A Strongly Convergent Primal-Dual Method for Nonoverlapping Domain Decomposition*

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Abstract

We propose a primal-dual parallel proximal splitting method for solving domain decomposition problems for partial differential equations. The problem is formulated via minimization of energy functions on the subdomains with coupling constraints which model various properties of the solution at the interfaces. The proposed method can handle a wide range of linear and nonlinear problems, with flexible, possibly nonlinear, transmission conditions across the interfaces. Strong convergence in the energy spaces is established in this general setting, and without any additional assumption on the energy functions or the geometry of the problem. Several examples are presented.

Keywords: domain decomposition for PDE's, obstacle problem, p-Laplacian, parallel splitting algorithm, primal-dual algorithm, proximal algorithm, Poisson problem, structured convex minimization methods, transmission condition.

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1 Introduction

One of the main objectives of domain decomposition is to solve partial differential equations and the associated boundary value problems on complex geometries by partitioning the original domain in smaller and simpler subdomains [10, 13, 18, 32, 34, 38, 40]. The objective of the present paper is to propose an original algorithm for solving variational formulations associated with partial differential equations posed on partitioned domains. Our analysis pertains to nonoverlapping domain decompositions, in which subdomains intersect only on their interfaces. The original domain Ω is partitioned into m subdomains $(\Omega_i)_{i\in I}$, the interface between two subdomains Ω_i and Ω_j is denoted by Υ_{ij}, and Υ_{ii} stands for the part of the boundary of Ω_i shared with the boundary of Ω (see Fig. 1, where $I=\{1,\ldots,m\}$).

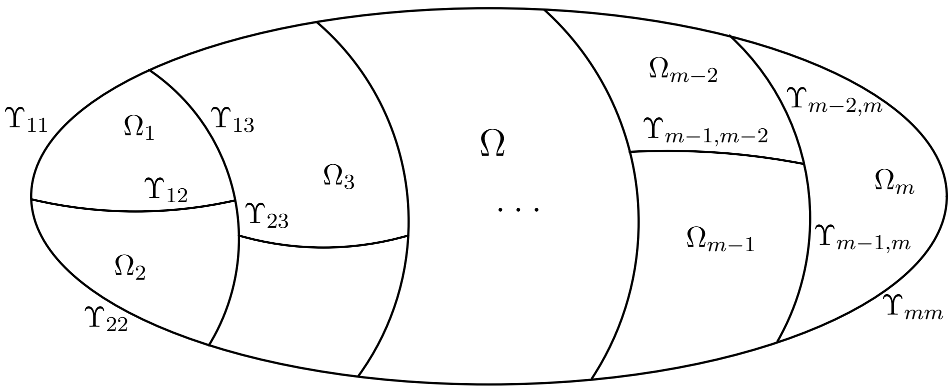

Figure 1: Decomposition of the domain Ω.

A sizable literature has been devoted to variational domain decomposition; see for instance [3, 5, 9, 10, 13, 17, 26, 28, 38, 40]. The novelty of our framework is to allow for the use of several subdomains with general convex energy functions on each of them, together with a broad range of transmission conditions on interfaces. More specifically, in our model the ith variable u_i lies in a suitable Sobolev space $\mathcal{H}_i$ and the structured minimization problem under consideration assumes the form

$$\underset{(u_i)_{i\in I}\in\bigoplus_{i\in I}\mathcal{H}_i}{\text{minimize}}\quad \sum_{i\in I}\varphi_i(u_i)+\sum_{(i,j)\in K}\psi_{ij}(\mathsf{T}_{ij}\,u_i-\mathsf{T}_{ji}\,u_j), \tag{1.1}$$

where K is the set indices of active interfaces, $\mathsf{T}_{ij}\colon\mathcal{H}_i\to L^2(\Upsilon_{ij})$ denotes the trace operator relative to the interface Υ_{ij}, and $\varphi_i\colon\mathcal{H}_i\to\,]-\infty,+\infty]$ and $\psi_{ij}\colon L^2(\Upsilon_{ij})\to\,]-\infty,+\infty]$ are lower semicontinuous convex functions. In applications, one is often interested in solving the Fenchel-Rockafellar dual problem associated with (1.1), the solutions of which model tensions (e.g., stresses or fluxes) at the interfaces. There are two main components in (1.1). The first component is the separable function $(u_i)_{i\in I}\mapsto\sum_{i\in I}\varphi_i(u_i)$ which incorporates the internal energy functions $(\varphi_i)_{i\in I}$ on each subdomain. The other component is a coupling term which models transmission conditions across the interfaces. Since the separable term needs not be smooth and may take on the value $+\infty$, hard constraints on $(u_i)_{i\in I}$ can be imposed in our formulation. It can also deal with non quadratic functions, capturing, for instance, p-Laplacian or obstacle problems.

On the other hand, the coupling function models transmission conditions, in particular continuity, through the interfaces. A major advantage of this approach is its flexibility, which makes it possible to treat in a unified fashion unilateral and/or nonlinear transmission conditions.

To solve (1.1) and its dual, we bring into play a multivariate primal-dual proximal splitting method recently proposed in [2] for structured convex minimization problems. The algorithm generates both primal and dual sequences which converge strongly to the unique solution satisfying the Kuhn-Tucker conditions, and lying closest to some initial point. At each iteration an outer approximation to the Kuhn-Tucker set is constructed as the intersection of two half-spaces, and the update is obtained by projecting the initial point onto this intersection. This method will be adapted to solve the variational problem (1.1) in a fully split fashion, in that each elementary step of the algorithm involves the constituents of the problem (namely u_i, φ_i, ψ_{ij}, and T_{ij}) separately. In addition, its structure lends it to implementations on parallel architectures. Let us note that typically, Lagrangian-based approaches [8, 28] do not achieve full splitting with respect to the linear operators, which complicates the numerical implementation and may require additional restrictions on these linear operators to ensure convergence. Another salient advantage of the proposed algorithm that distinguishes it from Lagrangian-based approaches as well as from splitting algorithms which could be considered for solving (1.1), such as those of [14, 16, 20, 21, 22, 41], is that these methods provide only weak convergence. In addition, the methods of [14, 16, 20, 21, 22, 41] require the computation of bounds on the range of certain parameters. In the case of (1.1), these bounds involve norms of combinations of trace operators, which are very hard to estimate. Altogether, the proposed algorithm provides significant advantages over the state of the art.

The paper is organized as follows. In Section 2, we present the notation and the abstract primal-dual splitting algorithm which is the basis of our method. In Section 3, we formally state the domain decomposition problem under investigation, define the functional setting, and introduce the main algorithm. Section 4 is devoted to applications to concrete domain decomposition problems. Finally, in Section 5, we briefly discuss some adaptations of our setting to other interesting problems.

2 Notation and preliminaries

Let $\mathcal{B}$ be a real Banach space. Weak and strong convergence in $\mathcal{B}$ are denoted by $\overset{\mathcal{B}}{\rightharpoonup}$ and $\overset{\mathcal{B}}{\to}$, respectively, and $\Gamma_0(\mathcal{B})$ is the class of lower semicontinuous convex functions $\varphi\colon \mathcal{B} \to \left]-\infty,+\infty\right]$ which are not identically equal to $+\infty$. A function $\varphi\colon \mathcal{B} \to \left]-\infty,+\infty\right]$ is coercive if $\lim_{\|u\|\to+\infty} \varphi(u) = +\infty$. The Hilbert direct sum of a finite family of Hilbert spaces $(\mathcal{H}_i)_{i\in I}$ is denoted by $\bigoplus_{i\in I} \mathcal{H}_i$.

