Variable Metric Quasi-Fejér Monotonicity*

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Abstract

The notion of quasi-Fejér monotonicity has proven to be an efficient tool to simplify and unify the convergence analysis of various algorithms arising in applied nonlinear analysis. In this paper, we extend this notion in the context of variable metric algorithms, whereby the underlying norm is allowed to vary at each iteration. Applications to convex optimization and inverse problems are demonstrated.

Keywords: convex feasibility problem, convex optimization, Hilbert space, inverse problems, proximal Landweber method, proximal point algorithm, quasi-Fejér sequence, variable metric.

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1 Introduction

Let C be a nonempty closed subset of the Euclidean space $\mathbb{R}^N$ and let y be a point in its complement. In 1922, Fejér [21] considered the problem of finding a point $x \in \mathbb{R}^N$ such that $(\forall z \in C)\ \|x - z\| < \|y - z\|$. Based on this work, the term Fejér-monotonicity was coined in [27] in connection with sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ that satisfy

$$(\forall z \in C)(\forall n \in \mathbb{N}) \quad \|x_{n+1} - z\| \leqslant \|x_n - z\|. \tag{1.1}$$

This concept was later broadened to that of quasi-Fejér monotonicity in [20] by relaxing (1.1) to

$$(\forall z \in C)(\forall n \in \mathbb{N}) \quad \|x_{n+1} - z\|^2 \leqslant \|x_n - z\|^2 + \varepsilon_n, \tag{1.2}$$

where $(\varepsilon_n)_{n\in\mathbb{N}}$ is a summable sequence in $[0, +\infty[$. These notions have proven to be remarkably useful in simplifying and unifying the convergence analysis of a large collection of algorithms

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arising in hilbertian nonlinear analysis, see for instance [2, 5, 12, 13, 14, 18, 19, 30, 31, 35] and the references therein. In recent years, there have been attempts to generalize standard algorithms such as those discussed in the above references by allowing the underlying metric to vary over the course of the iterations, e.g., [7, 10, 11, 16, 26, 29]. In order to better understand the convergence properties of such algorithms and lay the ground for further developments, we extend in the present paper the notion of quasi-Fejér monotonicity to the context of variable metric iterations in general Hilbert spaces and investigate its properties.

Our notation and preliminary results are presented in Section 2. The notion of variable metric quasi-Fejér monotonicity is introduced in Section 3, where weak and strong convergence results are also established. In Section 4, we focus on the special case when, as in (1.2), monotonicity is with respect to the squared norms. Finally, we illustrate the potential of these tools in the analysis of variable metric convex feasibility algorithms in Section 5 and in the design of algorithms for solving inverse problems in Section 6.

2 Notation and technical facts

Throughout, $\mathcal{H}$ is a real Hilbert space, $\langle \cdot \mid \cdot \rangle$ is its scalar product and $\|\cdot\|$ the associated norm. The symbols $\rightharpoonup$ and $\to$ denote respectively weak and strong convergence, Id denotes the identity operator, and $B(z;\rho)$ denotes the closed ball of center $z \in \mathcal{H}$ and radius $\rho \in \,]0,+\infty[\,$; $\mathcal{S}(\mathcal{H})$ is the space of self-adjoint bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. The Loewner partial ordering on $\mathcal{S}(\mathcal{H})$ is defined by

$$(\forall L_1 \in \mathcal{S}(\mathcal{H}))(\forall L_2 \in \mathcal{S}(\mathcal{H})) \quad L_1 \succcurlyeq L_2 \quad \Leftrightarrow \quad (\forall x \in \mathcal{H}) \quad \langle L_1 x \mid x\rangle \geqslant \langle L_2 x \mid x\rangle. \tag{2.1}$$

Now let $\alpha \in [0,+\infty[$, set

$$\mathcal{P}_\alpha(\mathcal{H}) = \big\{ L \in \mathcal{S}(\mathcal{H}) \mid L \succcurlyeq \alpha\,\mathrm{Id} \big\}, \tag{2.2}$$

and fix $W \in \mathcal{P}_\alpha(\mathcal{H})$. We define a semi-scalar product and a semi-norm (a scalar product and a norm if $\alpha > 0$) by

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \langle x \mid y\rangle_W = \langle Wx \mid y\rangle \quad \text{and} \quad \|x\|_W = \sqrt{\langle Wx \mid x\rangle}. \tag{2.3}$$

Let C be a nonempty subset of $\mathcal{H}$, let $\alpha \in \,]0,+\infty[$, and let $W \in \mathcal{P}_\alpha(\mathcal{H})$. The interior of C is $\operatorname{int} C$, the distance function of C is d_C, and the convex envelope of C is $\operatorname{conv} C$, with closure $\overline{\operatorname{conv}}\, C$. If C is closed and convex, the projection operator onto C relative to the metric induced by W in (2.3) is

$$P_C^W \colon \mathcal{H} \to C \colon x \mapsto \underset{y \in C}{\operatorname{argmin}} \|x-y\|_W. \tag{2.4}$$

We write $P_C^{\mathrm{Id}} = P_C$. Finally, $\ell_+^1(\mathbb{N})$ denotes the set of summable sequences in $[0,+\infty[$.

Lemma 2.1 *Let $\alpha \in \,]0,+\infty[$, let $\mu \in \,]0,+\infty[$, and let A and B be operators in $\mathcal{S}(\mathcal{H})$ such that $\mu\,\mathrm{Id} \succcurlyeq A \succcurlyeq B \succcurlyeq \alpha\,\mathrm{Id}$. Then the following hold.*

(i) $\alpha^{-1}\,\mathrm{Id} \succcurlyeq B^{-1} \succcurlyeq A^{-1} \succcurlyeq \mu^{-1}\,\mathrm{Id}$.

(ii) $(\forall x \in \mathcal{H})\ \langle A^{-1}x \mid x\rangle \geqslant \|A\|^{-1}\|x\|^2$.

(iii) $\|A^{-1}\| \leqslant \alpha^{-1}$.

Proof. These facts are known [24, Section VI.2.6]. We provide a simple convex-analytic proof.

(i): It suffices to show that $B^{-1} \succcurlyeq A^{-1}$. Set $(\forall x \in \mathcal{H})$ $f(x) = \langle Ax \mid x\rangle/2$ and $g(x) = \langle Bx \mid x\rangle/2$. The conjugate of f is $f^*\colon \mathcal{H} \to [-\infty, +\infty] : u \mapsto \sup_{x\in\mathcal{H}} \big(\langle x \mid u\rangle - f(x)\big) = \langle A^{-1}u \mid u\rangle/2$ [5, Proposition 17.28]. Likewise, $g^*\colon \mathcal{H} \to [-\infty, +\infty] : u \mapsto \langle B^{-1}u \mid u\rangle/2$. Since, $f \geqslant g$, we have $g^* \geqslant f^*$, hence the result.

(ii): Since $\|A\|\,\mathrm{Id} \succcurlyeq A$, (i) yields $A^{-1} \succcurlyeq \|A\|^{-1}\,\mathrm{Id}$.

(iii): We have $A^{-1} \in \mathcal{S}(\mathcal{H})$ and, by (i), $(\forall x \in \mathcal{H})$ $\|x\|^2/\alpha \geqslant \langle A^{-1}x \mid x\rangle$. Hence, upon taking the supremum over $B(0;1)$, we obtain $1/\alpha \geqslant \|A^{-1}\|$. ☐

Lemma 2.2 [30, Lemma 2.2.2] *Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $[0, +\infty[$, let $(\eta_n)_{n\in\mathbb{N}} \in \ell^1_+(\mathbb{N})$, and let $(\varepsilon_n)_{n\in\mathbb{N}} \in \ell^1_+(\mathbb{N})$ be such that $(\forall n \in \mathbb{N})$ $\alpha_{n+1} \leqslant (1 + \eta_n)\alpha_n + \varepsilon_n$. Then $(\alpha_n)_{n\in\mathbb{N}}$ converges.*

The following lemma extends the classical property that a uniformly bounded monotone sequence of operators in $\mathcal{S}(\mathcal{H})$ converges pointwise [33, Théorème 104.1].

Lemma 2.3 *Let $\alpha \in\ ]0, +\infty[$, let $(\eta_n)_{n\in\mathbb{N}} \in \ell^1_+(\mathbb{N})$, and let $(W_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{H})$ such that $\mu = \sup_{n\in\mathbb{N}} \|W_n\| < +\infty$. Suppose that one of the following holds.*

(i) $(\forall n \in \mathbb{N})$ $(1 + \eta_n)W_n \succcurlyeq W_{n+1}$.

(ii) $(\forall n \in \mathbb{N})$ $(1 + \eta_n)W_{n+1} \succcurlyeq W_n$.

Then there exists $W \in \mathcal{P}_\alpha(\mathcal{H})$ such that $W_n \to W$ pointwise.

Proof. (i): Set $\tau = \prod_{n\in\mathbb{N}}(1 + \eta_n)$, $\tau_0 = 1$, and, for every $n \in \mathbb{N} \smallsetminus \{0\}$, $\tau_n = \prod_{k=0}^{n-1}(1 + \eta_k)$. Then $\tau_n \to \tau < +\infty$ [25, Theorem 3.7.3] and

$$(\forall n \in \mathbb{N}) \quad \mu\,\mathrm{Id} \succcurlyeq W_n \succcurlyeq \alpha\,\mathrm{Id} \quad \text{and} \quad \tau_{n+1} = \tau_n(1 + \eta_n). \tag{2.5}$$

Now define

$$(\forall n \in \mathbb{N})(\forall m \in \mathbb{N}) \quad W_{n,m} = \frac{1}{\tau_n}W_n - \frac{1}{\tau_{n+m}}W_{n+m}. \tag{2.6}$$

Then we derive from (2.5) that

$$\begin{aligned}(\forall n \in \mathbb{N})(\forall m \in \mathbb{N} \smallsetminus \{0\})(\forall x \in \mathcal{H}) \quad 0 &= \frac{1}{\tau_n}\langle W_n x \mid x\rangle - \frac{1}{\tau_{n+m}}\prod_{k=n}^{n+m-1}(1 + \eta_k)\langle W_n x \mid x\rangle \\ &\leqslant \frac{1}{\tau_n}\langle W_n x \mid x\rangle - \frac{1}{\tau_{n+m}}\langle W_{n+m}x \mid x\rangle \\ &= \langle W_{n,m}x \mid x\rangle \\ &\leqslant \frac{1}{\tau_n}\langle W_n x \mid x\rangle \\ &\leqslant \langle W_n x \mid x\rangle \\ &\leqslant \mu\|x\|^2. \end{aligned} \tag{2.7}$$

Therefore

$$(\forall n \in \mathbb{N})(\forall m \in \mathbb{N}) \quad W_{n,m} \in \mathcal{P}_0(\mathcal{H}) \quad \text{and} \quad \|W_{n,m}\| \leqslant \mu. \tag{2.8}$$

