}}\big)\big(\forall (y_1, \ldots, y_m) \in \boldsymbol{\mathcal{H}}\big)\quad \sum_{i=1}^{m} \langle \nabla_i\, \boldsymbol{g}_i(x_1, \ldots, x_m) - \nabla_i\, \boldsymbol{g}_i(y_1, \ldots, y_m) \mid x_i - y_i \rangle \geq 0 \tag{9.46}$$

and set

$$(\forall (x_1, \ldots, x_m) \in \boldsymbol{\mathcal{H}})\quad \begin{cases} \boldsymbol{Q}_1(x_2, \ldots, x_m) = \big\{x \in \mathcal{H}_1 \;\big|\; (x, x_2, \ldots, x_m) \in \boldsymbol{C}\big\} \\ \qquad\vdots \\ \boldsymbol{Q}_m(x_1, \ldots, x_{m-1}) = \big\{x \in \mathcal{H}_m \;\big|\; (x_1, \ldots, x_{m-1}, x) \in \boldsymbol{C}\big\}. \end{cases} \tag{9.47}$$

The problem is to find $x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m$ such that

$$\begin{cases} x_1 \in \underset{x \in \boldsymbol{Q}_1(x_2, \ldots, x_m)}{\operatorname{Argmin}}\; \boldsymbol{g}_1(x, x_2, \ldots, x_m) \\ \quad\vdots \\ x_m \in \underset{x \in \boldsymbol{Q}_m(x_1, \ldots, x_{m-1})}{\operatorname{Argmin}}\; \boldsymbol{g}_m(x_1, \ldots, x_{m-1}, x). \end{cases} \tag{9.48}$$

The solutions to Example 9.4.4 are called generalized Nash equilibria [11], social equilibria [10], or equilibria of abstract economies [1], and their existence has been studied in [1, 10]. We deduce from Proposition 9.3.1 that we can find a solution to Example 9.4.4 by solving a variational inequality in $\boldsymbol{\mathcal{H}}$, provided the latter has solutions. This observation is also made in [11], which investigates a Euclidean setting in which additional smoothness properties are imposed on $(\boldsymbol{g}_i)_{1\leq i\leq m}$. An alternative approach for solving

Example 9.4.4 in Euclidean spaces is also proposed in [13] with stronger differentiability properties on $(\boldsymbol{g}_i)_{1\leq i\leq m}$ and a monotonicity assumption of the form (9.46). However, the convergence of the method is not guaranteed. Below we derive from Section 9.3.2 a weakly convergent method for solving Example 9.4.4.

Proposition 9.4.5 *In Example 9.4.4, suppose that there exist* $(z_1,\ldots,z_m)\in\boldsymbol{\mathcal{H}}$ *such that*

$$-\big(\nabla_1\,\boldsymbol{g}_1(z_1,\ldots,z_m),\ldots,\nabla_m\,\boldsymbol{g}_m(z_1,\ldots,z_m)\big)\in N_{\boldsymbol{C}}(z_1,\ldots,z_m) \tag{9.49}$$

and $\chi\in\,]0,+\infty[$ *such that*

$$\begin{aligned}&(\forall(x_1,\ldots,x_m)\in\boldsymbol{\mathcal{H}})(\forall(y_1,\ldots,y_m)\in\boldsymbol{\mathcal{H}})\\ &\quad\sum_{i=1}^m\|\nabla_i\,\boldsymbol{g}_i(x_1,\ldots,x_m)-\nabla_i\,\boldsymbol{g}_i(y_1,\ldots,y_m)\|^2\leq\chi^2\sum_{i=1}^m\|x_i-y_i\|^2.\end{aligned} \tag{9.50}$$