$\mathbb{R}^N$ denotes the usual N-dimensional Euclidean space and $|\cdot|$ its norm. Let Ω be a nonempty open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary $\operatorname{bdry}\Omega$. We denote by x a generic element of Ω, and by dx the restriction to Ω of the Lebesgue measure on $\mathbb{R}^N$. All the functional spaces considered throughout the paper involve real-valued functions. For every $p \in \left]1,+\infty\right[$, $W^{1,p}(\Omega) = \big\{v \in L^p(\Omega) \;\big|\; Dv \in (L^p(\Omega))^N\big\}$, where D denotes the weak gradient (derivatives in the sense of distributions). In particular, we set $H^1(\Omega) = W^{1,2}(\Omega)$, which is a Hilbert space with scalar product $\langle\cdot \mid \cdot\rangle_{H^1(\Omega)}\colon (u,v) \mapsto \int_\Omega uv + \int_\Omega (Du)^\top Dv$. We denote by S the surface measure on $\operatorname{bdry}\Omega$ [37, Section 1.1.3]. Now let Υ be a nonempty open subset of $\text{bdry}\,\Omega$ and let $L^2(\Upsilon)$ be the space of square S-integrable functions on Υ. Endowed with the scalar product $(v,w) \mapsto \int_\Upsilon vw\,dS$, $L^2(\Upsilon)$ is a Hilbert space. The Sobolev trace operator $\mathsf{T}\colon H^1(\Omega) \to L^2(\text{bdry}\,\Omega)$ is the unique bounded linear operator such that $(\forall v \in \mathcal{C}^1(\overline{\Omega}))\ \mathsf{T}v = v|_{\text{bdry}\,\Omega}$. Endowed with the scalar product

$$\langle \cdot \mid \cdot \rangle \colon (u,v) \mapsto \int_\Omega (Du)^\top Dv, \tag{2.1}$$

the space $H^1_{0,\Upsilon}(\Omega) = \big\{u \in H^1(\Omega) \;\big|\; \mathsf{T}\,u = 0 \ \text{ on}\,\Upsilon\big\}$ is a Hilbert space [44, Section 25.10]. For every $\alpha \in \left]0,1\right]$, $\mathcal{C}^{1,\alpha}(\overline{\Omega})$ is the subspace of $\mathcal{C}^1(\overline{\Omega})$ consisting of those functions u such that

$$(\exists\,\mu \in \left]0,+\infty\right[)(\forall (x,y) \in \Omega^2) \quad |u(x)-u(y)| \leqslant \mu|x-y|^\alpha \quad \text{and} \quad |Du(x)-Du(y)| \leqslant \mu|x-y|^\alpha. \tag{2.2}$$

Finally, for S-almost every $\omega \in \text{bdry}\,\Omega$, there exists a unit outward normal vector $\nu(\omega)$. For details and complements, see [1, 4, 23, 29, 37, 43, 44].

Let $\mathcal{H}$ be a real Hilbert space with scalar product $\langle \cdot \mid \cdot \rangle$ and associated norm $\|\cdot\|$, and let $\varphi \in \Gamma_0(\mathcal{H})$. The subdifferential of φ is

$$\partial\varphi\colon \mathcal{H} \to 2^{\mathcal{H}}\colon u \mapsto \big\{u^* \in \mathcal{H} \;\big|\; (\forall v \in \mathcal{H})\ \varphi(u) + \langle v-u \mid u^*\rangle \leqslant \varphi(v)\big\}, \tag{2.3}$$

the conjugate of φ is the function $\varphi^* \in \Gamma_0(\mathcal{H})$ defined by

$$\varphi^*\colon u^* \mapsto \sup_{u\in\mathcal{H}} \big(\langle u \mid u^*\rangle - \varphi(u)\big), \tag{2.4}$$

and the proximity operator of $\varphi \in \Gamma_0(\mathcal{H})$ is [36]

$$\text{prox}_\varphi\colon \mathcal{H} \to \mathcal{H}\colon u \mapsto \underset{v\in\mathcal{H}}{\text{argmin}} \left(\varphi(v) + \frac{1}{2}\|u-v\|^2\right). \tag{2.5}$$

Let C be a nonempty closed convex subset of $\mathcal{H}$. The indicator function of C is

$$\iota_C\colon \mathcal{H} \to \left]-\infty,+\infty\right]\colon u \mapsto \begin{cases} 0, & \text{if } \ u \in C;\\ +\infty, & \text{otherwise,} \end{cases} \tag{2.6}$$

and the projection (or best approximation) operator onto C is

$$P_C = \text{prox}_{\iota_C}\colon \mathcal{H} \to C\colon u \mapsto \underset{v\in C}{\text{argmin}}\ \|u-v\|. \tag{2.7}$$

For background on convex analysis in Hilbert spaces the reader is referred to [11].

The backbone of our model will be the following abstract primal-dual saddle problem.

Problem 2.1 Let I and K be nonempty finite index sets, and let $(\mathcal{H}_i)_{i\in I}$ and $(\mathcal{G}_k)_{k\in K}$ be real Hilbert spaces. For every $i \in I$ and $k \in K$, let $\Phi_i \in \Gamma_0(\mathcal{H}_i)$, let $\Psi_k \in \Gamma_0(\mathcal{G}_k)$, let $\Lambda_{ki} \colon \mathcal{H}_i \to \mathcal{G}_k$ be a bounded linear operator, and let $\Lambda_{ki}^* \colon \mathcal{G}_k \to \mathcal{H}_i$ be its adjoint. It is assumed that

$$(\forall i \in I) \quad 0 \in \operatorname{range}\Bigg(\partial\Phi_i + \sum_{k\in K} \Lambda_{ki}^* \circ (\partial\Psi_k) \circ \sum_{j\in I} \Lambda_{kj}\Bigg). \tag{2.8}$$

Let $\boldsymbol{u}_0 = (u_{i,0})_{i\in I} \in \boldsymbol{\mathcal{H}} = \bigoplus_{i\in I} \mathcal{H}_i$ and let $\boldsymbol{w}_0 = (w_{k,0})_{k\in K} \in \boldsymbol{\mathcal{G}} = \bigoplus_{k\in K} \mathcal{G}_k$. The problem is to find the best approximation in $\boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}}$ to $(\boldsymbol{u}_0, \boldsymbol{w}_0)$ from the Kuhn-Tucker set

$$\boldsymbol{Z} = \Bigg\{ \boldsymbol{u} = (u_i)_{i\in I} \in \boldsymbol{\mathcal{H}},\ \boldsymbol{w} = (w_k)_{k\in K} \in \boldsymbol{\mathcal{G}} \;\Bigg|\; (\forall i \in I) \quad -\sum_{k\in K} \Lambda_{ki}^* w_k \in \partial\Phi_i(u_i) \\ \text{and} \quad (\forall k \in K) \quad \sum_{i\in I} \Lambda_{ki} u_i \in \partial\Psi_k^*(w_k) \Bigg\}. \tag{2.9}$$

Proposition 2.2 *Problem 2.1 has a unique solution $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{w}})$. Moreover, $\overline{\boldsymbol{u}} = (\overline{u}_i)_{i\in I}$ solves the primal problem*

$$\underset{(u_i)_{i\in I} \in \bigoplus_{i\in I} \mathcal{H}_i}{\text{minimize}} \quad \sum_{i\in I} \Phi_i(u_i) + \sum_{k\in K} \Psi_k\Bigg(\sum_{i\in I} \Lambda_{ki} u_i\Bigg), \tag{2.10}$$

and $\overline{\boldsymbol{w}} = (\overline{w}_k)_{k\in K}$ solves the dual problem

$$\underset{(w_k)_{k\in K} \in \bigoplus_{k\in K} \mathcal{G}_k}{\text{minimize}} \quad \sum_{i\in I} \Phi_i^*\Bigg(-\sum_{k\in K} \Lambda_{ki}^* w_k\Bigg) + \sum_{k\in K} \Psi_k^*(w_k). \tag{2.11}$$

Proof. Since $\boldsymbol{Z}$ in (2.9) is nonempty, closed, and convex [16, Proposition 2.8], the projection $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{w}})$ of $(\boldsymbol{u}_0, \boldsymbol{w}_0)$ onto $\boldsymbol{Z}$ is uniquely defined. The remaining claims follow from [2, Corollary 4.5(i)]. ☐

To solve Problem 2.1, we shall use the following splitting algorithm from [2]. This algorithm generates a sequence $(\boldsymbol{u}_n, \boldsymbol{w}_n)_{n\in\mathbb{N}}$ that converges strongly to the unique solution to Problem 2.1. It exploits a convergence principle that goes back in its simplest form to the work of Haugazeau [30] (see [19] for historical comments). Let us note that existing methods for solving (2.10)–(2.11) [14, 16, 20, 21, 22, 41] guarantee only weak convergence to an unspecified primal-dual solution and, in addition, require the knowledge of bounds on certain compositions of the linear operators involved in the model. In our setting, such bounds would be extremely hard to obtain. Moreover, the proposed method solves Problem 2.1 in a fully split fashion in that each elementary step of the algorithm activates the functions and operators of the problem separately.