Let us fix $x \in \mathcal{H}$. By assumption, $(\forall n \in \mathbb{N})\ \|x\|^2_{W_{n+1}} \leqslant (1+\eta_n)\|x\|^2_{W_n}$. Hence, by Lemma 2.2, $(\|x\|^2_{W_n})_{n\in\mathbb{N}}$ converges. In turn, $(\tau_n^{-1}\|x\|^2_{W_n})_{n\in\mathbb{N}}$ converges, which implies that

$$\|x\|^2_{W_{n,m}} = \langle W_{n,m}x \mid x\rangle = \frac{1}{\tau_n}\|x\|^2_{W_n} - \frac{1}{\tau_{n+m}}\|x\|^2_{W_{n+m}} \to 0 \quad \text{as} \quad n,m \to +\infty. \tag{2.9}$$

Therefore, using (2.8), Cauchy-Schwarz for the semi-norms $(\|\cdot\|_{W_{n,m}})_{(n,m)\in\mathbb{N}^2}$, and (2.9), we obtain

$$\begin{aligned} \|W_{n,m}x\|^4 &= \langle x \mid W_{n,m}x\rangle^2_{W_{n,m}} \\ &\leqslant \|x\|^2_{W_{n,m}}\, \|W_{n,m}x\|^2_{W_{n,m}} \\ &\leqslant \|x\|^2_{W_{n,m}}\, \mu^3\|x\|^2 \\ &\to 0 \quad \text{as} \quad n,m \to +\infty. \end{aligned} \tag{2.10}$$

Thus, we derive from (2.6) that $(\tau_n^{-1}W_nx)_{n\in\mathbb{N}}$ is a Cauchy sequence. Hence, it converges strongly, and so does $(W_nx)_{n\in\mathbb{N}}$. If we call Wx the limit of $(W_nx)_{n\in\mathbb{N}}$, the above construction yields the desired operator $W \in \mathcal{P}_\alpha(\mathcal{H})$.

(ii): Set $(\forall n \in \mathbb{N})\ L_n = W_n^{-1}$. It follows from Lemma 2.1(i)&(iii) that $(L_n)_{n\in\mathbb{N}}$ lies in $\mathcal{P}_{1/\mu}(\mathcal{H})$, $\sup_{n\in\mathbb{N}}\|L_n\| \leqslant 1/\alpha$, and $(\forall n \in \mathbb{N})\ (1+\eta_n)L_n \succcurlyeq L_{n+1}$. Hence, appealing to (i), there exists $L \in \mathcal{P}_{1/\mu}(\mathcal{H})$ such that $\|L\| \leqslant 1/\alpha$ and $L_n \to L$ pointwise. Now let $x \in \mathcal{H}$, and set $W = L^{-1}$ and $(\forall n \in \mathbb{N})\ x_n = L_n(Wx)$. Then $W \in \mathcal{P}_\alpha(\mathcal{H})$ and $x_n \to L(Wx) = x$. Moreover, $\|W_nx - Wx\| = \|W_n(x-x_n)\| \leqslant \mu\|x_n - x\| \to 0$. ◻

3 Variable metric quasi-Fejér monotone sequences

Our paper hinges on the following extension of (1.2).

Definition 3.1 *Let $\alpha \in\ ]0,+\infty[$, let $\phi\colon [0,+\infty[\ \to [0,+\infty[$, let $(W_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{H})$, let C be a nonempty subset of $\mathcal{H}$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$. Then $(x_n)_{n\in\mathbb{N}}$ is:*

(i) *ϕ-quasi-Fejér monotone with respect to the target set C relative to $(W_n)_{n\in\mathbb{N}}$ if*

$$\begin{aligned} &\big(\exists\,(\eta_n)_{n\in\mathbb{N}} \in \ell^1_+(\mathbb{N})\big)\big(\forall z \in C\big)\big(\exists\,(\varepsilon_n)_{n\in\mathbb{N}} \in \ell^1_+(\mathbb{N})\big)(\forall n \in \mathbb{N}) \\ &\qquad\qquad \phi(\|x_{n+1}-z\|_{W_{n+1}}) \leqslant (1+\eta_n)\phi(\|x_n - z\|_{W_n}) + \varepsilon_n; \end{aligned} \tag{3.1}$$

(ii) *stationarily ϕ-quasi-Fejér monotone with respect to the target set C relative to $(W_n)_{n\in\mathbb{N}}$ if*

$$\begin{aligned} &\big(\exists\,(\varepsilon_n)_{n\in\mathbb{N}} \in \ell^1_+(\mathbb{N})\big)\big(\exists\,(\eta_n)_{n\in\mathbb{N}} \in \ell^1_+(\mathbb{N})\big)(\forall z \in C)(\forall n \in \mathbb{N}) \\ &\qquad\qquad \phi(\|x_{n+1}-z\|_{W_{n+1}}) \leqslant (1+\eta_n)\phi(\|x_n - z\|_{W_n}) + \varepsilon_n. \end{aligned} \tag{3.2}$$

We start with basic properties.

Proposition 3.2 *Let $\alpha \in \left]0,+\infty\right[$, let $\phi\colon \left[0,+\infty\right[\to \left[0,+\infty\right[$ be strictly increasing and such that $\lim_{t\to+\infty}\phi(t)=+\infty$, let $(W_n)_{n\in\mathbb{N}}$ be in $\mathcal{P}_\alpha(\mathcal{H})$, let C be a nonempty subset of $\mathcal{H}$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that (3.1) is satisfied. Then the following hold.*

(i) *Let $z \in C$. Then $(\|x_n - z\|_{W_n})_{n\in\mathbb{N}}$ converges.*

(ii) *$(x_n)_{n\in\mathbb{N}}$ is bounded.*

Proof. (i): Set $(\forall n\in\mathbb{N})$ $\xi_n = \|x_n - z\|_{W_n}$. It follows from (3.1) and Lemma 2.2 that $(\phi(\xi_n))_{n\in\mathbb{N}}$ converges, say $\phi(\xi_n)\to\lambda$. In turn, since $\lim_{t\to+\infty}\phi(t)=+\infty$, $(\xi_n)_{n\in\mathbb{N}}$ is bounded and, to show that it converges, it suffices to show that it cannot have two distinct cluster points. Suppose to the contrary that we can extract two subsequences $(\xi_{k_n})_{n\in\mathbb{N}}$ and $(\xi_{l_n})_{n\in\mathbb{N}}$ such that $\xi_{k_n}\to\eta$ and $\xi_{l_n}\to\zeta>\eta$, and fix $\varepsilon\in\left]0,(\zeta-\eta)/2\right[$. Then, for n sufficiently large, $\xi_{k_n}\leqslant\eta+\varepsilon<\zeta-\varepsilon\leqslant\xi_{l_n}$ and, since ϕ is strictly increasing, $\phi(\xi_{k_n})\leqslant\phi(\eta+\varepsilon)<\phi(\zeta-\varepsilon)\leqslant\phi(\xi_{l_n})$. Taking the limit as $n\to+\infty$ yields $\lambda\leqslant\phi(\eta+\varepsilon)<\phi(\zeta-\varepsilon)\leqslant\lambda$, which is impossible.

(ii): Let $z\in C$. Since $(W_n)_{n\in\mathbb{N}}$ lies in $\mathcal{P}_\alpha(\mathcal{H})$, we have

$$(\forall n\in\mathbb{N})\quad \alpha\|x_n-z\|^2 \leqslant \langle x_n - z \mid W_n(x_n-z)\rangle = \|x_n-z\|^2_{W_n}. \tag{3.3}$$

Hence, since (i) asserts that $(\|x_n-z\|_{W_n})_{n\in\mathbb{N}}$ is bounded, so is $(x_n)_{n\in\mathbb{N}}$. ☐

The next result concerns weak convergence. In the case of standard Fejér monotonicity (1.1), it appears in [9, Lemma 6] and, in the case of quasi-Fejér monotonicity (1.2), it appears in [1, Proposition 1.3].

Theorem 3.3 *Let $\alpha \in \left]0,+\infty\right[$, let $\phi\colon \left[0,+\infty\right[\to \left[0,+\infty\right[$ be strictly increasing and such that $\lim_{t\to+\infty}\phi(t)=+\infty$, let $(W_n)_{n\in\mathbb{N}}$ and W be operators in $\mathcal{P}_\alpha(\mathcal{H})$ such that $W_n\to W$ pointwise, let C be a nonempty subset of $\mathcal{H}$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that (3.1) is satisfied. Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point in C if and only if every weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$ is in C.*

Proof. Necessity is clear. To show sufficiency, suppose that every weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$ is in C, and let x and y be two such points, say $x_{k_n}\rightharpoonup x$ and $x_{l_n}\rightharpoonup y$. Then it follows from Proposition 3.2(i) that $(\|x_n-x\|_{W_n})_{n\in\mathbb{N}}$ and $(\|x_n-y\|_{W_n})_{n\in\mathbb{N}}$ converge. Moreover, $\|x\|^2_{W_n}=\langle W_n x\mid x\rangle\to\langle Wx\mid x\rangle$ and, likewise, $\|y\|^2_{W_n}\to\langle Wy\mid y\rangle$. Therefore, since

$$(\forall n\in\mathbb{N})\quad \langle W_n x_n \mid x-y\rangle = \frac{1}{2}\big(\|x_n-y\|^2_{W_n}-\|x_n-x\|^2_{W_n}+\|x\|^2_{W_n}-\|y\|^2_{W_n}\big), \tag{3.4}$$

the sequence $(\langle W_n x_n\mid x-y\rangle)_{n\in\mathbb{N}}$ converges, say $\langle W_n x_n\mid x-y\rangle\to\lambda\in\mathbb{R}$, which implies that

$$\langle x_n \mid W_n(x-y)\rangle \to \lambda\in\mathbb{R}. \tag{3.5}$$

However, since $x_{k_n}\rightharpoonup x$ and $W_{k_n}(x-y)\to W(x-y)$, it follows from (3.5) and [5, Lemma 2.41(iii)] that $\langle x\mid W(x-y)\rangle=\lambda$. Likewise, passing to the limit along the subsequence $(x_{l_n})_{n\in\mathbb{N}}$ in (3.5) yields $\langle y\mid W(x-y)\rangle=\lambda$. Thus,

$$0=\langle x\mid W(x-y)\rangle-\langle y\mid W(x-y)\rangle=\langle x-y\mid W(x-y)\rangle\geqslant\alpha\|x-y\|^2. \tag{3.6}$$

This shows that $x = y$. Upon invoking Proposition 3.2(ii) and [5, Lemma 2.38], we conclude that $x_n \rightharpoonup x$. $\square$

Lemma 2.3 provides instances in which the conditions imposed on $(W_n)_{n\in\mathbb{N}}$ in Theorem 3.3 are satisfied. Next, we present a characterization of strong convergence which can be found in [12, Theorem 3.11] in the special case of quasi-Fejér monotonicity (1.2).