Let $\varepsilon\in\,]0,1/(\chi+1)[$ *and let* $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence in* $[\varepsilon,(1-\varepsilon)/\chi]$. *Moreover, for every* $i\in\{1,\ldots,m\}$, *let* $x_{i,0}\in\mathcal{H}_i$, *and let* $(a_{i,n})_{n\in\mathbb{N}}$, $(b_{i,n})_{n\in\mathbb{N}}$, *and* $(c_{i,n})_{n\in\mathbb{N}}$ *be absolutely summable sequences in* $\mathcal{H}_i$. *Now consider the following routine.*

$$(\forall n\in\mathbb{N})\quad\left\lfloor\begin{array}{l}\text{For } i=1,\ldots,m\\ \lfloor\; y_{i,n}=x_{i,n}-\gamma_n(\nabla_i\,\boldsymbol{g}_i(x_{1,n},\ldots,x_{m,n})+a_{i,n})\\ (p_{1,n},\ldots,p_{m,n})=P_{\boldsymbol{C}}(y_{1,n},\ldots,y_{m,n})+(b_{1,n},\ldots,b_{m,n})\\ \text{For } i=1,\ldots,m\\ \left\lfloor\begin{array}{l}q_{i,n}=p_{i,n}-\gamma_n(\nabla_i\,\boldsymbol{g}_i(p_{1,n},\ldots,p_{m,n})+c_{i,n})\\ x_{i,n+1}=x_{i,n}-y_{i,n}+q_{i,n}.\end{array}\right.\end{array}\right. \tag{9.51}$$

Then there exists a solution $(\overline{x}_1,\ldots,\overline{x}_m)$ *to Example 9.4.4 such that, for every* $i\in\{1,\ldots,m\}$, $x_{i,n}\rightharpoonup\overline{x}_i$ *and* $p_{i,n}\rightharpoonup\overline{x}_i$.

Proof. Example 9.4.4 corresponds to the particular instance of Problem 9.1.1 in which $\boldsymbol{f}=\iota_{\boldsymbol{C}}$. Since $P_{\boldsymbol{C}}=\operatorname{prox}_{\boldsymbol{f}}$, the result follows from Theorem 9.3.3.

9.4.3 Cyclic proximation problem

We consider the following problem in $\boldsymbol{\mathcal{H}}=\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_m$.

Example 9.4.6 Let $\mathcal{G}$ be a real Hilbert space, let $\boldsymbol{f}\in\Gamma_0(\boldsymbol{\mathcal{H}})$, and, for every $i\in\{1,\ldots,m\}$, let $L_i\colon\mathcal{H}_i\to\mathcal{G}$ be a bounded linear operator. The problem is to find $x_1\in\mathcal{H}_1,\ldots,x_m\in\mathcal{H}_m$ such that

$$
\begin{cases}
x_1 \in \underset{x\in\mathcal{H}_1}{\operatorname{Argmin}} \;\; \boldsymbol{f}(x,x_2,\ldots,x_m)+\dfrac{1}{2}\|L_1x-L_2x_2\|^2\\
x_2 \in \underset{x\in\mathcal{H}_2}{\operatorname{Argmin}} \;\; \boldsymbol{f}(x_1,x,\ldots,x_m)+\dfrac{1}{2}\|L_2x-L_3x_3\|^2\\
\quad\vdots\\
x_m \in \underset{x\in\mathcal{H}_m}{\operatorname{Argmin}} \;\; \boldsymbol{f}(x_1,\ldots,x_{m-1},x)+\dfrac{1}{2}\|L_mx-L_1x_1\|^2.
\end{cases}
\tag{9.52}
$$

For every $i \in \{1,\ldots,m\}$, the individual penalty function of player i models his desire to keep some linear transformation L_i of his strategy close to some linear transformation of that of the next player $i+1$. In the particular case when $\boldsymbol{f}\colon (x_i)_{1\leq i\leq m} \mapsto \sum_{i=1}^m f_i(x_i)$, a similar formulation is studied in [2, Section 3.1], where an algorithm is proposed for solving (9.52). However, each step of the algorithm involves the proximity operator of a sum of convex functions, which is extremely difficult to implement numerically. The method described below circumvents this difficulty.

Proposition 9.4.7 *In Example 9.4.6, suppose that there exists* $(z_1,\ldots,z_m)\in\boldsymbol{\mathcal{H}}$ *such that*

$$
\big(L_1^*(L_2z_2-L_1z_1),\ldots,L_m^*(L_1z_1-L_mz_m)\big)\in\partial\boldsymbol{f}(z_1,\ldots,z_m). \tag{9.53}
$$