In geometrical terms, the algorithm is executed as follows [2] (see Fig. 2). Set $\boldsymbol{x}_0 = (\boldsymbol{u}_0, \boldsymbol{w}_0)$ and, given two points $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\boldsymbol{\mathcal{K}} = \boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}}$, denote by $H(\boldsymbol{a}, \boldsymbol{b})$ the closed affine half-space of $\boldsymbol{\mathcal{K}}$ onto which $\boldsymbol{b}$ is the projection of $\boldsymbol{a}$. At iteration n, the current iterate is $\boldsymbol{x}_n = \big((u_{i,n})_{i\in I}, (w_{k,n})_{k\in K}\big) \in \boldsymbol{\mathcal{K}}$ and we find $\boldsymbol{x}_{n+1/2} = \big((u_{i,n+1/2})_{i\in I}, (w_{k,n+1/2})_{k\in K}\big) \in \boldsymbol{\mathcal{K}}$ such that

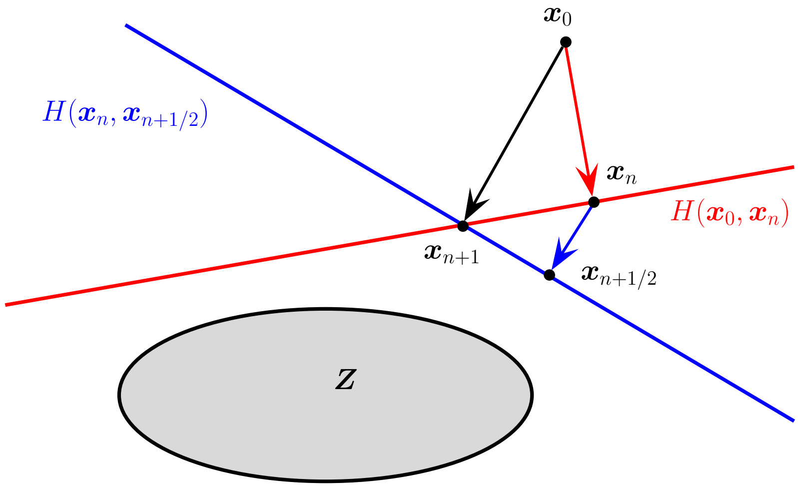

Figure 2: Geometrical interpretation of a generic iteration of (2.14) for computing the projection of $\boldsymbol{x}_0$ onto the Kuhn-Tucker set $\boldsymbol{Z}$ in the primal-dual space $\boldsymbol{\mathcal{K}}$. At iteration n, the current iterate is $\boldsymbol{x}_n$ and $\boldsymbol{Z}$ is contained in the half-space $H(\boldsymbol{x}_0, \boldsymbol{x}_n)$ onto which $\boldsymbol{x}_n$ is the projection of $\boldsymbol{x}_0$. A point $\boldsymbol{x}_{n+1/2}$ is constructed so that the half-space $H(\boldsymbol{x}_n, \boldsymbol{x}_{n+1/2})$ contains $\boldsymbol{Z}$. The update $\boldsymbol{x}_{n+1}$ is the projection of $\boldsymbol{x}_0$ onto $H(\boldsymbol{x}_0, \boldsymbol{x}_n) \cap H(\boldsymbol{x}_n, \boldsymbol{x}_{n+1/2})$.

$\boldsymbol{Z} \subset H(\boldsymbol{x}_n, \boldsymbol{x}_{n+1/2})$. The computation of $\boldsymbol{x}_{n+1/2}$ involves proximal steps with respect to the functions $(\Phi_i)_{i\in I}$ and $(\Psi_k)_{k\in K}$, as well as applications of the linear operators $(\Lambda_{ki})_{i\in I,k\in K}$ and their adjoints. The update $\boldsymbol{x}_{n+1} = \big((u_{i,n+1})_{i\in I}, (w_{k,n+1})_{k\in K}\big)$ is then obtained as the projection of $\boldsymbol{x}_0$ onto $H(\boldsymbol{x}_0, \boldsymbol{x}_n) \cap H(\boldsymbol{x}_n, \boldsymbol{x}_{n+1/2})$, which can be computed explicitly in terms of $(\boldsymbol{x}_0, \boldsymbol{x}_n, \boldsymbol{x}_{n+1/2})$ as [11, Corollary 28.21]

$$\boldsymbol{x}_{n+1} = \begin{cases} \boldsymbol{x}_{n+1/2}, & \text{if } \rho_n = 0 \text{ and } \chi_n \geqslant 0; \\ \boldsymbol{x}_0 + (1 + \chi_n/\nu_n)(\boldsymbol{x}_{n+1/2} - \boldsymbol{x}_n), & \text{if } \rho_n > 0 \text{ and } \chi_n\nu_n \geqslant \rho_n; \\ \boldsymbol{x}_n + (\nu_n/\rho_n)\big(\chi_n(\boldsymbol{x}_0 - \boldsymbol{x}_n) + \mu_n(\boldsymbol{x}_{n+1/2} - \boldsymbol{x}_n)\big), & \text{if } \rho_n > 0 \text{ and } \chi_n\nu_n < \rho_n, \end{cases} \tag{2.12}$$

where

$$\begin{cases} \chi_n = \big\langle \boldsymbol{x}_0 - \boldsymbol{x}_n \mid \boldsymbol{x}_n - \boldsymbol{x}_{n+1/2} \big\rangle \\ \mu_n = \|\boldsymbol{x}_0 - \boldsymbol{x}_n\|^2 \\ \nu_n = \|\boldsymbol{x}_n - \boldsymbol{x}_{n+1/2}\|^2 \\ \rho_n = \mu_n\nu_n - \chi_n^2. \end{cases} \tag{2.13}$$

The sequence $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ thus constructed converges strongly to $P_{\boldsymbol{Z}}\boldsymbol{x}_0$.

Theorem 2.3 [2, Corollary 4.5(ii)–(iii)] *Let* $\varepsilon \in]0,1[$, *let* $(\gamma_n)_{n\in\mathbb{N}}$ *and* $(\mu_n)_{n\in\mathbb{N}}$ *be sequences in* $[\varepsilon, 1/\varepsilon]$, *let* $(\lambda_n)_{n\in\mathbb{N}}$ *be a sequence in* $[\varepsilon, 1]$, *and iterate*

$$
\begin{array}{l}
\textit{for } n = 0,1,\ldots \\
\left\lfloor
\begin{array}{l}
\textit{for every } i \in I \\
\left\lfloor
\begin{array}{l}
v_{i,n} = u_{i,n} - \gamma_n \sum_{k\in K} \Lambda_{ki}^* w_{k,n} \\
p_{i,n} = \text{prox}_{\gamma_n \Phi_i} v_{i,n}
\end{array}
\right. \\
\textit{for every } k \in K \\
\left\lfloor
\begin{array}{l}
l_{k,n} = \sum_{i\in I} \Lambda_{ki} u_{i,n} \\
q_{k,n} = \text{prox}_{\mu_n \Psi_k} \big(l_{k,n} + \mu_n w_{k,n}\big) \\
t_{k,n} = q_{k,n} - \sum_{i\in I} \Lambda_{ki} p_{i,n}
\end{array}
\right. \\
\textit{for every } i \in I \\
\left\lfloor\; s_{i,n} = \gamma_n^{-1}(u_{i,n} - p_{i,n}) + \mu_n^{-1} \sum_{k\in K} \Lambda_{ki}^* (l_{k,n} - q_{k,n}) \right. \\
\tau_n = \sum_{i\in I} \|s_{i,n}\|^2 + \sum_{k\in K} \|t_{k,n}\|^2 \\
\textit{if } \tau_n = 0 \\
\left\lfloor\; \theta_n = 0 \right. \\
\textit{if } \tau_n > 0 \\
\left\lfloor\; \theta_n = \lambda_n \big(\gamma_n^{-1} \sum_{i\in I} \|u_{i,n} - p_{i,n}\|^2 + \mu_n^{-1} \sum_{k\in K} \|l_{k,n} - q_{k,n}\|^2\big)/\tau_n \right. \\
\textit{for every } i \in I \\
\left\lfloor\; u_{i,n+1/2} = u_{i,n} - \theta_n s_{i,n} \right. \\
\textit{for every } k \in K \\
\left\lfloor\; w_{k,n+1/2} = w_{k,n} - \theta_n t_{k,n} \right. \\
\chi_n = \sum_{i\in I} \big\langle u_{i,0} - u_{i,n} \mid u_{i,n} - u_{i,n+1/2} \big\rangle + \sum_{k\in K} \big\langle w_{k,0} - w_{k,n} \mid w_{k,n} - w_{k,n+1/2} \big\rangle \\
\mu_n = \sum_{i\in I} \|u_{i,0} - u_{i,n}\|^2 + \sum_{k\in K} \|w_{k,0} - w_{k,n}\|^2 \\
\nu_n = \sum_{i\in I} \|u_{i,n} - u_{i,n+1/2}\|^2 + \sum_{k\in K} \|w_{k,n} - w_{k,n+1/2}\|^2 \\
\rho_n = \mu_n \nu_n - \chi_n^2 \\
\textit{if } \rho_n = 0 \textit{ and } \chi_n \geqslant 0 \\
\left\lfloor
\begin{array}{l}
\textit{for every } i \in I \\
\left\lfloor\; u_{i,n+1} = u_{i,n+1/2} \right. \\
\textit{for every } k \in K \\
\left\lfloor\; w_{k,n+1} = w_{k,n+1/2} \right.
\end{array}
\right. \\
\textit{if } \rho_n > 0 \textit{ and } \chi_n \nu_n \geqslant \rho_n \\
\left\lfloor
\begin{array}{l}
\textit{for every } i \in I \\
\left\lfloor\; u_{i,n+1} = u_{i,0} + (1 + \chi_n/\nu_n)(u_{i,n+1/2} - u_{i,n}) \right. \\
\textit{for every } k \in K \\
\left\lfloor\; w_{k,n+1} = w_{k,0} + (1 + \chi_n/\nu_n)(w_{k,n+1/2} - w_{k,n}) \right.
\end{array}
\right. \\
\textit{if } \rho_n > 0 \textit{ and } \chi_n \nu_n < \rho_n \\
\left\lfloor
\begin{array}{l}
\textit{for every } i \in I \\
\left\lfloor\; u_{i,n+1} = u_{i,n} + (\nu_n/\rho_n)\big(\chi_n(u_{i,0} - u_{i,n}) + \mu_n(u_{i,n+1/2} - u_{i,n})\big) \right. \\
\textit{for every } k \in K \\
\left\lfloor\; w_{k,n+1} = w_{k,n} + (\nu_n/\rho_n)\big(\chi_n(w_{k,0} - w_{k,n}) + \mu_n(w_{k,n+1/2} - w_{k,n})\big). \right.
\end{array}
\right.
\end{array}
\right.
\end{array}
\tag{2.14}
$$