Proposition 3.4 *Let $\alpha \in \left]0,+\infty\right[$, let $\chi \in \left[1,+\infty\right[$, and let $\phi\colon \left[0,+\infty\right[\to \left[0,+\infty\right[$ be an increasing upper semicontinuous function vanishing only at 0 and such that*

$$\big(\forall(\xi_1,\xi_2) \in \left[0,+\infty\right[^2\big) \quad \phi(\xi_1+\xi_2) \leqslant \chi\big(\phi(\xi_1)+\phi(\xi_2)\big). \tag{3.7}$$

Let $(W_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{H})$ such that $\mu = \sup_{n\in\mathbb{N}} \|W_n\| < +\infty$, let C be a nonempty closed subset of $\mathcal{H}$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that (3.2) is satisfied. Then $(x_n)_{n\in\mathbb{N}}$ converges strongly to a point in C if and only if $\underline{\lim}\, d_C(x_n) = 0$.

Proof. Necessity is clear. For sufficiency, suppose that $\underline{\lim}\, d_C(x_n) = 0$ and set $(\forall n \in \mathbb{N})$ $\xi_n = \inf_{z\in C} \|x_n - z\|_{W_n}$. For every $n \in \mathbb{N}$, let $(z_{n,k})_{k\in\mathbb{N}}$ be a sequence in C such that $\|x_n - z_{n,k}\|_{W_n} \to \xi_n$. Then, since ϕ is increasing, (3.2) yields

$$(\forall n \in \mathbb{N})(\forall k \in \mathbb{N}) \quad \phi(\xi_{n+1}) \leqslant \phi(\|x_{n+1} - z_{n,k}\|_{W_{n+1}}) \leqslant (1+\eta_n)\phi(\|x_n - z_{n,k}\|_{W_n}) + \varepsilon_n. \tag{3.8}$$

Hence, it follows from the upper semicontinuity of ϕ that

$$\begin{aligned}(\forall n \in \mathbb{N}) \quad \phi(\xi_{n+1}) &\leqslant (1+\eta_n) \varlimsup_{k\to+\infty} \phi(\|x_n - z_{n,k}\|_{W_n}) + \varepsilon_n \\ &\leqslant (1+\eta_n)\phi(\xi_n) + \varepsilon_n.\end{aligned} \tag{3.9}$$

Therefore, by Lemma 2.2,

$$\big(\phi(\xi_n)\big)_{n\in\mathbb{N}} \quad \text{converges.} \tag{3.10}$$

Moreover, since

$$(\forall n \in \mathbb{N})(\forall m \in \mathbb{N})(\forall x \in \mathcal{H}) \quad \alpha\|x_n - x\|^2 \leqslant \|x_n - x\|^2_{W_m} \leqslant \mu\|x_n - x\|^2, \tag{3.11}$$

we have

$$(\forall n \in \mathbb{N}) \quad \sqrt{\alpha} d_C(x_n) \leqslant \xi_n \leqslant \sqrt{\mu} d_C(x_n). \tag{3.12}$$

Consequently, since $\underline{\lim}\, d_C(x_n) = 0$, we derive from (3.12) that $\underline{\lim}\, \xi_n = 0$. Let us extract a subsequence $(\xi_{k_n})_{n\in\mathbb{N}}$ such that $\xi_{k_n} \to 0$. Since ϕ is upper semicontinuous, we have $0 \leqslant \underline{\lim}\, \phi(\xi_{k_n}) \leqslant \overline{\lim}\, \phi(\xi_{k_n}) \leqslant \phi(0) = 0$. In view of (3.10), we therefore obtain $\phi(\xi_n) \to 0$ and, in turn, $\xi_n \to 0$. Hence, we deduce from (3.12) that

$$d_C(x_n) \to 0. \tag{3.13}$$

Next, let N be the smallest integer such that $N > \sqrt{\mu}$, and set $\rho = \chi^{N-1} + \sum_{k=1}^{N-1} \chi^k$ if $N > 1$; $\rho = 1$ if $N = 1$. Moreover, let $x \in C$ and let m and n be strictly positive integers. Using (3.11), the monotonicity of ϕ, and (3.7), we obtain

$$\phi\big(\|x_n - x\|_{W_m}\big) \leqslant \phi\big(\sqrt{\mu}\|x_n - x\|\big) \leqslant \phi\big(N\|x_n - x\|\big) \leqslant \rho\phi\big(\|x_n - x\|\big). \tag{3.14}$$

Now set $\tau = \prod_{k\in\mathbb{N}}(1+\eta_k)$. Then $\tau < +\infty$ [25, Theorem 3.7.3] and we derive from (3.7), (3.2), and (3.14) that

$$
\begin{aligned}
\chi^{-1}\phi\big(\|x_{n+m}-x_n\|_{W_{n+m}}\big) &\leqslant \chi^{-1}\phi\big(\|x_{n+m}-x\|_{W_{n+m}} + \|x_n - x\|_{W_{n+m}}\big)\\
&\leqslant \phi\big(\|x_{n+m}-x\|_{W_{m+n}}\big) + \phi\big(\|x_n-x\|_{W_{m+n}}\big)\\
&\leqslant \tau\left(\phi\big(\|x_n-x\|_{W_n}\big) + \sum_{k=n}^{n+m-1}\varepsilon_k\right) + \phi\big(\|x_n-x\|_{W_{m+n}}\big)\\
&\leqslant \rho(1+\tau)\phi\big(\|x_n-x\|\big) + \tau\sum_{k\geqslant n}\varepsilon_k. \qquad (3.15)
\end{aligned}
$$

Therefore, upon taking the infimum over $x\in C$, we obtain by upper semicontinuity of ϕ

$$
\phi\big(\|x_{n+m}-x_n\|_{W_{n+m}}\big) \leqslant \chi\rho(1+\tau)\phi\big(d_C(x_n)\big) + \chi\tau\sum_{k\geqslant n}\varepsilon_k. \qquad (3.16)
$$

Hence, appealing to (3.13) and the summability of $(\varepsilon_k)_{k\in\mathbb{N}}$, we deduce from (3.16) that, as $n\to+\infty$, $\phi(\|x_{n+m}-x_n\|_{W_{n+m}})\to 0$ and, hence, $\alpha\|x_{n+m}-x_n\|^2 \leqslant \|x_{n+m}-x_n\|^2_{W_{n+m}} \to 0$. Thus, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$ and there exists $\overline{x}\in\mathcal{H}$ such that $x_n\to\overline{x}$. By continuity of d_C and (3.13), we obtain $d_C(\overline{x})=0$ and, since C is closed, $\overline{x}\in C$. ☐

4 The quadratic case

In this section, we focus on the important case when $\phi = |\cdot|^2$ in Definition 3.1. Our first result states that variable metric quasi-Fejér monotonicity "spreads" to the convex hull of the target set.

Proposition 4.1 *Let $\alpha\in\,]0,+\infty[$, let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence in $\ell^1_+(\mathbb{N})$, and let $(W_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{H})$ such that*

$$
\mu = \sup_{n\in\mathbb{N}}\|W_n\| < +\infty \quad \textit{and} \quad (\forall n\in\mathbb{N})\quad (1+\eta_n)W_n \succcurlyeq W_{n+1}. \qquad (4.1)
$$

Let C be a nonempty subset of $\mathcal{H}$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that

$$
\big(\exists\,(\eta_n)_{n\in\mathbb{N}}\in\ell^1_+(\mathbb{N})\big)\big(\forall z\in C\big)\big(\exists\,(\varepsilon_n)_{n\in\mathbb{N}}\in\ell^1_+(\mathbb{N})\big)(\forall n\in\mathbb{N})
\quad \|x_{n+1}-z\|^2_{W_{n+1}} \leqslant (1+\eta_n)\|x_n-z\|^2_{W_n} + \varepsilon_n. \qquad (4.2)
$$

Then the following hold.

(i) *$(x_n)_{n\in\mathbb{N}}$ is $|\cdot|^2$-quasi-Fejér monotone with respect to* $\operatorname{conv} C$ *relative to $(W_n)_{n\in\mathbb{N}}$.*

(ii) *For every $y\in\overline{\operatorname{conv}}\,C$, $(\|x_n-y\|_{W_n})_{n\in\mathbb{N}}$ converges.*

Proof. Let us fix $z\in\operatorname{conv} C$. There exist finite sets $\{z_i\}_{i\in I}\subset C$ and $\{\lambda_i\}_{i\in I}\subset\,]0,1]$ such that

$$
\sum_{i\in I}\lambda_i = 1 \quad \text{and} \quad z = \sum_{i\in I}\lambda_i z_i. \qquad (4.3)
$$

For every $i \in I$, it follows from (4.2) that there exists a sequence $(\varepsilon_{i,n})_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})$ such that

$$(\forall n \in \mathbb{N}) \quad \|x_{n+1} - z_i\|_{W_{n+1}}^2 \leqslant (1 + \eta_n)\|x_n - z_i\|_{W_n}^2 + \varepsilon_{i,n}. \tag{4.4}$$

Now set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \alpha_n = \dfrac{1}{2}\displaystyle\sum_{i\in I}\sum_{j\in I} \lambda_i\lambda_j \|z_i - z_j\|_{W_n}^2 \\ \varepsilon_n = (1 + \eta_n)\alpha_n - \alpha_{n+1} + \max\{\varepsilon_{1,n}, \ldots, \varepsilon_{m,n}\}. \end{cases} \tag{4.5}$$

Then $(\max\{\varepsilon_{1,n}, \ldots, \varepsilon_{m,n}\})_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})$ and, by (4.1), $(\forall n \in \mathbb{N})$ $(1 + \eta_n)\alpha_n \geqslant \alpha_{n+1}$. Hence, Lemma 2.2 asserts that $(\alpha_n)_{n\in\mathbb{N}}$ converges, which implies that $(\varepsilon_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})$.