Set $\chi = 2\max_{1\leq i\leq m}\|L_i\|^2$, *let* $\varepsilon \in \left]0,2/(\chi+1)\right[$, *and let* $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence in* $[\varepsilon,(2-\varepsilon)/\chi]$. *For every* $i\in\{1,\ldots,m\}$, *let* $x_{i,0}\in\mathcal{H}_i$, *and let* $(a_{i,n})_{n\in\mathbb{N}}$ *and* $(b_{i,n})_{n\in\mathbb{N}}$ *be absolutely summable sequences in* $\mathcal{H}_i$. *Now set* $L_{m+1}=L_1$, *for every* $n\in\mathbb{N}$, *set* $x_{m+1,n}=x_{1,n}$, *and consider the following routine.*

$$
(\forall n\in\mathbb{N})\quad
\left\lfloor
\begin{array}{l}
\text{For } i=1,\ldots,m\\
\lfloor\; y_{i,n}=x_{i,n}-\gamma_n\big(L_i^*(L_ix_{i,n}-L_{i+1}x_{i+1,n})+a_{i,n}\big)\\
(x_{1,n+1},\ldots,x_{m,n+1})=\operatorname{prox}_{\gamma_n\boldsymbol{f}}(y_{1,n},\ldots,y_{m,n})+(b_{1,n},\ldots,b_{m,n}).
\end{array}
\right.
\tag{9.54}
$$

Then there exists a solution $(\overline{x}_1,\ldots,\overline{x}_m)$ *to Example 9.4.6 such that, for every* $i\in\{1,\ldots,m\}$, $x_{i,n}\rightharpoonup\overline{x}_i$ *and* $L_i^*\big(L_i(x_{i,n}-\overline{x}_i)-L_{i+1}(x_{i+1,n}-\overline{x}_{i+1})\big)\to 0$.

Proof. Note that Example 9.4.6 corresponds to the particular instance of Problem 9.1.1 in which, for every $i\in\{1,\ldots,m\}$, $\boldsymbol{g}_i\colon (x_i)_{1\leq i\leq m}\mapsto\|L_ix_i-L_{i+1}x_{i+1}\|^2/2$, where we set $x_{m+1}=x_1$. Indeed, since

$$
(\forall (x_1,\ldots,x_m)\in\boldsymbol{\mathcal{H}})\qquad
\begin{cases}
\nabla_1\,\boldsymbol{g}_1(x_1,\ldots,x_m)=L_1^*(L_1x_1-L_2x_2)\\
\qquad\vdots\\
\nabla_m\,\boldsymbol{g}_m(x_1,\ldots,x_m)=L_m^*(L_mx_m-L_1x_1),
\end{cases}
\tag{9.55}
$$

the operator $(x_i)_{1\leq i\leq m} \mapsto (\nabla_i \boldsymbol{g}_i(x_1,\ldots,x_m))_{1\leq i\leq m}$ is linear and bounded. Thus, for every $(x_1,\ldots,x_m) \in \boldsymbol{\mathcal{H}}$,

$$
\begin{aligned}
\sum_{i=1}^m \langle \nabla_i \boldsymbol{g}_i(x_1,\ldots,x_m) \mid x_i\rangle
&= \sum_{i=1}^m \langle L_i^*(L_ix_i - L_{i+1}x_{i+1}) \mid x_i\rangle\\
&= \sum_{i=1}^m \langle L_ix_i - L_{i+1}x_{i+1} \mid L_ix_i\rangle\\
&= \sum_{i=1}^m \|L_ix_i\|^2 - \sum_{i=1}^m \langle L_{i+1}x_{i+1} \mid L_ix_i\rangle\\
&= \frac{1}{2}\sum_{i=1}^m \|L_ix_i\|^2 + \frac{1}{2}\sum_{i=1}^m \|L_{i+1}x_{i+1}\|^2 - \sum_{i=1}^m \langle L_{i+1}x_{i+1} \mid L_ix_i\rangle\\
&= \sum_{i=1}^m \frac{1}{2}\|L_ix_i - L_{i+1}x_{i+1}\|^2\\
&= \sum_{i=1}^m \frac{1}{2\|L_i\|^2}\|L_i\|^2\|L_ix_i - L_{i+1}x_{i+1}\|^2\\
&\geq \chi^{-1}\sum_{i=1}^m \|L_i^*(L_ix_i - L_{i+1}x_{i+1})\|^2\\
&= \chi^{-1}\sum_{i=1}^m \|\nabla_i \boldsymbol{g}_i(x_1,\ldots,x_m)\|^2,
\end{aligned}
\tag{9.56}
$$

and hence (9.33) and (9.2) hold. In addition, (9.53) yields (9.32). Altogether, since (9.34) reduces to (9.54), the result follows from Theorem 9.3.4.