Then, for every $i \in I$ *and every* $k \in K$, (2.14) *generates infinite sequences* $(u_{i,n})_{n\in\mathbb{N}}$ *and* $(w_{k,n})_{n\in\mathbb{N}}$ *such that* $u_{i,n} \xrightarrow{\mathcal{H}_i} \overline{u}_i$ *and* $w_{k,n} \xrightarrow{\mathcal{G}_k} \overline{w}_k$.

3 Problem formulation and algorithm

The problem under consideration is the following.

Problem 3.1 Let Ω be a nonempty open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary $\operatorname{bdry}\Omega$, let $m \geqslant 2$ be an integer, and set $I = \{1, \ldots, m\}$. Suppose that the following hold:

(i) $(\Omega_i)_{i\in I}$ are disjoint open subsets of Ω (see Fig. 1) with Lipschitz boundaries $(\operatorname{bdry}\Omega_i)_{i\in I}$, $\overline{\Omega} = \bigcup_{i\in I}\overline{\Omega_i}$, and

$$(\forall i \in I) \quad \Upsilon_{ii} = \operatorname{int}_{\operatorname{bdry}\Omega}(\operatorname{bdry}\Omega_i \cap \operatorname{bdry}\Omega) \neq \varnothing, \tag{3.1}$$

where $\operatorname{int}_{\operatorname{bdry}\Omega}$ denotes the interior relative to $\operatorname{bdry}\Omega$.

(ii) For every $i \in I$,

$$J(i) = \big\{ j \in I \smallsetminus \{i\} \;\big|\; \Upsilon_{ij} \neq \varnothing \big\} \neq \varnothing, \tag{3.2}$$

where

$$(\forall j \in \{i+1, \ldots, m\}) \quad \Upsilon_{ij} = \Upsilon_{ji} = \operatorname{int}_{\operatorname{bdry}\Omega_i}(\operatorname{bdry}\Omega_i \cap \operatorname{bdry}\Omega_j). \tag{3.3}$$

Moreover, $J(i-) = J(i)\cap\{1, \ldots, i-1\}$ and $J(i+) = J(i)\cap\{i+1, \ldots, m\}$, with the convention $J(1-) = J(m+) = \varnothing$.

(iii) The set of indices of interfaces is

$$K = \big\{ (i,j) \;\big|\; i \in \{1, \ldots, m-1\} \;\text{ and }\; j \in J(i+) \big\}. \tag{3.4}$$

(iv) For every $i \in I$, $\mathsf{T}_i \colon H^1(\Omega_i) \to L^2(\operatorname{bdry}\Omega_i)$ is the trace operator. Moreover,

$$\mathcal{H}_i = H^1_{0,\Upsilon_{ii}}(\Omega_i) = \big\{ u \in H^1(\Omega_i) \;\big|\; \mathsf{T}_i\, u = 0 \;\text{ on } \Upsilon_{ii} \big\}, \tag{3.5}$$

endowed with the scalar product

$$\langle u \mid v \rangle = \int_{\Omega_i} (Du)^\top Dv, \tag{3.6}$$

is a Hilbert space, and, for every $j \in J(i)$, $\mathsf{T}_{ij} \colon \mathcal{H}_i \to L^2(\Upsilon_{ij}) \colon u \mapsto (\mathsf{T}_i u)|_{\Upsilon_{ij}}$.

(v) For every $i \in I$,

$$\boldsymbol{\mathcal{G}}_i = \bigoplus_{j\in J(i)} L^2(\Upsilon_{ij}), \tag{3.7}$$

$\nu_i(\omega)$ is the unit outward normal vector at $\omega \in \operatorname{bdry}\Omega_i$, and

$$Q_i \colon L^2(\Omega_i) \times \boldsymbol{\mathcal{G}}_i \to \mathcal{H}_i \tag{3.8}$$

is the operator that maps every $(f, (h_j)_{j\in J(i)})$ in $L^2(\Omega_i) \times \boldsymbol{\mathcal{G}}_i$ into the weak solution in $\mathcal{H}_i$ of the Dirichlet-Neumann boundary value problem

$$\begin{cases} -\Delta u = f & \text{on } \Omega_i, \\ u = 0 & \text{on } \Upsilon_{ii}, \\ \nu_i^\top Du = h_j & \text{on } \Upsilon_{ij}, \;\text{ for every } j \in J(i+), \\ \nu_i^\top Du = -h_j & \text{on } \Upsilon_{ij}, \;\text{ for every } j \in J(i-). \end{cases} \tag{3.9}$$

(vi) For every $(i,j) \in K$, $\varphi_i \in \Gamma_0(\mathcal{H}_i)$ and $\psi_{ij} \in \Gamma_0(L^2(\Upsilon_{ij}))$.

(vii) There exist $\widetilde{\boldsymbol{u}} = (\widetilde{u}_i)_{i\in I} \in \bigoplus_{i\in I} \mathcal{H}_i$ and $\widetilde{\boldsymbol{g}} = (\widetilde{g}_{ij})_{(i,j)\in K} \in \bigoplus_{(i,j)\in K} L^2(\Upsilon_{ij})$ such that

$$(\forall (i,j) \in K) \quad \begin{cases} \widetilde{g}_{ij} \in \partial\psi_{ij}(\mathsf{T}_{ij}\,\widetilde{u}_i - \mathsf{T}_{ji}\,\widetilde{u}_j) \\ -Q_i\big(0, (\widetilde{g}_{ij})_{j\in J(i+)}, (\widetilde{g}_{ji})_{j\in J(i-)}\big) \in \partial\varphi_i(\widetilde{u}_i). \end{cases} \tag{3.10}$$

(viii) Let $\boldsymbol{u}_0 = (u_{i,0})_{i\in I} \in \bigoplus_{i\in I} \mathcal{H}_i$ and let $\boldsymbol{g}_0 = (g_{ij,0})_{(i,j)\in K} \in \bigoplus_{(i,j)\in K} L^2(\Upsilon_{ij})$.

The problem is to find the closest point $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{g}})$ to $(\boldsymbol{u}_0, \boldsymbol{g}_0)$ in $\bigoplus_{i\in I} \mathcal{H}_i \oplus \bigoplus_{(i,j)\in K} L^2(\Upsilon_{ij})$ that satisfies (3.10).

Let us illustrate our problem via a simple example.