(i): Using (4.3), [5, Lemma 2.13(ii)], and (4.4), we obtain

$$\begin{aligned}(\forall n \in \mathbb{N}) \quad \|x_{n+1} - z\|_{W_{n+1}}^2 &= \sum_{i\in I} \lambda_i \|x_{n+1} - z_i\|_{W_{n+1}}^2 - \alpha_{n+1} \\ &\leqslant (1 + \eta_n)\sum_{i\in I} \lambda_i \|x_n - z_i\|_{W_n}^2 - \alpha_{n+1} + \max\{\varepsilon_{1,n}, \ldots, \varepsilon_{m,n}\} \\ &= (1 + \eta_n)\|x_n - z\|_{W_n}^2 + (1 + \eta_n)\alpha_n - \alpha_{n+1} + \max\{\varepsilon_{1,n}, \ldots, \varepsilon_{m,n}\} \\ &= (1 + \eta_n)\|x_n - z\|_{W_n}^2 + \varepsilon_n. \end{aligned} \tag{4.6}$$

(ii): It follows from [5, Lemma 2.13(ii)] that

$$(\forall n \in \mathbb{N}) \quad \|x_n - z\|_{W_n}^2 = \sum_{i\in I} \lambda_i \|x_n - z_i\|_{W_n}^2 - \alpha_n. \tag{4.7}$$

However, $(\alpha_n)_{n\in\mathbb{N}}$ converges and, for every $i \in I$, Proposition 3.2(i) asserts that $(\|x_n - z_i\|_{W_n})_{n\in\mathbb{N}}$ converges. Hence, $(\|x_n - z\|_{W_n})_{n\in\mathbb{N}}$ converges. Now let $y \in \overline{\operatorname{conv}}\, C$. Then there exists a sequence $(y_k)_{k\in\mathbb{N}}$ in $\operatorname{conv} C$ such that $y_k \to y$. It follows from (i) and Proposition 3.2(i) that, for every $k \in \mathbb{N}$, $(\|x_n - y_k\|_{W_n})_{n\in\mathbb{N}}$ converges. Moreover, we have

$$\begin{aligned}(\forall k \in \mathbb{N})(\forall n \in \mathbb{N}) \quad -\sqrt{\mu}\|y_k - y\| &\leqslant -\|y_k - y\|_{W_n} \\ &\leqslant \|x_n - y\|_{W_n} - \|x_n - y_k\|_{W_n} \\ &\leqslant \|y_k - y\|_{W_n} \\ &\leqslant \sqrt{\mu}\|y_k - y\|. \end{aligned} \tag{4.8}$$

Consequently,

$$\begin{aligned}(\forall k \in \mathbb{N}) \quad -\sqrt{\mu}\|y_k - y\| &\leqslant \varliminf \|x_n - y\|_{W_n} - \lim \|x_n - y_k\|_{W_n} \\ &\leqslant \varlimsup \|x_n - y\|_{W_n} - \lim \|x_n - y_k\|_{W_n} \\ &\leqslant \sqrt{\mu}\|y_k - y\|. \end{aligned} \tag{4.9}$$

Taking the limit as $k \to +\infty$ yields $\lim_{n\to+\infty} \|x_n - y\|_{W_n} = \lim_{k\to+\infty}\lim_{n\to+\infty} \|x_n - y_k\|_{W_n}$. ☐

Standard Fejér monotone sequences may fail to converge weakly and, even when they converge weakly, strong convergence may fail [12, 23]. However, if the target set C is closed and convex in (1.1), the projected sequence $(P_C x_n)_{n\in\mathbb{N}}$ converges strongly; see [2, Theorem 2.16(iv)] and [32, Remark 1]. This property, which remains true in the quasi-Fejérian case [12, Proposition 3.6(iv)], is extended below.

Proposition 4.2 *Let $\alpha \in \,]0,+\infty[$, let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence in $\ell_+^1(\mathbb{N})$, let $(W_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence in $\mathcal{P}_\alpha(\mathcal{H})$, let C be a nonempty closed convex subset of $\mathcal{H}$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that*

$$\begin{aligned}&\big(\exists\,(\varepsilon_n)_{n\in\mathbb{N}}\in\ell_+^1(\mathbb{N})\big)\big(\exists\,(\eta_n)_{n\in\mathbb{N}}\in\ell_+^1(\mathbb{N})\big)(\forall z\in C)(\forall n\in\mathbb{N})\\ &\qquad\qquad\qquad\qquad \|x_{n+1}-z\|_{W_{n+1}}^2\leqslant(1+\eta_n)\|x_n-z\|_{W_n}^2+\varepsilon_n.\end{aligned} \tag{4.10}$$

Then $(P_C^{W_n}x_n)_{n\in\mathbb{N}}$ converges strongly.

Proof. Set $(\forall n\in\mathbb{N})$ $z_n=P_C^{W_n}x_n$. For every $(m,n)\in\mathbb{N}^2$, since $z_n\in C$ and $z_{m+n}=P_C^{W_{n+m}}x_{n+m}$, the well-known convex projection theorem [5, Theorem 3.14] yields

$$\langle z_n-z_{n+m}\mid x_{n+m}-z_{n+m}\rangle_{W_{n+m}}\leqslant 0, \tag{4.11}$$

which implies that

$$\begin{aligned}\langle z_n-x_{n+m}\mid x_{n+m}-z_{n+m}\rangle_{W_{n+m}}&=\langle z_n-z_{n+m}\mid x_{n+m}-z_{n+m}\rangle_{W_{n+m}}-\|x_{n+m}-z_{n+m}\|_{W_{n+m}}^2\\ &\leqslant-\|x_{n+m}-z_{n+m}\|_{W_{n+m}}^2.\end{aligned} \tag{4.12}$$

Therefore, for every $(m,n)\in\mathbb{N}^2$,

$$\begin{aligned}\|z_n-z_{n+m}\|_{W_{n+m}}^2&=\|z_n-x_{n+m}\|_{W_{n+m}}^2+2\langle z_n-x_{n+m}\mid x_{n+m}-z_{n+m}\rangle_{W_{n+m}}\\ &\quad+\|x_{n+m}-z_{n+m}\|_{W_{n+m}}^2\\ &\leqslant\|z_n-x_{n+m}\|_{W_{n+m}}^2-\|x_{n+m}-z_{n+m}\|_{W_{n+m}}^2.\end{aligned} \tag{4.13}$$

Now fix $z\in C$, and set $\mu=\sup_{n\in\mathbb{N}}\|W_n\|$ and $\rho=\sup_{n\in\mathbb{N}}\|x_n-z\|_{W_n}^2$. Then $\mu<+\infty$ and, in view of Proposition 3.2(i), $\rho<+\infty$. It follows from (4.10) that, for every $n\in\mathbb{N}$ and every $m\in\mathbb{N}\smallsetminus\{0\}$, since $P_C^{W_n}$ is nonexpansive with respect to $\|\cdot\|_{W_n}$ [5, Proposition 4.8], we have

$$\begin{aligned}\|x_{n+m}-z_n\|_{W_{n+m}}^2&\leqslant\|x_n-z_n\|_{W_n}^2+\sum_{k=n}^{n+m-1}\big(\eta_k\|x_k-z_n\|_{W_k}^2+\varepsilon_k\big)\\ &\leqslant\|x_n-z_n\|_{W_n}^2+\sum_{k=n}^{n+m-1}\Big(2\eta_k\big(\|x_k-z\|_{W_k}^2+\|z_n-z\|_{W_k}^2\big)+\varepsilon_k\Big)\\ &\leqslant\|x_n-z_n\|_{W_n}^2+\sum_{k=n}^{n+m-1}\Big(2\eta_k\Big(\rho+\frac{\mu}{\alpha}\|P_C^{W_n}x_n-P_C^{W_n}z\|_{W_n}^2\Big)+\varepsilon_k\Big)\\ &\leqslant\|x_n-z_n\|_{W_n}^2+\sum_{k=n}^{n+m-1}\Big(2\eta_k\Big(\rho+\frac{\mu}{\alpha}\|x_n-z\|_{W_n}^2\Big)+\varepsilon_k\Big)\\ &\leqslant\|x_n-z_n\|_{W_n}^2+\sum_{k=n}^{n+m-1}\Big(2\rho\eta_k\Big(1+\frac{\mu}{\alpha}\Big)+\varepsilon_k\Big).\end{aligned} \tag{4.14}$$

Combining (4.13) and (4.14), we obtain that for every $n\in\mathbb{N}$ and every $m\in\mathbb{N}\smallsetminus\{0\}$,

$$\begin{aligned}\alpha\|z_{n+m}-z_n\|^2&\leqslant\|z_{n+m}-z_n\|_{W_{n+m}}^2\\ &\leqslant\|x_n-z_n\|_{W_n}^2-\|x_{n+m}-z_{n+m}\|_{W_{n+m}}^2+\sum_{k\geqslant n}\Big(2\rho\eta_k\Big(1+\frac{\mu}{\alpha}\Big)+\varepsilon_k\Big).\end{aligned} \tag{4.15}$$

On the other hand, (4.10) yields

$$(\forall n \in \mathbb{N}) \quad \|x_{n+1} - z_{n+1}\|^2_{W_{n+1}} \leqslant \|x_{n+1} - z_n\|^2_{W_{n+1}} \\ \leqslant (1+\eta_n)\|x_n - z_n\|^2_{W_n} + \varepsilon_n, \tag{4.16}$$

which, by Lemma 2.2, implies that $(\|x_n - z_n\|_{W_n})_{n\in\mathbb{N}}$ converges. Consequently, since $(\eta_k)_{k\in\mathbb{N}}$ and $(\varepsilon_k)_{k\in\mathbb{N}}$ are in $\ell^1_+(\mathbb{N})$, we derive from (4.15) that $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and hence that it converges strongly. ☐

In the case of classical Fejér monotone sequences, it has been known since [31] that strong convergence is achieved when the interior of the target set is nonempty (see also [12, Proposition 3.10] for the case of quasi-Fejér monotonicity). The following result extends this fact in the context of variable metric quasi-Fejér sequences.

Proposition 4.3 *Let $\alpha \in \left]0,+\infty\right[$, let $(\nu_n)_{n\in\mathbb{N}} \in \ell^1_+(\mathbb{N})$, and let $(W_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{H})$ such that*

$$\mu = \sup_{n\in\mathbb{N}} \|W_n\| < +\infty \quad \textit{and} \quad (\forall n \in \mathbb{N}) \quad (1+\nu_n)W_{n+1} \succcurlyeq W_n. \tag{4.17}$$

Furthermore, let C be a subset of $\mathcal{H}$ such that $\operatorname{int} C \neq \varnothing$, let $z \in C$ and $\rho \in \left]0,+\infty\right[$ be such that $B(z;\rho) \subset C$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that

$$\big(\exists\, (\varepsilon_n)_{n\in\mathbb{N}} \in \ell^1_+(\mathbb{N})\big)\big(\exists\, (\eta_n)_{n\in\mathbb{N}} \in \ell^1_+(\mathbb{N})\big)(\forall x \in B(z;\rho))(\forall n \in \mathbb{N}) \\ \|x_{n+1} - x\|^2_{W_{n+1}} \leqslant (1+\eta_n)\|x_n - x\|^2_{W_n} + \varepsilon_n. \tag{4.18}$$

Then $(x_n)_{n\in\mathbb{N}}$ converges strongly.