We present below an application of Proposition 9.4.7 to cyclic proximation problems and, in particular, to cyclic projection problems.

Example 9.4.8 We apply Example 9.4.6 to cyclic evaluations of proximity operators. For every $i \in \{1,\ldots,m\}$, let $\mathcal{H}_i = \mathcal{H}$, let $f_i \in \Gamma_0(\mathcal{H})$, let $L_i = \mathrm{Id}$, and set $\boldsymbol{f}\colon (x_i)_{1\leq i\leq m} \mapsto \sum_{i=1}^m f_i(x_i)$. In view of (9.12), Example 9.4.6 reduces to finding $x_1 \in \mathcal{H},\ldots,x_m \in \mathcal{H}$ such that

$$
\begin{cases}
x_1 = \mathrm{prox}_{f_1} x_2\\
x_2 = \mathrm{prox}_{f_2} x_3\\
\quad\vdots\\
x_m = \mathrm{prox}_{f_m} x_1.
\end{cases}
\tag{9.57}
$$

It is assumed that (9.57) has at least one solution. Since $\operatorname{prox}_{\boldsymbol{f}}\colon (x_i)_{1\leq i\leq m} \mapsto (\operatorname{prox}_{f_i} x_i)_{1\leq i\leq m}$ [5, Proposition 23.30], (9.54) becomes (we set errors to zero for simplicity)

$$(\forall n\in\mathbb{N})\quad \left\lfloor \begin{array}{l} \text{For } i=1,\ldots,m \\ \lfloor\, x_{i,n+1} = \operatorname{prox}_{\gamma_n f_i}\big((1-\gamma_n)x_{i,n}+\gamma_n x_{i+1,n}\big), \end{array}\right. \tag{9.58}$$

where $(x_{i,0})_{1\leq i\leq m}\in\mathcal{H}^m$ and $(\gamma_n)_{n\in\mathbb{N}}$ is a sequence in $[\varepsilon,1-\varepsilon]$ for some arbitrary $\varepsilon\in\,]0,1/2[$. Proposition 9.4.7 asserts that the sequences $(x_{1,n})_{n\in\mathbb{N}}$, $\ldots$, $(x_{m,n})_{n\in\mathbb{N}}$ generated by (9.58) converge weakly to points $\overline{x}_1\in\mathcal{H}$, $\ldots$, $\overline{x}_m\in\mathcal{H}$, respectively, such that $(\overline{x}_1,\ldots,\overline{x}_m)$ is a solution to (9.57).

In the particular case when for every $i\in\{1,\ldots,m\}$, $f_i=\iota_{C_i}$, a solution of (9.57) represents a cycle of points in $C_1,\ldots,C_m$. It can be interpreted as a Nash equilibrium of the game in which, for every $i\in\{1,\ldots,m\}$, the strategies of player i, belong to C_i and its penalty function is $(x_i)_{1\leq i\leq m}\mapsto\|x_i-x_{i+1}\|^2$, that is, player i wants to have strategies as close as possible to the strategies of player $i+1$. Such schemes go back at least to [12]. It has recently been proved [4] that, in this case, if $m>2$, the cycles are not minimizers of any potential, from which we infer that this problem cannot be reduced to a potential game. Note that (9.58) becomes

$$(\forall n\in\mathbb{N})\quad \left\lfloor \begin{array}{l} \text{For } i=1,\ldots,m \\ \lfloor\, x_{i,n+1} = P_{C_i}\big((1-\gamma_n)x_{i,n}+\gamma_n x_{i+1,n}\big), \end{array}\right. \tag{9.59}$$

and the sequences $(x_{1,n})_{n\in\mathbb{N}},\ldots,(x_{m,n})_{n\in\mathbb{N}}$ thus generated converge weakly to points $\overline{x}_1\in\mathcal{H},\ldots,\overline{x}_m\in\mathcal{H}$, respectively, such that $(\overline{x}_1,\ldots,\overline{x}_m)$ is a cycle. The existence of cycles has been proved in [12] when one of the sets $C_1,\ldots,C_m$ is bounded. Thus, (9.59) is an alternative parallel algorithm to the method of successive projections [12].

References

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