Example 3.2 In Problem 3.1 set, for every $i \in I$ and every $j \in J(i+)$, $\psi_{ij} = \iota_{\{0\}}$ and $\varphi_i\colon u_i \mapsto \frac{1}{2}\int_{\Omega_i} |Du_i|^2 - \int_{\Omega_i} fu$. Then Problem 3.1 reduces to

$$\underset{\substack{(u_i)_{i\in I}\in\bigoplus_{i\in I}\mathcal{H}_i \\ (\forall (i,j)\in K)\ \mathsf{T}_{ij}u_i = \mathsf{T}_{ji}u_j}}{\text{minimize}} \quad \sum_{i=1}^m \frac{1}{2}\int_{\Omega_i} |Du_i|^2 - \int_{\Omega_i} fu_i, \tag{3.11}$$

which is the domain decomposition associated to the Poisson problem

$$\begin{cases} -\Delta u = f, & \text{on } \Omega; \\ u = 0, & \text{on } \operatorname{bdry}\Omega. \end{cases} \tag{3.12}$$

As will be seen in Section 4, the flexibility of our setting allows for more elaborated structures and conditions on the interfaces. This example will be studied in detail in Section 4.1.

Remark 3.3 In Problem 3.1, (i)–(iii) describe the geometrical setting, and (iv)–(viii) fix the functional Hilbert setting. In particular, item (vii) will ensure the existence of a solution. For every $i \in I$, since $\operatorname{bdry}\Omega_i = \overline{\Upsilon_{ii}} \cup \overline{\bigcup_{j\in J(i)} \Upsilon_{ij}}$, the existence and uniqueness of the solution to (3.9) is guaranteed by condition (i) in Problem 3.1 and [44, Theorem 25.I], from which we deduce that Q_i is linear and continuous.

In order to analyze and solve Problem 3.1, we shall exploit the following connection.

Proposition 3.4 *Problem 3.1 is a special case of Problem 2.1.*

Proof. Let us set

$$\big(\forall k = (i,j) \in K\big) \quad \Psi_k = \psi_{ij}, \quad \mathcal{G}_k = L^2(\Upsilon_{ij}), \quad (\forall \ell \in I) \quad \Lambda_{k\ell} = \begin{cases} \mathsf{T}_{ij}, & \text{if } \ell = i; \\ -\mathsf{T}_{ji}, & \text{if } \ell = j; \\ 0, & \text{otherwise}, \end{cases} \tag{3.13}$$

and define

$$(\forall i \in I) \quad \Phi_i = \varphi_i. \tag{3.14}$$

For every $i \in I$, it follows from Poincaré's inequality, that the embedding $\mathcal{H}_i \hookrightarrow H^1(\Omega_i)$ is continuous [44, p. 1033] and therefore, for every $j \in J(i)$, the trace operators $\mathsf{T}_i \colon H^1(\Omega_i) \to L^2(\text{bdry}\,\Omega_i)$ and $\mathsf{T}_{ij} \colon \mathcal{H}_i \to L^2(\Upsilon_{ij})$ are linear and bounded. Moreover, for every $i \in I$, every $(u_i)_{i\in I} \in \bigoplus_{i\in I} \mathcal{H}_i$, and every

$$(w_k)_{k\in K} = (g_{ij})_{(i,j)\in K} \in \bigoplus_{(i,j)\in K} L^2(\Upsilon_{ij}), \tag{3.15}$$

it follows from (v) in Problem 3.1 that

$$\begin{aligned}\left\langle u_i \,\middle|\, \sum_{k\in K} \Lambda_{k,i}^* w_k \right\rangle &= \left\langle u_i \,\middle|\, \sum_{j\in J(i+)} \mathsf{T}_{ij}^* g_{ij} - \sum_{j\in J(i-)} \mathsf{T}_{ji}^* g_{ji} \right\rangle \\ &= \sum_{j\in J(i+)} \langle \mathsf{T}_{ij} u_i \mid g_{ij}\rangle - \sum_{j\in J(i-)} \langle \mathsf{T}_{ji} u_i \mid g_{ji}\rangle \\ &= \sum_{j\in J(i+)} \int_{\Upsilon_{ij}} (\mathsf{T}_{ij} u_i) g_{ij}\, dS - \sum_{j\in J(i-)} \int_{\Upsilon_{ij}} (\mathsf{T}_{ij} u_i) g_{ji}\, dS \\ &= \int_{\text{bdry}\,\Omega_i} (\mathsf{T}_i u_i)\big(\nu_i^\top DQ_i(0, (g_{ij})_{j\in J(i+)}, (g_{ji})_{j\in J(i-)})\big)\, dS \\ &= \int_{\Omega_i} (Du_i)^\top DQ_i(0, (g_{ij})_{j\in J(i+)}, (g_{ji})_{j\in J(i-)}) \\ &= \big\langle u_i \mid Q_i\big(0, (g_{ij})_{j\in J(i+)}, (g_{ji})_{j\in J(i-)}\big)\big\rangle,\end{aligned} \tag{3.16}$$

which yields

$$(\forall i \in I) \quad Q_i\big(0, (g_{ij})_{j\in J(i+)}, (g_{ji})_{j\in J(i-)}\big) = \sum_{k\in K} \Lambda_{ki}^* w_k. \tag{3.17}$$

It remains to check that (2.8) is satisfied. It follows from (vii) that there exist $(\widetilde{u}_i)_{i\in I} \in \bigoplus_{i\in I} \mathcal{H}_i$ and $(\widetilde{w}_k)_{k\in K} = (\widetilde{g}_{ij})_{(i,j)\in K} \in \bigoplus_{(i,j)\in K} L^2(\Upsilon_{ij})$ such that (3.10) holds. Combining (vii), (3.13), (3.14), and (3.17) we obtain

$$\begin{aligned}(3.10) \;&\Leftrightarrow\; \begin{cases} (\forall i \in I) & -\sum_{k\in K} \Lambda_{ki}^* \widetilde{w}_k \in \partial\Phi_i(\widetilde{u}_i) \\ (\forall k \in K) & \widetilde{w}_k \in \partial\Psi_k\big(\sum_{\ell\in I} \Lambda_{k\ell}\widetilde{u}_\ell\big) \end{cases} \\ &\Rightarrow\; 0 \in \partial\Phi_i(\widetilde{u}_i) + \sum_{k\in K} \Lambda_{ki}^* \left(\partial\Psi_k\left(\sum_{\ell\in I} \Lambda_{k\ell}\widetilde{u}_\ell\right)\right) \\ &\Rightarrow\; (2.8),\end{aligned} \tag{3.18}$$

which completes the proof. □

The following proposition clarifies the interplay between Problem 3.1, (1.1), and its dual.

Proposition 3.5 *Problem 3.1 has a unique solution* $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{g}})$. *Moreover,* $\overline{\boldsymbol{u}} = (\overline{u}_i)_{i\in I}$ *solves*

$$\underset{(u_i)_{i\in I} \in \bigoplus_{i\in I} \mathcal{H}_i}{\text{minimize}} \; \sum_{i\in I} \varphi_i(u_i) + \sum_{(i,j)\in K} \psi_{ij}(\mathsf{T}_{ij}\, u_i - \mathsf{T}_{ji}\, u_j) \tag{3.19}$$

and $\overline{\boldsymbol{g}} = (\overline{g}_{ij})_{(i,j)\in K}$ *solves*

$$\underset{(g_{ij})_{(i,j)\in K}\in \bigoplus_{(i,j)\in K} L^2(\Upsilon_{ij})}{\text{minimize}} \;\; \sum_{i\in I} \varphi_i^*\Big(- Q_i\big(0, (g_{ij})_{j\in J(i+)}, (g_{ji})_{j\in J(i-)}\big)\Big) + \sum_{(i,j)\in K} \psi_{ij}^*(g_{ij}). \tag{3.20}$$

Proof. This follows from Proposition 3.4 and Proposition 2.2 applied with (3.13), (3.14), (3.15), and (3.17). ☐

Our objective is to provide a flexible method for solving Problem 3.1 (and hence (3.19) and (3.20)) in which each elementary step involves the constituents of the problem, i.e., the trace operators and the functions, separately.