Proof. We derive from (4.17) and Proposition 3.2(ii) that

$$\zeta = \sup_{x\in B(z;\rho)} \sup_{n\in\mathbb{N}} \|x_n - x\|^2_{W_n} \leqslant 2\mu\bigg(\sup_{n\in\mathbb{N}} \|x_n - z\|^2 + \sup_{x\in B(z;\rho)} \|x - z\|^2\bigg) < +\infty. \tag{4.19}$$

It follows from (4.18) and (4.19) that

$$(\forall n \in \mathbb{N})(\forall x \in B(z;\rho)) \quad \|x_{n+1} - x\|^2_{W_{n+1}} \leqslant \|x_n - x\|^2_{W_n} + \xi_n, \quad \text{where} \quad \xi_n = \zeta\eta_n + \varepsilon_n. \tag{4.20}$$

Now set

$$(\forall n \in \mathbb{N}) \quad v_n = W_{n+1}(x_{n+1} - z) - W_n(x_n - z), \tag{4.21}$$

and define a sequence $(z_n)_{n\in\mathbb{N}}$ in $B(z;\rho)$ by

$$(\forall n \in \mathbb{N}) \quad z_n = z - \rho u_n, \quad \text{where} \quad u_n = \begin{cases} 0, & \text{if } v_n = 0; \\ v_n/\|v_n\|, & \text{if } v_n \neq 0. \end{cases} \tag{4.22}$$

Then

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \|x_{n+1} - z_n\|^2_{W_{n+1}} & = \|x_{n+1} - z\|^2_{W_{n+1}} + 2\rho\langle W_{n+1}(x_{n+1} - z) \mid u_n\rangle \\ & \quad + \rho^2\|u_n\|^2_{W_{n+1}}; \\ \|x_n - z_n\|^2_{W_n} & = \|x_n - z\|^2_{W_n} + 2\rho\langle W_n(x_n - z) \mid u_n\rangle + \rho^2\|u_n\|^2_{W_n}. \end{cases} \tag{4.23}$$

On the other hand, (4.20) yields $(\forall n\in\mathbb{N})$ $\|x_{n+1}-z_n\|^2_{W_{n+1}}\leqslant\|x_n-z_n\|^2_{W_n}+\xi_n$. Therefore, it follows from (4.23), (4.21), and (4.17) that

$$\begin{aligned}(\forall n\in\mathbb{N})\quad \|x_{n+1}-z\|^2_{W_{n+1}}&\leqslant\|x_n-z\|^2_{W_n}-2\rho\|v_n\|+\rho^2\big(\|u_n\|^2_{W_n}-\|u_n\|^2_{W_{n+1}}\big)+\xi_n\\&\leqslant\|x_n-z\|^2_{W_n}-2\rho\|v_n\|+\rho^2\mu\nu_n+\xi_n.\end{aligned}\tag{4.24}$$

Since $(\rho^2\mu\nu_n+\xi_n)_{n\in\mathbb{N}}\in\ell^1_+(\mathbb{N})$, this implies that

$$\sum_{n\in\mathbb{N}}\|w_{n+1}-w_n\|=\sum_{n\in\mathbb{N}}\|v_n\|<+\infty,\quad\text{where}\quad(\forall n\in\mathbb{N})\quad w_n=W_n(x_n-z).\tag{4.25}$$

Hence, $(w_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$ and, therefore, there exists $w\in\mathcal{H}$ such that $w_n\to w$. On the other hand, we deduce from (4.17) and Lemma 2.3(ii) that there exists $W\in\mathcal{P}_\alpha(\mathcal{H})$ such that $W_n\to W$. Now set $x=z+W^{-1}w$. Then, since $(W_n)_{n\in\mathbb{N}}$ lies in $\mathcal{P}_\alpha(\mathcal{H})$, it follows from Cauchy-Schwarz that

$$\alpha\|x_n-x\|\leqslant\|W_nx_n-W_nx\|=\|w_n-W_nW^{-1}w\|\leqslant\|w_n-w\|+\|w-W_nW^{-1}w\|\to 0,\tag{4.26}$$

which concludes the proof. ☐

5 Application to convex feasibility

We illustrate our results through an application to the convex feasibility problem, i.e., the generic problem of finding a common point of a family of closed convex sets. As in [4], given $\alpha\in\,]0,+\infty[$ and $W\in\mathcal{P}_\alpha(\mathcal{H})$, we say that an operator $T\colon\mathcal{H}\to\mathcal{H}$ with fixed point set $\operatorname{Fix}T$ belongs to $\mathfrak{T}(W)$ if

$$(\forall x\in\mathcal{H})(\forall y\in\operatorname{Fix}T)\quad\langle y-Tx\mid x-Tx\rangle_W\leqslant 0.\tag{5.1}$$

If $T\in\mathfrak{T}(W)$, then [12, Proposition 2.3(ii)] yields

$$\begin{aligned}(\forall x\in\mathcal{H})(\forall y\in\operatorname{Fix}T)(\forall\lambda\in[0,2])\quad&\|(\mathrm{Id}+\lambda(T-\mathrm{Id}))x-y\|^2_W\\&\leqslant\|x-y\|^2_W-\lambda(2-\lambda)\|Tx-x\|^2_W.\end{aligned}\tag{5.2}$$

The usefulness of the class $\mathfrak{T}(W)$ stems from the fact that it contains many of the operators commonly encountered in nonlinear analysis: firmly nonexpansive operators (in particular resolvents of maximally monotone operators and proximity operators of proper lower semicontinuous convex functions), subgradient projection operators, projection operators, averaged quasi-nonexpansive operators, and several combinations thereof [4, 6, 12].

Theorem 5.1 *Let $\alpha\in\,]0,+\infty[$, let $(C_i)_{i\in I}$ be a finite or countably infinite family of closed convex subsets of $\mathcal{H}$ such that $C=\bigcap_{i\in I}C_i\neq\varnothing$, let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that $\sum_{n\in\mathbb{N}}\|a_n\|<+\infty$, let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence in $\ell^1_+(\mathbb{N})$, and let $(W_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{H})$ such that*

$$\mu=\sup_{n\in\mathbb{N}}\|W_n\|<+\infty\quad\textit{and}\quad(\forall n\in\mathbb{N})\quad(1+\eta_n)W_n\succcurlyeq W_{n+1}.\tag{5.3}$$

Let $\mathrm{i}\colon\mathbb{N}\to I$ be such that

$$(\forall j\in I)(\exists\,M_j\in\mathbb{N}\smallsetminus\{0\})(\forall n\in\mathbb{N})\quad j\in\{\mathrm{i}(n),\ldots,\mathrm{i}(n+M_j-1)\}.\tag{5.4}$$

For every $i \in I$, let $(T_{i,n})_{n\in\mathbb{N}}$ be a sequence of operators such that

$$(\forall n \in \mathbb{N}) \quad T_{i,n} \in \mathfrak{T}(W_n) \quad \textit{and} \quad \operatorname{Fix} T_{i,n} = C_i. \tag{5.5}$$

Fix $\varepsilon \in]0,1[$ and $x_0 \in \mathcal{H}$, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, 2-\varepsilon]$, and set

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = x_n + \lambda_n\big(T_{\mathrm{i}(n),n}x_n + a_n - x_n\big). \tag{5.6}$$

Suppose that, for every strictly increasing sequence $(p_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$, every $x \in \mathcal{H}$, and every $j \in I$,

$$\begin{cases} x_{p_n} \rightharpoonup x \\ T_{j,p_n}x_{p_n} - x_{p_n} \to 0 \\ (\forall n \in \mathbb{N}) \;\; j = \mathrm{i}(p_n) \end{cases} \quad \Rightarrow \quad x \in C_j. \tag{5.7}$$

Then the following hold for some $\overline{x} \in C$.

(i) $x_n \rightharpoonup \overline{x}$.

(ii) *Suppose that* $\operatorname{int} C \neq \varnothing$ *and that there exists $(\nu_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})$ such that $(\forall n \in \mathbb{N})$ $(1+\nu_n)W_{n+1} \succcurlyeq W_n$. Then $x_n \to \overline{x}$.*

(iii) *Suppose that $\underline{\lim}\, d_C(x_n) = 0$. Then $x_n \to \overline{x}$.*

(iv) *Suppose that there exists an index $j \in I$ of demicompact regularity: for every strictly increasing sequence $(p_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$,*

$$\begin{cases} \sup_{n\in\mathbb{N}} \|x_{p_n}\| < +\infty \\ T_{j,p_n}x_{p_n} - x_{p_n} \to 0 \\ (\forall n \in \mathbb{N}) \;\; j = \mathrm{i}(p_n) \end{cases} \quad \Rightarrow \quad (x_{p_n})_{n\in\mathbb{N}} \textit{ has a strong sequential cluster point.} \tag{5.8}$$

Then $x_n \to \overline{x}$.

Proof. Fix $z \in C$ and set

$$(\forall n \in \mathbb{N}) \quad y_n = x_n + \lambda_n\big(T_{\mathrm{i}(n),n}x_n - x_n\big). \tag{5.9}$$

Appealing to (5.2) and the fact that, by virtue of (5.4), $z \in \bigcap_{i\in I} C_i = \bigcap_{n\in\mathbb{N}} \operatorname{Fix} T_{\mathrm{i}(n),n}$, we obtain,

$$\begin{aligned}(\forall n \in \mathbb{N}) \quad \|y_n - z\|_{W_n}^2 &\leqslant \|x_n - z\|_{W_n}^2 - \lambda_n(2-\lambda_n)\|T_{\mathrm{i}(n),n}x_n - x_n\|_{W_n}^2 \\ &\leqslant \|x_n - z\|_{W_n}^2 - \varepsilon^2\|T_{\mathrm{i}(n),n}x_n - x_n\|_{W_n}^2. \end{aligned} \tag{5.10}$$

Moreover, it follows from (5.3) that

$$(\forall n \in \mathbb{N}) \quad \|y_n - z\|_{W_{n+1}}^2 \leqslant (1+\eta_n)\|y_n - z\|_{W_n}^2. \tag{5.11}$$

Thus,

$$\begin{aligned}(\forall n \in \mathbb{N}) \quad \|y_n - z\|_{W_{n+1}}^2 &\leqslant (1+\eta_n)\|x_n - z\|_{W_n}^2 - \varepsilon^2(1+\eta_n)\|T_{\mathrm{i}(n),n}x_n - x_n\|_{W_n}^2 \\ &\leqslant (1+\eta_n)\|x_n - z\|_{W_n}^2 - \varepsilon^2\|T_{\mathrm{i}(n),n}x_n - x_n\|_{W_n}^2 \qquad (5.12) \\ &\leqslant (1+\eta_n)\|x_n - z\|_{W_n}^2. \qquad (5.13)\end{aligned}$$

Using (5.6), (5.9), and (5.13), we get

$$(\forall n\in\mathbb{N})\quad \begin{aligned}\|x_{n+1}-z\|_{W_{n+1}} &\leqslant \|y_n-z\|_{W_{n+1}}+\lambda_n\|a_n\|_{W_{n+1}}\\ &\leqslant \sqrt{1+\eta_n}\|x_n-z\|_{W_n}+\sqrt{\mu}\lambda_n\|a_n\|\\ &\leqslant (1+\eta_n)\|x_n-z\|_{W_n}+2\sqrt{\mu}\|a_n\|,\end{aligned}\tag{5.14}$$

which shows that

$$(x_n)_{n\in\mathbb{N}} \text{ satisfies (3.2) – and hence (3.1) – with } \phi=|\cdot|.\tag{5.15}$$

It follows from (5.15) and Proposition 3.2(i) that $(\|x_n-z\|_{W_n})_{n\in\mathbb{N}}$ converges, say

$$\|x_n-z\|_{W_n}\to\xi\in\mathbb{R}.\tag{5.16}$$

We therefore derive from (5.14) that $\|y_n-z\|_{W_{n+1}}\to\xi$ and then from (5.12) that

$$\alpha\varepsilon^2\|T_{\mathrm{i}(n),n}x_n-x_n\|^2\leqslant\varepsilon^2\|T_{\mathrm{i}(n),n}x_n-x_n\|^2_{W_n}\leqslant(1+\eta_n)\|x_n-z\|^2_{W_n}-\|y_n-z\|^2_{W_{n+1}}\to 0.\tag{5.17}$$

(i): It follows from (5.6) and (5.17) that

$$\begin{aligned}\|x_{n+1}-x_n\| &= \lambda_n\big\|T_{\mathrm{i}(n),n}x_n+a_n-x_n\big\|\\ &\leqslant 2\big(\|T_{\mathrm{i}(n),n}x_n-x_n\|+\|a_n\|\big)\\ &\leqslant 2\big(\|T_{\mathrm{i}(n),n}x_n-x_n\|_{W_n}/\sqrt{\alpha}+\|a_n\|\big)\\ &\to 0.\end{aligned}\tag{5.18}$$

Now, fix $j\in I$ and let x be a weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$. According to (5.4), there exist strictly increasing sequences $(k_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $x_{k_n}\rightharpoonup x$ and

$$(\forall n\in\mathbb{N})\quad \begin{cases}k_n\leqslant p_n\leqslant k_n+M_j-1<k_{n+1}\leqslant p_{n+1},\\ j=\mathrm{i}(p_n).\end{cases}\tag{5.19}$$

Therefore, we deduce from (5.18) that

$$\begin{aligned}\|x_{p_n}-x_{k_n}\| &\leqslant \sum_{l=k_n}^{k_n+M_j-2}\|x_{l+1}-x_l\|\\ &\leqslant (M_j-1)\max_{k_n\leqslant l\leqslant k_n+M_j-2}\|x_{l+1}-x_l\|\\ &\to 0,\end{aligned}\tag{5.20}$$

which implies that $x_{p_n}\rightharpoonup x$. We also derive from (5.17) and (5.19) that $T_{j,p_n}x_{p_n}-x_{p_n}=T_{\mathrm{i}(p_n),p_n}x_{p_n}-x_{p_n}\to 0$. Altogether, it follows from (5.7) that $x\in C_j$. Since j was arbitrarily chosen in I, we obtain $x\in C$ and, in view of Lemma 2.3(i) and Theorem 3.3, we conclude that $x_n\rightharpoonup x$.

(ii): Suppose that $z\in\operatorname{int}C$ and fix $\rho\in\left]0,+\infty\right[$ such that $B(z;\rho)\subset C$. Set $\eta=\sup_{n\in\mathbb{N}}\eta_n$, $\zeta=\sup_{x\in B(z;\rho)}\sup_{n\in\mathbb{N}}\|x_n-x\|_{W_n}$, and

$$(\forall n\in\mathbb{N})\quad \varepsilon_n=4\big(\zeta\sqrt{\mu(1+\eta)}\|a_n\|+\mu\|a_n\|^2\big).\tag{5.21}$$

Then $\eta < +\infty$ and, as in (4.19), $\zeta < +\infty$. Therefore $(\varepsilon_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})$. Furthermore, we derive from (5.6), (5.9), and (5.13) that, for every $x \in B(z;\rho)$ and every $n \in \mathbb{N}$,

$$\begin{aligned}\|x_{n+1} - x\|_{W_{n+1}}^2 &\leqslant \|y_n - x\|_{W_{n+1}}^2 + 2\lambda_n \|y_n - x\|_{W_{n+1}} \|a_n\|_{W_{n+1}} + \lambda_n^2 \|a_n\|_{W_{n+1}}^2 \\ &\leqslant (1+\eta_n)\|x_n - x\|_{W_n}^2 + 4\sqrt{\mu(1+\eta_n)}\|x_n - x\|_{W_n} \|a_n\| + 4\mu\|a_n\|^2 \\ &\leqslant (1+\eta_n)\|x_n - x\|_{W_n}^2 + \varepsilon_n. \end{aligned} \tag{5.22}$$

Altogether, the assertion follows from (i) and Proposition 4.3.

(iii): This follows from (5.15), Proposition 3.4, and (i).

(iv): Let $j \in I$ be an index of demicompact regularity and let $(p_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence such that $(\forall n \in \mathbb{N})$ $j = \mathrm{i}(p_n)$. Then $(x_{p_n})_{n\in\mathbb{N}}$ is bounded, while (5.17) asserts that $T_{j,p_n} x_{p_n} - x_{p_n} \to 0$. In turn, (5.8) and (i) imply that $x_{p_n} \to \overline{x} \in C$. Therefore $\underline{\lim}\, d_C(x_n) \leqslant \|x_{p_n} - \overline{x}\| \to 0$ and (iii) yields the result. ☐

Condition (5.4) first appeared in [9, Definition 5]. Property (5.7) was introduced in [2, Definition 3.7] and property (5.8) in [12, Definition 6.5]. Examples of sequences of operators that satisfy (5.7) can be found in [2, 6, 12]. Here is a simple application of Theorem 5.1 to a variable metric periodic projection method.

Corollary 5.2 *Let $\alpha \in \left]0,+\infty\right[$, let m be a strictly positive integer, let $I = \{1,\ldots,m\}$, let $(C_i)_{i\in I}$ be family of closed convex subsets of $\mathcal{H}$ such that $C = \bigcap_{i\in I} C_i \neq \varnothing$, let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that $\sum_{n\in\mathbb{N}} \|a_n\| < +\infty$, let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence in $\ell_+^1(\mathbb{N})$, and let $(W_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{H})$ such that $\sup_{n\in\mathbb{N}} \|W_n\| < +\infty$ and $(\forall n \in \mathbb{N})$ $(1+\eta_n)W_n \succcurlyeq W_{n+1}$. Fix $\varepsilon \in \left]0,1\right[$ and $x_0 \in \mathcal{H}$, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, 2-\varepsilon]$, and set*

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = x_n + \lambda_n \Big(P_{C_{1+\mathrm{rem}(n,m)}}^{W_n} x_n + a_n - x_n \Big), \tag{5.23}$$

where $\mathrm{rem}(\cdot, m)$ *is the remainder function of the division by m. Then the following hold for some $\overline{x} \in C$.*

(i) $x_n \rightharpoonup \overline{x}$.

(ii) *Suppose that* $\mathrm{int}\, C \neq \varnothing$ *and that there exists $(\nu_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})$ such that $(\forall n \in \mathbb{N})$ $(1+\nu_n)W_{n+1} \succcurlyeq W_n$. Then $x_n \to \overline{x}$.*

(iii) *Suppose that there exists $j \in I$ such that C_j is boundedly compact, i.e., its intersection with every closed ball of $\mathcal{H}$ is compact. Then $x_n \to \overline{x}$.*

Proof. The function $\mathrm{i}\colon \mathbb{N} \to I \colon n \mapsto 1+\mathrm{rem}(n,m)$ satisfies (5.4) with $(\forall j \in I)$ $M_j = m$. Now, set $(\forall i \in I)(\forall n \in \mathbb{N})$ $T_{i,n} = P_{C_i}^{W_n}$. Then $(\forall i \in I)(\forall n \in \mathbb{N})$ $T_{i,n} \in \mathfrak{T}(W_n)$ and $\mathrm{Fix}\, T_{i,n} = C_i$. Hence, (5.23) is a special case of (5.6).

(i)–(ii): Fix $j \in I$ and let $(x_{p_n})_{n\in\mathbb{N}}$ be a weakly convergent subsequence of $(x_n)_{n\in\mathbb{N}}$, say $x_{p_n} \rightharpoonup x$, such that $T_{j,p_n} x_{p_n} - x_{p_n} \to 0$ and $(\forall n \in \mathbb{N})$ $j = \mathrm{i}(p_n)$. Then $C_j \ni P_{C_j}^{W_{p_n}} x_{p_n} = T_{j,p_n} x_{p_n} \rightharpoonup x$ and, since C_j is weakly closed [5, Theorem 3.32], we have $x \in C_j$. This shows that (5.7) holds. Altogether, the claims follow from Theorem 5.1(i)–(ii).

(iii): Let $(p_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$ such that $P^{W_{p_n}}_{C_j}x_{p_n} - x_{p_n} = T_{j,p_n}x_{p_n} - x_{p_n} \to 0$ and $(\forall n \in \mathbb{N})\ j = \mathrm{i}(p_n)$. Then

$$\|P_{C_j}x_{p_n} - x_{p_n}\| \leqslant \|P^{W_{p_n}}_{C_j}x_{p_n} - x_{p_n}\| \to 0. \tag{5.24}$$

On the other hand, since $(x_{p_n})_{n\in\mathbb{N}}$ is bounded and P_{C_j} is nonexpansive, $(P_{C_j}x_{p_n})_{n\in\mathbb{N}}$ is a bounded sequence in the boundedly compact set C_j. Hence, $(P_{C_j}x_{p_n})_{n\in\mathbb{N}}$ admits a strong sequential cluster point and so does $(x_{p_n})_{n\in\mathbb{N}}$ since $P_{C_j}x_{p_n} - x_{p_n} \to 0$. Thus, $j \in I$ is an index of demicompact regularity and the claim therefore follows from Theorem 5.1(iv). ☐

Remark 5.3 In the special case when, for every $n \in \mathbb{N}$, $W_n = \mathrm{Id}$ and $\eta_n = 0$, Corollary 5.2(i) was established in [8] (with $(\forall n \in \mathbb{N})\ \lambda_n = 1$), and Corollary 5.2(ii) in [22].

Next is an application of Corollary 5.2 to the problem of solving linear inequalities. In Euclidean spaces, the use of periodic projection methods to solve this problem goes back to [27].