Theorem 3.6 *Let $\varepsilon \in \left]0,1\right[$, let $(\gamma_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ be sequences in $[\varepsilon, 1/\varepsilon]$, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, 1]$, and iterate*

$$
\begin{array}{l}
\textit{for } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
\textit{for every } i \in I\\
\left\lfloor
\begin{array}{l}
v_{i,n} = u_{i,n} - \gamma_n Q_i\big(0, (g_{ij,n})_{j\in J(i+)}, (g_{ji,n})_{j\in J(i-)}\big)\\
p_{i,n} = \operatorname{prox}_{\gamma_n\varphi_i} v_{i,n}
\end{array}
\right.\\
\textit{for every } i \in I\\
\left\lfloor
\begin{array}{l}
\textit{for every } j \in J(i+)\\
\left\lfloor
\begin{array}{l}
l_{ij,n} = \mathsf{T}_{ij} u_{i,n} - \mathsf{T}_{ji} u_{j,n}\\
q_{ij,n} = \operatorname{prox}_{\mu_n\psi_{ij}}(l_{ij,n} + \mu_n g_{ij,n})\\
t_{ij,n} = q_{ij,n} - \mathsf{T}_{ij} p_{i,n} + \mathsf{T}_{ji} p_{j,n}
\end{array}
\right.
\end{array}
\right.\\
\textit{for every } i \in I\\
\left\lfloor\; s_{i,n} = \gamma_n^{-1}(u_{i,n} - p_{i,n}) + \mu_n^{-1} Q_i\big(0, (l_{ij,n} - q_{ij,n})_{j\in J(i+)}, (l_{ji,n} - q_{ji,n})_{j\in J(i-)}\big)\right.\\
\tau_n = \sum_{i\in I} \|s_{i,n}\|^2 + \sum_{(i,j)\in K} \|t_{ij,n}\|^2\\
\textit{if } \tau_n = 0\\
\left\lfloor\; \theta_n = 0\right.\\
\textit{if } \tau_n > 0\\
\left\lfloor\; \theta_n = \lambda_n\big(\gamma_n^{-1} \sum_{i\in I} \|u_{i,n} - p_{i,n}\|^2 + \mu_n^{-1} \sum_{(i,j)\in K} \|l_{ij,n} - q_{ij,n}\|^2\big)/\tau_n\right.\\
\textit{for every } i \in I\\
\left\lfloor
\begin{array}{l}
u_{i,n+1/2} = u_{i,n} - \theta_n s_{i,n}\\
\textit{for every } j \in J(i+)\\
\left\lfloor\; g_{ij,n+1/2} = g_{ij,n} - \theta_n t_{ij,n}\right.
\end{array}
\right.\\
\chi_n = \sum_{i\in I} \big\langle u_{i,0} - u_{i,n} \mid u_{i,n} - u_{i,n+1/2}\big\rangle + \sum_{(i,j)\in K} \big\langle g_{ij,0} - g_{ij,n} \mid g_{ij,n} - g_{ij,n+1/2}\big\rangle\\
\mu_n = \sum_{i\in I} \|u_{i,0} - u_{i,n}\|^2 + \sum_{(i,j)\in K} \|g_{ij,0} - g_{ij,n}\|^2\\
\nu_n = \sum_{i\in I} \|u_{i,n} - u_{i,n+1/2}\|^2 + \sum_{(i,j)\in K} \|g_{ij,n} - g_{ij,n+1/2}\|^2\\
\rho_n = \mu_n\nu_n - \chi_n^2\\
\textit{if } \rho_n = 0 \textit{ and } \chi_n \geqslant 0\\
\left\lfloor
\begin{array}{l}
\textit{for every } i \in I\\
\left\lfloor
\begin{array}{l}
u_{i,n+1} = u_{i,n+1/2}\\
\textit{for every } j \in J(i+)\\
\left\lfloor\; g_{ij,n+1} = g_{ij,n+1/2}\right.
\end{array}
\right.
\end{array}
\right.\\
\textit{if } \rho_n > 0 \textit{ and } \chi_n\nu_n \geqslant \rho_n\\
\left\lfloor
\begin{array}{l}
\textit{for every } i \in I\\
\left\lfloor
\begin{array}{l}
u_{i,n+1} = u_{i,0} + (1 + \chi_n/\nu_n)(u_{i,n+1/2} - u_{i,n})\\
\textit{for every } j \in J(i+)\\
\left\lfloor\; g_{ij,n+1} = g_{ij,0} + (1 + \chi_n/\nu_n)(g_{ij,n+1/2} - g_{ij,n})\right.
\end{array}
\right.
\end{array}
\right.\\
\textit{if } \rho_n > 0 \textit{ and } \chi_n\nu_n < \rho_n\\
\left\lfloor
\begin{array}{l}
\textit{for every } i \in I\\
\left\lfloor
\begin{array}{l}
u_{i,n+1} = u_{i,n} + (\nu_n/\rho_n)\big(\chi_n(u_{i,0} - u_{i,n}) + \mu_n(u_{i,n+1/2} - u_{i,n})\big)\\
\textit{for every } j \in J(i+)\\
\left\lfloor\; g_{ij,n+1} = g_{ij,n} + (\nu_n/\rho_n)\big(\chi_n(g_{ij,0} - g_{ij,n}) + \mu_n(g_{ij,n+1/2} - g_{ij,n})\big).\right.
\end{array}
\right.
\end{array}
\right.
\end{array}
\right.
\end{array}
\tag{3.21}
$$

Then, for every $i \in I$ and $j \in J(i+)$, $u_{i,n} \xrightarrow{\mathcal{H}_i} \overline{u}_i$ and $g_{ij,n} \xrightarrow{L^2(\Upsilon_{ij})} \overline{g}_{ij}$.

Proof. Using (3.13), (3.14), and (3.15), it follows from (3.17) that (3.21) is a special case of (2.14). In view of Proposition 3.4 and Theorem 2.3, the proof is complete. □

Remark 3.7 Algorithm (3.21) is mainly organized as a series of loops indexed by the variables i and j that can be executed simultaneously and, therefore, implemented on parallel processors. The first loop computes $v_{i,n}$ as well as $p_{i,n} = \text{prox}_{\gamma_n \varphi_i} v_{i,n}$ for each subdomain $i \in I$. The computation of $v_{i,n}$ involves the operator Q_i which, in view of Problem 3.1(v), amounts to solving the Dirichlet-Neumann boundary problem

$$\begin{cases} -\Delta u = 0 & \text{on } \Omega_i, \\ u = 0 & \text{on } \Upsilon_{ii}, \\ \nu_i^\top Du = g_{ij,n} & \text{on } \Upsilon_{ij}, \ \text{for every } j \in J(i+), \\ \nu_i^\top Du = -g_{ji,n} & \text{on } \Upsilon_{ij}, \ \text{for every } j \in J(i-). \end{cases} \tag{3.22}$$

On the other hand, it follows from (2.5) that

$$p_{i,n} = \underset{w \in \mathcal{H}_i}{\text{argmin}} \ \gamma_n \varphi_i(w) + \frac{1}{2} \int_{\Omega_i} \left| Dw - Dv_{i,n} \right|^2. \tag{3.23}$$

Likewise, the proximity operation across interface Υ_{ij} in the next loop is computed as

$$q_{ij,n} = \underset{w \in L^2(\Upsilon_{ij})}{\text{argmin}} \ \mu_n \psi_{ij}(w) + \frac{1}{2} \int_{\Upsilon_{ij}} \left| w - l_{ij,n} - \mu_n g_{ij,n} \right|^2 dS. \tag{3.24}$$

The remaining steps involve straightforward computations.

Remark 3.8 The variational formulation of Problem 3.1 can be modified to include domain decomposition problems with overlapping subdomains. Indeed, for every $i \in I$ and $j \in J(i+)$, it is necessary to consider a projection operator $P_{ij} \colon \mathcal{H}_i \to H^1(\Omega_i \cap \Omega_j)$ instead of the trace operator $\mathsf{T}_{ij} \colon \mathcal{H}_i \to L^2(\Upsilon_{ij})$. An application of the overlapping framework to image processing with total variation and ℓ^1 minimization can be found in [25].

Remark 3.9 An alternative approach in order to guarantee condition (vii) in Problem 3.1 is to replace the Hilbert spaces $(L^2(\Upsilon_{ij}))_{(i,j)\in K}$ by $(H^{1/2}(\Upsilon_{ij}))_{(i,j)\in K}$, in which case the trace operators are surjective [29, Theorem 1.5.1.2]. The difficulty of this approach resides in the computation of the proximity operators $(\text{prox}_{\psi_{ij}})_{(i,j)\in K}$ in (3.21), which is not easy because of the complexity of the metric of $(H^{1/2}(\Upsilon_{ij}))_{(i,j)\in K}$.

4 Special cases

We illustrate the potential use of algorithm (3.21) through a few applications to domain decomposition in the context of the Poisson, p–Laplacian, and obstacle problems with Dirichlet conditions and continuity at the interfaces. We start with a couple of technical facts. First, define

$$(\forall i \in I) \quad E_i^p = \left\{ u \in W^{1,p}(\Omega_i) \ \middle| \ \mathsf{T}_i u = 0 \ \text{on } \Upsilon_{ii} \right\}. \tag{4.1}$$

Proposition 4.1 *Consider the setting of Problem 3.1. Let* $p \in \,]1,+\infty[$*, for every* $i \in I$ *let* $\phi_i \in \Gamma_0(W^{1,p}(\Omega_i))$ *be a strictly convex coercive function with respect to the* $W^{1,p}(\Omega_i)$ *norm, and set*

$$\varphi\colon W^{1,p}(\Omega) \to \,]-\infty,+\infty] : u \mapsto \sum_{i\in I} \phi_i(u|_{\Omega_i}). \tag{4.2}$$

Then φ *is a strictly convex coercive function in* $\Gamma_0(W^{1,p}(\Omega))$ *which is coercive with respect to the* $W^{1,p}(\Omega)$ *norm, and the optimization problems*

$$\underset{u\in W_0^{1,p}(\Omega)}{\text{minimize}}\;\; \varphi(u) \tag{4.3}$$

and

$$\underset{\substack{(u_i)_{i\in I}\in \times_{i\in I} E_i^p\\ (\forall (i,j)\in K)\ \mathsf{T}_{ij}u_i=\mathsf{T}_{ji}u_j}}{\text{minimize}}\;\; \sum_{i\in I}\phi_i(u_i) \tag{4.4}$$

have unique solutions $\overline{u} \in W_0^{1,p}(\Omega)$ *and* $(\overline{u}_i)_{i\in I} \in E_1^p \times \cdots \times E_m^p$*, respectively. Moreover,*

$$(\forall i \in I) \quad \overline{u}(x) = \overline{u}_i(x) \quad \textit{for almost every } x \in \Omega_i. \tag{4.5}$$

Proof. Let u and v be functions in $W^{1,p}(\Omega)$ such that $u \neq v$, and let $\alpha \in \,]0,1[$. There exists a measurable set $U \subset \Omega$ of nonzero Lebesgue measure such that $(\forall x \in U)\ u(x) \neq v(x)$. For every $i \in I$, set $U_i = U \cap \Omega_i$. By assumption (i) in Problem 3.1, and the additivity property of the Lebesgue measure, there exists $j \in I$ such that U_j has nonzero measure, which yields $u|_{\Omega_j} \neq v|_{\Omega_j}$. It then follows from the strict convexity of the functions $(\phi_i)_{i\in I}$ that

$$\sum_{i\in I}\phi_i\big((\alpha u+(1-\alpha)v)|_{\Omega_i}\big) = \sum_{i\in I}\phi_i\big(\alpha u|_{\Omega_i}+(1-\alpha)v|_{\Omega_i}\big) < \alpha\sum_{i\in I}\phi_i(u|_{\Omega_i})+(1-\alpha)\sum_{i\in I}\phi_i(v|_{\Omega_i}), \tag{4.6}$$

which shows that φ is strictly convex. On the other hand, since assumption (i) in Problem 3.1 yields, for every $u \in W^{1,p}(\Omega)$,

$$\|u\|^p_{W^{1,p}(\Omega)} = \int_\Omega |u|^p + \int_\Omega |Du|^p = \sum_{i\in I}\int_{\Omega_i}|u|^p + \int_{\Omega_i}|Du|^p = \sum_{i\in I}\|u|_{\Omega_i}\|^p_{W^{1,p}(\Omega_i)}, \tag{4.7}$$

the coercivity of φ follows from the coercivity of the functions $(\phi_i)_{i\in I}$.

The existence of solutions $\overline{u} \in W_0^{1,p}(\Omega)$ and $(\overline{u}_i)_{i\in I} \in E_1^p \times \cdots \times E_m^p$, respectively to (4.3) and (4.4), follows from the classical theorems for the minimization of closed convex coercive functions on reflexive Banach spaces (see, e.g., [4, Theorem 3.3.4], [42, Theorem 2.5.1(ii)]). The uniqueness is a consequence of the strict convexity of the objective functions. Set

$$(\forall i \in I) \quad \widetilde{u}(x) = \overline{u}_i(x) \quad \text{for almost every } x \in \Omega_i. \tag{4.8}$$

Since $\Omega \smallsetminus \bigcup_{i\in I}\Omega_i$ has zero Lebesgue measure, it follows from condition (i) in Problem 3.1 that the function $\widetilde{u}$ is well defined in $L^p(\Omega)$. Let us prove that $\widetilde{u} = \overline{u}$, which will complete the proof. Arguing as in [4, Lemma 6.4.1], we deduce that, for every $u \in L^p(\Omega)$,

$$u \in W^{1,p}(\Omega) \quad \Leftrightarrow \quad (\forall (i,j) \in K) \quad u|_{\Omega_i} \in W^{1,p}(\Omega_i) \text{ and } \mathsf{T}_{ij}(u|_{\Omega_i}) = \mathsf{T}_{ji}(u|_{\Omega_j}). \tag{4.9}$$

The characterization (4.9) expresses the fact that the jumps of every $u \in W^{1,p}(\Omega)$ across the interfaces $(\Upsilon_{ij})_{(i,j)\in K}$ are zero. Correspondingly, taking into account the Dirichlet boundary condition [24, Section 2.1], we deduce that, for every $u \in L^p(\Omega)$,

$$u \in W_0^{1,p}(\Omega) \quad \Leftrightarrow \quad (\forall (i,j) \in K) \quad u|_{\Omega_i} \in E_i^p \text{ and } \mathsf{T}_{ij}(u|_{\Omega_i}) = \mathsf{T}_{ji}(u|_{\Omega_j}). \tag{4.10}$$

It then follows from (4.8) that, for every $i \in I$, $\widetilde{u}|_{\Omega_i} = \overline{u}_i \in E_i^p$, and, for every $(i,j) \in K$, $\mathsf{T}_{ij}(\widetilde{u}|_{\Omega_i}) = \mathsf{T}_{ij}\overline{u}_i = \mathsf{T}_{ji}\overline{u}_j = \mathsf{T}_{ji}(\widetilde{u}|_{\Omega_j})$. Hence, (4.10) yields $\widetilde{u} \in W_0^{1,p}(\Omega)$ and, for every $u \in W_0^{1,p}(\Omega)$, (4.2) yields (the sets $(\Omega_i)_{i\in I}$ are disjoint, and the Lebesgue measure of the interfaces is zero)

$$\varphi(\widetilde{u}) = \sum_{i\in I} \phi_i(\overline{u}_i) \leqslant \sum_{i\in I} \phi_i(u|_{\Omega_i}) = \varphi(u), \tag{4.11}$$

which, by uniqueness of the solution, yields $\widetilde{u} = \overline{u}$. ☐

Proposition 4.2 *Consider the setting of Problem 3.1. Let $\gamma \in \left]0, +\infty\right[$, let $f \in L^2(\Omega)$, and, for every $i \in I$, let C_i be a nonempty closed convex subset of $\mathcal{H}_i$. Suppose that*

$$\varphi_i \colon \mathcal{H}_i \to \left]-\infty, +\infty\right] : u_i \mapsto \iota_{C_i}(u_i) + \frac{1}{2}\int_{\Omega_i} |Du_i|^2 - \int_{\Omega_i} f u_i. \tag{4.12}$$

Then the following hold for every $i \in I$:

(i) *We have*

$$\begin{cases} \varphi_i \colon u_i \mapsto \iota_{C_i}(u_i) + \dfrac{1}{2}\|u_i\|^2 - \langle Q_i(f, 0, \ldots, 0) \mid u_i\rangle \\ \partial\varphi_i = N_{C_i} + \mathrm{Id} - Q_i(f, 0, \ldots, 0) \\ \mathrm{prox}_{\gamma\varphi_i} = P_{C_i}\left(\dfrac{1}{1+\gamma}\,\mathrm{Id} + \dfrac{\gamma}{1+\gamma} Q_i(f, 0, \ldots, 0)\right). \end{cases} \tag{4.13}$$

(ii) *Suppose that $C_i = \mathcal{H}_i$. Then φ_i is Gâteaux–differentiable on $\mathcal{H}_i$ and*

$$\begin{cases} \varphi_i \colon u_i \mapsto \dfrac{1}{2}\|u_i\|^2 - \langle Q_i(f, 0, \ldots, 0) \mid u_i\rangle \\ \nabla\varphi_i = \mathrm{Id} - Q_i(f, 0, \ldots, 0) \\ \mathrm{prox}_{\gamma\varphi_i} = \dfrac{1}{1+\gamma}\,\mathrm{Id} + \dfrac{\gamma}{1+\gamma} Q_i(f, 0, \ldots, 0). \end{cases} \tag{4.14}$$