Example 5.4 Let $\alpha \in \left]0,+\infty\right[$, let m be a strictly positive integer, let $I = \{1,\ldots,m\}$, let $(\eta_i)_{i\in I}$ be real numbers, and suppose that $(u_i)_{i\in I}$ are nonzero vectors in $\mathcal{H}$ such that

$$C = \big\{x \in \mathcal{H} \mid (\forall i \in I)\ \ \langle x \mid u_i\rangle \leqslant \eta_i\big\} \neq \varnothing. \tag{5.25}$$

Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence in $\ell^1_+(\mathbb{N})$, and let $(W_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{H})$ such that $\sup_{n\in\mathbb{N}}\|W_n\| < +\infty$ and $(\forall n \in \mathbb{N})\ (1+\eta_n)W_n \succcurlyeq W_{n+1}$. Fix $\varepsilon \in \left]0,1\right[$ and $x_0 \in \mathcal{H}$, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, 2-\varepsilon]$, and set

$$(\forall n \in \mathbb{N})\quad \left\lfloor
\begin{array}{l}
\mathrm{i}(n) = 1 + \mathrm{rem}(n,m)\\
\text{if } \langle x_n \mid u_{\mathrm{i}(n)}\rangle \leqslant \eta_{\mathrm{i}(n)}\\
\left\lfloor\ y_n = x_n\right.\\
\text{if } \langle x_n \mid u_{\mathrm{i}(n)}\rangle > \eta_{\mathrm{i}(n)}\\
\left\lfloor\ y_n = x_n + \dfrac{\eta_{\mathrm{i}(n)} - \langle x_n \mid u_{\mathrm{i}(n)}\rangle}{\langle u_{\mathrm{i}(n)} \mid W_n^{-1}u_{\mathrm{i}(n)}\rangle} W_n^{-1}u_{\mathrm{i}(n)}\right.\\
x_{n+1} = x_n + \lambda_n(y_n - x_n).
\end{array}\right. \tag{5.26}$$

Then there exists $\overline{x} \in C$ such that $x_n \rightharpoonup \overline{x}$.

Proof. Set $(\forall i \in I)\ C_i = \big\{x \in \mathcal{H} \mid \langle x \mid u_i\rangle \leqslant \eta_i\big\}$. Then it follows from [5, Example 28.16(iii)] that (5.26) can be rewritten as

$$(\forall n \in \mathbb{N})\quad x_{n+1} = x_n + \lambda_n\Big(P^{W_n}_{C_{1+\mathrm{rem}(n,m)}}x_n - x_n\Big). \tag{5.27}$$

The claim is therefore a consequence of Corollary 5.2(i). ☐

We now turn our attention to the problem of finding a zero of a maximally monotone operator $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ (see [5] for background) via a variable metric proximal point algorithm. Let $\alpha \in \left]0,+\infty\right[$, let $\gamma \in \left]0,+\infty\right[$, let $W \in \mathcal{P}_\alpha(\mathcal{H})$, and let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone with graph $\operatorname{gra} A$. It follows from [3, Corollary 3.14(ii)] (applied with $f\colon x \mapsto \langle Wx \mid x\rangle/2$) that

$$J^W_{\gamma A}\colon \mathcal{H} \to \mathcal{H}\colon x \mapsto (W + \gamma A)^{-1}(Wx) \tag{5.28}$$

is well-defined, and that

$$J_{\gamma A}^{W} \in \mathfrak{T}(W) \quad \text{and} \quad \operatorname{Fix} J_{\gamma A}^{W} = \big\{ z \in \mathcal{H} \mid 0 \in Az \big\}. \tag{5.29}$$

We write $J_{\gamma A}^{\mathrm{Id}} = J_{\gamma A}$.

Corollary 5.5 *Let $\alpha \in \,]0,+\infty[$, let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a maximally monotone operator such that $C = \big\{ z \in \mathcal{H} \mid 0 \in Az \big\} \neq \varnothing$, let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that $\sum_{n\in\mathbb{N}} \|a_n\| < +\infty$, let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence in $\ell_+^1(\mathbb{N})$, and let $(W_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{H})$ such that $\mu = \sup_{n\in\mathbb{N}} \|W_n\| < +\infty$ and $(\forall n \in \mathbb{N})$ $(1+\eta_n)W_n \succcurlyeq W_{n+1}$. Fix $\varepsilon \in \,]0,1[$ and $x_0 \in \mathcal{H}$, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, 2-\varepsilon]$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon,+\infty[$, and set*

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = x_n + \lambda_n \Big(J_{\gamma_n A}^{W_n} x_n + a_n - x_n \Big). \tag{5.30}$$

Then the following hold for some $\overline{x} \in C$.

(i) $x_n \rightharpoonup \overline{x}$.

(ii) *Suppose that $\operatorname{int} C \neq \varnothing$ and that there exists $(\nu_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})$ such that $(\forall n \in \mathbb{N})$ $(1+\nu_n)W_{n+1} \succcurlyeq W_n$. Then $x_n \to \overline{x}$.*

(iii) *Suppose that A is pointwise uniformly monotone on C, i.e., for every $x \in C$ there exists an increasing function $\phi\colon\, [0,+\infty[\, \to [0,+\infty]$ vanishing only at 0 such that*

$$(\forall u \in Ax)(\forall (y,v) \in \operatorname{gra} A) \;\; \langle x - y \mid u - v \rangle \geqslant \phi(\|x-y\|). \tag{5.31}$$

Then $x_n \to \overline{x}$.

Proof. In view of (5.29), (5.30) is a special case of (5.6) with $I = \{1\}$ and $(\forall n \in \mathbb{N})$ $T_{1,n} = J_{\gamma_n A}^{W_n}$. Hence, using Theorem 5.1(i)–(ii), to show (i)–(ii), it suffices to prove that (5.7) holds. To this end, let $(x_{p_n})_{n\in\mathbb{N}}$ be a weakly convergent subsequence of $(x_n)_{n\in\mathbb{N}}$, say $x_{p_n} \rightharpoonup x$, such that $J_{\gamma_{p_n} A}^{W_{p_n}} x_{p_n} - x_{p_n} \to 0$. To show that $0 \in Ax$, let us set

$$(\forall n \in \mathbb{N}) \quad y_n = J_{\gamma_n A}^{W_n} x_n \quad \text{and} \quad v_n = \frac{1}{\gamma_n} W_n(x_n - y_n). \tag{5.32}$$

Then (5.28) yields $(\forall n \in \mathbb{N})$ $v_n \in Ay_n$. On the other hand, since $y_{p_n} - x_{p_n} \to 0$, we have

$$\|v_{p_n}\| = \frac{\|W_{p_n}(x_{p_n} - y_{p_n})\|}{\gamma_{p_n}} \leqslant \frac{\mu}{\varepsilon} \|x_{p_n} - y_{p_n}\| \to 0. \tag{5.33}$$

Thus, $y_{p_n} \rightharpoonup x$ and $Ay_{p_n} \ni v_{p_n} \to 0$. Since $\operatorname{gra} A$ is sequentially closed in $\mathcal{H}^{\mathrm{weak}} \times \mathcal{H}^{\mathrm{strong}}$ [5, Proposition 20.33(ii)], we conclude that $0 \in Ax$. Let us now show (iii). We have $0 \in A\overline{x}$ and $(\forall n \in \mathbb{N})$ $v_{p_n} \in Ay_{p_n}$. Hence, it follows from (5.31) that there exists an increasing function $\phi\colon\, [0,+\infty[\, \to [0,+\infty]$ vanishing only at 0 such that

$$(\forall n \in \mathbb{N}) \quad \langle y_{p_n} - \overline{x} \mid v_{p_n} \rangle \geqslant \phi(\|y_{p_n} - \overline{x}\|). \tag{5.34}$$

Since $v_{p_n} \to 0$, we get $\phi(\|y_{p_n} - \overline{x}\|) \to 0$ and, in turn, $\|y_{p_n} - \overline{x}\| \to 0$. It follows that $\|x_{p_n} - \overline{x}\| \to 0$ and hence that $\underline{\lim}\, d_C(x_n) = 0$. In view of Theorem 5.1(iii), we conclude that $x_n \to \overline{x}$. ☐

Remark 5.6 Corollary 5.5(i) reduces to the classical result of [34, Theorem 1] when $(\forall n \in \mathbb{N})$ $W_n = \mathrm{Id}$, $\eta_n = 0$, and $\lambda_n = 1$. In this context, Corollary 5.5(ii) appears in [28, Section 6]. In a finite-dimensional setting, an alternative variable metric proximal point algorithm is proposed in [29], which also uses the above conditions on $(W_n)_{n\in\mathbb{N}}$ but alternative error terms and relaxation parameters.

6 Application to inverse problems

In this section, we consider an application to a structured variational inverse problem. Henceforth, $\Gamma_0(\mathcal{H})$ denotes the class of proper lower semicontinuous convex functions from $\mathcal{H}$ to $\left]-\infty,+\infty\right]$.

Problem 6.1 Let $f \in \Gamma_0(\mathcal{H})$ and let I be a nonempty finite index set. For every $i \in I$, let $(\mathcal{G}_i, \|\cdot\|_i)$ be a real Hilbert space, let $L_i \colon \mathcal{H} \to \mathcal{G}_i$ be a nonzero bounded linear operator, let $r_i \in \mathcal{G}_i$, and let $\mu_i \in \left]0,+\infty\right[$. The problem is to

$$\underset{x\in\mathcal{H}}{\text{minimize}}\;\; f(x) + \frac{1}{2}\sum_{i\in I}\mu_i\|L_i x - r_i\|_i^2. \tag{6.1}$$

This formulation covers many inverse problems (see [17, Section 5] and the references therein) and it can be interpreted as follows: an ideal object $\widetilde{x} \in \mathcal{H}$ is to be recovered from noisy linear measurements $r_i = L_i\widetilde{x} + w_i \in \mathcal{G}_i$, where w_i represents noise ($i \in I$), and the function f penalizes the violation of prior information on $\widetilde{x}$. Thus, (6.1) attempts to strike a balance between the observation model, represented by the data fitting term $x \mapsto (1/2)\sum_{i\in I}\mu_i\|L_i x - r_i\|_i^2$, and a priori knowledge, represented by f. To solve this problem within our framework, we require the following facts.

Let $\alpha \in \left]0,+\infty\right[$, let $W \in \mathcal{P}_\alpha(\mathcal{H})$, and let $\varphi \in \Gamma_0(\mathcal{H})$. The proximity operator of φ relative to the metric induced by W is

$$\mathrm{prox}_\varphi^W \colon \mathcal{H} \to \mathcal{H} \colon x \mapsto \underset{y\in\mathcal{H}}{\mathrm{argmin}}\Big(\varphi(y) + \frac{1}{2}\|x-y\|_W^2\Big). \tag{6.2}$$

Now, let $\partial\varphi$ be the subdifferential of φ [5, Chapter 16]. Then, in connection with (5.28), $\partial\varphi$ is maximally monotone and we have [16, Section 3.3]

$$(\forall \gamma \in \left]0,+\infty\right[)\quad \mathrm{prox}_{\gamma\varphi}^W = J_{\gamma\partial\varphi}^W = (W + \gamma\partial\varphi)^{-1} \circ W. \tag{6.3}$$

We write $\mathrm{prox}_{\gamma\varphi}^{\mathrm{Id}} = \mathrm{prox}_{\gamma\varphi}$.

Lemma 6.2 *Let $A \colon \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone, let U be a nonzero operator in $\mathcal{P}_0(\mathcal{H})$, let $\gamma \in \left]0, 1/\|U\|\right[$, let $u \in \mathcal{H}$, set $W = \mathrm{Id} - \gamma U$, and set $B = A + U + \{u\}$. Then*

$$(\forall x \in \mathcal{H})\quad J_{\gamma B}^W x = J_{\gamma A}\big(Wx - \gamma u\big). \tag{6.4}$$

Proof. Since $U \in \mathcal{P}_0(\mathcal{H})$, U is maximally monotone [5, Example 20.29]. In turn, it follows from [5, Corollary 24.4(i)] that B is maximally monotone. Moreover, $W \in \mathcal{P}_\alpha(\mathcal{H})$, where $\alpha = 1 - \gamma\|U\|$. Now, let x and p be in $\mathcal{H}$. Then it follows from (5.28) that

$$p = J^W_{\gamma B} x \Leftrightarrow Wx \in Wp + \gamma Bp \Leftrightarrow Wx - \gamma u \in p + \gamma Ap \Leftrightarrow p = J_{\gamma A}\big(Wx - \gamma u\big), \tag{6.5}$$

which completes the proof. ☐

Proposition 6.3 *Let* $\varepsilon \in \left]0, 1/(1 + \sum_{i\in I} \mu_i\|L_i\|^2)\right[$, *let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{H}$ *such that* $\sum_{n\in\mathbb{N}} \|a_n\| < +\infty$, *let* $(\eta_n)_{n\in\mathbb{N}}$ *be a sequence in* $\ell^1_+(\mathbb{N})$, *and let* $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathbb{R}$ *such that*

$$(\forall n \in \mathbb{N}) \quad \varepsilon \leqslant \gamma_n \leqslant \frac{1-\varepsilon}{\sum_{i\in I} \mu_i\|L_i\|^2} \quad \textit{and} \quad (1+\eta_n)\gamma_n - \gamma_{n+1} \leqslant \frac{\eta_n}{\sum_{i\in I} \mu_i\|L_i\|^2}. \tag{6.6}$$

Furthermore, let C *be the set of solutions to Problem 6.1, let* $x_0 \in \mathcal{H}$, *let* $(\lambda_n)_{n\in\mathbb{N}}$ *be a sequence in* $[\varepsilon, 2-\varepsilon]$, *and set*

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = x_n + \lambda_n\Bigg(\operatorname{prox}_{\gamma_n f}\bigg(x_n + \gamma_n \sum_{i\in I} \mu_i L_i^*\big(r_i - L_i x_n\big)\bigg) + a_n - x_n\Bigg). \tag{6.7}$$

Then the following hold for some $\overline{x} \in C$.

(i) *Suppose that*

$$\lim_{\|x\|\to+\infty} f(x) + \frac{1}{2}\sum_{i\in I} \mu_i \|L_i x - r_i\|_i^2 = +\infty. \tag{6.8}$$

Then $x_n \rightharpoonup \overline{x}$.

(ii) *Suppose that there exists* $j \in I$ *such that* L_j *is bounded below, say,*

$$(\exists\, \beta \in \left]0, +\infty\right[)(\forall x \in \mathcal{H}) \quad \|L_j x\|_j \geqslant \beta\|x\|. \tag{6.9}$$

Then $C = \{\overline{x}\}$ *and* $x_n \to \overline{x}$.

Proof. Set $U = \sum_{i\in I} \mu_i L_i^* L_i$ and $u = -\sum_{i\in I} \mu_i L_i^* r_i$. Then

$$\|U\| \leqslant \sum_{i\in I} \mu_i \|L_i\|^2, \tag{6.10}$$

and the assumptions imply that $0 \neq U \in \mathcal{P}_0(\mathcal{H})$ and that $(\forall n \in \mathbb{N})\ \varepsilon \leqslant \gamma_n \leqslant (1-\varepsilon)/\|U\|$. Now set

$$g \colon \mathcal{H} \to \left]-\infty, +\infty\right] : x \mapsto f(x) + \frac{1}{2}\langle Ux \mid x\rangle + \langle x \mid u\rangle \tag{6.11}$$

and

$$(\forall n \in \mathbb{N}) \quad W_n = \operatorname{Id} - \gamma_n U. \tag{6.12}$$

Then (6.1) is equivalent to minimizing g. Furthermore, it follows from (6.6) that $(W_n)_{n\in\mathbb{N}}$ lies in $\mathcal{P}_\varepsilon(\mathcal{H})$ and that $\sup_{n\in\mathbb{N}} \|W_n\| \leqslant 2 - \varepsilon$. In addition, we have

$$(\forall n \in \mathbb{N}) \quad \eta_n \geqslant \big((1+\eta_n)\gamma_n - \gamma_{n+1}\big)\|U\|. \tag{6.13}$$

Indeed if, for some $n \in \mathbb{N}$, $(1+\eta_n)\gamma_n \leqslant \gamma_{n+1}$ then $\eta_n \geqslant 0 \geqslant ((1+\eta_n)\gamma_n - \gamma_{n+1})\|U\|$; otherwise we deduce from (6.6) and (6.10) that $\eta_n \geqslant ((1+\eta_n)\gamma_n - \gamma_{n+1}) \sum_{i\in I} \mu_i \|L_i\|^2 \geqslant ((1+\eta_n)\gamma_n - \gamma_{n+1})\|U\|$. Thus, since $U \in \mathcal{P}_0(\mathcal{H})$, we have $\|U\| = \sup_{\|x\|\leqslant 1}\langle Ux \mid x\rangle$ and therefore

$$\begin{aligned}
(6.13) \quad &\Rightarrow \quad (\forall n \in \mathbb{N})(\forall x \in \mathcal{H}) \quad \eta_n\|x\|^2 \geqslant \big((1+\eta_n)\gamma_n - \gamma_{n+1}\big)\langle Ux \mid x\rangle \\
&\Rightarrow \quad (\forall n \in \mathbb{N})(\forall x \in \mathcal{H}) \quad (1+\eta_n)(\|x\|^2 - \gamma_n\langle Ux \mid x\rangle) \geqslant \|x\|^2 - \gamma_{n+1}\langle Ux \mid x\rangle \\
&\Rightarrow \quad (\forall n \in \mathbb{N}) \quad (1+\eta_n)W_n \succcurlyeq W_{n+1}.
\end{aligned} \tag{6.14}$$

Now set $A = \partial f$ and $B = A + U + \{u\}$. Then we derive from [5, Corollary 16.38(iii)] that $B = \partial g$. Hence, using (6.3), (6.12), and Lemma 6.2, (6.7) can be rewritten as

$$\begin{aligned}
(\forall n \in \mathbb{N}) \quad x_{n+1} &= x_n + \lambda_n\Big(\operatorname{prox}_{\gamma_n f}\big(x_n - \gamma_n(Ux_n + u)\big) + a_n - x_n\Big) \\
&= x_n + \lambda_n\Big(J_{\gamma_n A}\big(W_n x_n - \gamma_n u\big) + a_n - x_n\Big) \\
&= x_n + \lambda_n\Big(J^{W_n}_{\gamma_n B} x_n + a_n - x_n\Big).
\end{aligned} \tag{6.15}$$

On the other hand, it follows from Fermat's rule [5, Theorem 16.2] that

$$\big\{z \in \mathcal{H} \mid 0 \in Bz\big\} = \operatorname{Argmin} g = C. \tag{6.16}$$

(i): Since $f \in \Gamma_0(\mathcal{H})$ and $U \in \mathcal{P}_0(\mathcal{H})$, it follows from [5, Proposition 11.14(i)] that Problem 6.1 admits at least one solution. Altogether, the result follows from Corollary 5.5(i).

(ii): It follows from (6.9) that $L_j^* L_j \in \mathcal{P}_{\beta^2}(\mathcal{H})$. Therefore, $U \in \mathcal{P}_{\mu_j\beta^2}(\mathcal{H})$ and, since $f \in \Gamma_0(\mathcal{H})$, we derive from (6.11) that $g \in \Gamma_0(\mathcal{H})$ is strongly convex. Hence, [5, Corollary 11.16] asserts that (6.1) possesses a unique solution, while [5, Example 22.3(iv)] asserts that B is strongly – hence uniformly – monotone. Altogether, the claim follows from Corollary 5.5(iii). ☐

Remark 6.4 In Problem 6.1 suppose that $I = \{1\}$, $\mu_1 = 1$, $L_1 = L$, and $r_1 = r$, and that $\lim_{\|x\|\to+\infty} f(x) + \|Lx - r\|_1^2/2 = +\infty$. Then (6.7) reduces to the proximal Landweber method

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = x_n + \lambda_n\Big(\operatorname{prox}_{\gamma_n f}\big(x_n + \gamma_n L^*(r - Lx_n)\big) + a_n - x_n\Big), \tag{6.17}$$

and we derive from Proposition 6.3(i) that $(x_n)_{n\in\mathbb{N}}$ converges weakly to a minimizer of $x \mapsto f(x) + \|Lx - r\|_1^2/2$ if

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \varepsilon \leqslant \gamma_n \leqslant (1-\varepsilon)/\|L\|^2 \\ (1+\eta_n)\gamma_n \leqslant \gamma_{n+1} + \eta_n/\|L\|^2 \\ \varepsilon \leqslant \lambda_n \leqslant 2 - \varepsilon. \end{cases} \tag{6.18}$$

This result complements [17, Theorem 5.5(i)], which establishes weak convergence under alternative conditions on the parameters $(\gamma_n)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$, namely

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \varepsilon \leqslant \gamma_n \leqslant (2-\varepsilon)/\|L\|^2 \\ \varepsilon \leqslant \lambda_n \leqslant 1. \end{cases} \tag{6.19}$$

In particular, suppose that $\mathcal{H}$ is separable, let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$, and set $f\colon x \mapsto \sum_{k\in\mathbb{N}} \phi_k(\langle x \mid e_k\rangle)$, where $(\forall k \in \mathbb{N})$ $\Gamma_0(\mathbb{R}) \ni \phi_k \geqslant \phi_k(0) = 0$. Moreover, for every $n \in \mathbb{N}$, let $(\alpha_{n,k})_{k\in\mathbb{N}}$ be a sequence in $\ell^2(\mathbb{N})$ and suppose that $\sum_{n\in\mathbb{N}}\sqrt{\sum_{k\in\mathbb{N}}|\alpha_{n,k}|^2} < +\infty$. Now set $(\forall n\in\mathbb{N})$ $a_n = \sum_{k\in\mathbb{N}}\alpha_{n,k}e_k$. Then, arguing as in [17, Section 5.4], (6.17) becomes

$$(\forall n\in\mathbb{N})\quad x_{n+1} = x_n + \lambda_n\Bigg(\sum_{k\in\mathbb{N}}\big(\alpha_{n,k} + \operatorname{prox}_{\gamma_n\phi_k}\langle x_n + \gamma_n L^*(r - Lx_n)\mid e_k\rangle\big)e_k - x_n\Bigg), \tag{6.20}$$

and we obtain convergence under the new condition (6.18) (see also [15] for potential signal and image processing applications of this result).

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