Proof. Fix $i \in I$. First note that

$$\phi_i \colon \mathcal{H}_i \to \mathbb{R} \colon u_i \mapsto \int_{\Omega_i} f u_i \tag{4.15}$$

is linear. Moreover, since Ω_i bounded, the Cauchy-Schwarz and Poincaré's inequalities [44, Appendix (53c)], and (2.1) yield

$$(\exists\, \delta \in \left]0, +\infty\right[)(\forall u_i \in \mathcal{H}_i) \quad |\phi_i(u_i)| \leqslant \|f\|_{L^2(\Omega_i)}\|u_i\|_{L^2(\Omega_i)} \leqslant \delta\|f\|_{L^2(\Omega_i)}\|u_i\|. \tag{4.16}$$

Hence, the Riesz-Fréchet representation theorem asserts that there exists a unique $v_i \in \mathcal{H}_i$ such that

$$(\forall u_i \in \mathcal{H}_i) \quad \phi_i(u_i) = \int_{\Omega_i} f u_i = \int_{\Omega_i} (Dv_i)^\top Du_i = \langle v_i \mid u_i \rangle. \tag{4.17}$$

Thus, it follows from [44, Proposition 25.28] and (3.9) that $v_i = Q_i(f, 0, \ldots, 0)$. Using (2.1), we can therefore write (4.12) as

$$\varphi_i \colon u_i \mapsto \frac{1}{2}\|u_i\|^2 - \langle Q_i(f, 0, \ldots, 0) \mid u_i \rangle + \iota_{C_i}(u_i). \tag{4.18}$$

Moreover, we deduce from standard subdifferential calculus [11, Section 16.4] that

$$\partial\varphi_i = \mathrm{Id}\, - Q_i(f, 0, \ldots, 0) + N_{C_i}, \tag{4.19}$$

where N_{C_i} is the normal cone operator to C_i. Hence, it follows from (4.19) that, for every u_i and p_i in $\mathcal{H}_i$,

$$\begin{aligned} p_i = \mathrm{prox}_{\gamma\varphi_i} u_i \quad &\Leftrightarrow \quad u_i - p_i \in \gamma\partial\varphi_i(p_i) \\ &\Leftrightarrow \quad u_i \in (1+\gamma)p_i - \gamma Q_i(f, 0, \ldots, 0) + N_{C_i} p_i \\ &\Leftrightarrow \quad \frac{1}{1+\gamma} u_i + \frac{\gamma}{1+\gamma} Q_i(f, 0, \ldots, 0) \in p_i + N_{C_i} p_i \\ &\Leftrightarrow \quad p_i = P_{C_i}\left(\frac{1}{1+\gamma} u_i + \frac{\gamma}{1+\gamma} Q_i(f, 0, \ldots, 0)\right). \end{aligned} \tag{4.20}$$

(ii): Since $N_{C_i} \equiv \{0\}$ and $P_{C_i} = \mathrm{Id}$, the result follows from (i). ☐

4.1 Poisson problem

Let $f \in L^2(\Omega)$, and consider the Poisson problem with an homogeneous Dirichlet boundary condition

$$\begin{cases} -\Delta u = f, & \text{on } \Omega; \\ u = 0, & \text{on } \operatorname{bdry} \Omega. \end{cases} \tag{4.21}$$

Classically, this problem has a unique weak solution $\overline{u} \in H_0^1(\Omega)$, which can be obtained by solving the strongly convex minimization problem (see [24, Chapter IV.2.1] or [44, Chapter 25.9])

$$\underset{u \in H_0^1(\Omega)}{\text{minimize}} \;\; \frac{1}{2}\int_\Omega |Du|^2 - \int_\Omega f u. \tag{4.22}$$

As a simple example of the flexibility of our framework, we solve (4.22) by decomposing the domain Ω into subdomains satisfying the hypotheses in Problem 3.1, and by imposing continuity conditions at the interfaces.

Problem 4.3 Consider the setting of Problem 3.1. Let $f \in L^2(\Omega)$ and, for every $(i, j) \in K$, assume that Υ_{ij} and $\operatorname{bdry} \Omega$ are of class $\mathcal{C}^2$. The problem is to

$$\underset{\substack{(u_i)_{i \in I} \in \bigoplus_{i \in I} \mathcal{H}_i \\ (\forall (i,j) \in K)\ \mathsf{T}_{ij} u_i = \mathsf{T}_{ji} u_j}}{\text{minimize}} \quad \sum_{i=1}^m \frac{1}{2}\int_{\Omega_i} |Du_i|^2 - \int_{\Omega_i} f u_i. \tag{4.23}$$

We first show the equivalence between Problem 4.3 and (4.22).

Proposition 4.4 *The optimization problem in* (4.23) *has a unique solution* $(\overline{u}_i)_{i\in I}$. *Moreover, the function defined in* (4.5) *is the unique solution to* (4.22).

Proof. This is a consequence of Proposition 4.1 with $p = 2$ and, for every $i \in I$, $\phi_i\colon u \mapsto \frac{1}{2}\int_{\Omega_i} |Du|^2 - \int_{\Omega_i} fu$, which are strongly convex. In this case $\varphi\colon u \mapsto \frac{1}{2}\int_{\Omega} |Du|^2 - \int_{\Omega} fu$. □

Our method for solving Problem 4.3 is a particular case of (3.21). Hence, the following convergence result is an application of Theorem 3.6.

Theorem 4.5 *In algorithm* (3.21) *of Theorem 3.6, replace the steps defining* $p_{i,n}$ *and* $q_{ij,n}$ *by*

$$p_{i,n} = \frac{1}{1+\gamma_n} v_{i,n} + \frac{\gamma_n}{1+\gamma_n} Q_i(f, 0, \ldots, 0) \quad \text{and} \quad q_{ij,n} = 0, \tag{4.24}$$

respectively. Then, for every $i \in I$*, the sequence* $(u_{i,n})_{n\in\mathbb{N}}$ *generated by* (3.21) *converges strongly to* $\overline{u}_i$ *in* $\mathcal{H}_i$.

Proof. Set

$$\begin{cases} (\forall i \in I) \quad \varphi_i\colon u_i \mapsto \dfrac{1}{2}\displaystyle\int_{\Omega_i} |Du_i|^2 - \int_{\Omega_i} fu_i \\ (\forall (i,j) \in K) \quad \psi_{ij} = \iota_{\{0\}}. \end{cases} \tag{4.25}$$

Since, for every $(i, j) \in K$, $\varphi_i \in \Gamma_0(\mathcal{H}_i)$ and $\psi_{ij} \in \Gamma_0(L^2(\Upsilon_{ij}))$, Problem 4.3 is a particular case of (3.19). Let us verify that condition (3.10) holds. Let $(\overline{u}_i)_{i\in I} \in \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$ be the solution to (4.23) guaranteed by Proposition 4.4 and let $\overline{u} \in H_0^1(\Omega)$ be as in (4.5). Since $\psi_{ij} = \iota_{\{0\}}$, we have $\partial\psi_{ij}(0) = L^2(\Upsilon_{ij})$ and, hence, the first condition in (3.10) is satisfied. Since $\operatorname{bdry}\Omega$ and $(\Upsilon_{ij})_{(i,j)\in K}$ are of class $\mathcal{C}^2$, [29, Theorem 2.2.2.3] yields $\overline{u} \in H^2(\Omega)$. Therefore, we deduce from [29, Theorem 1.5.1.2] that, for every $i \in I$ and $j \in J(i)$, $\nu_i^\top D\overline{u}_i$ and $\nu_j^\top D\overline{u}_j$ belong to $L^2(\Upsilon_{ij})$. Now let us show that the second condition in (3.10) holds with

$$(\forall (i,j) \in K) \quad \overline{g}_{ij} = \nu_j^\top D\overline{u}_j \in L^2(\Upsilon_{ij}). \tag{4.26}$$

We note that the solution $(\overline{u}_i)_{i\in I}$ to Problem 4.3 satisfies (see, e.g., [4, Theorem 6.4.1])

$$(\forall i \in I) \quad \begin{cases} -\Delta\overline{u}_i = f, & \text{on } \Omega_i; \\ \overline{u}_i = 0, & \text{on } \Upsilon_{ii}; \\ \mathsf{T}_{ij}\,\overline{u}_i = \mathsf{T}_{ji}\overline{u}_j, & \text{on } \Upsilon_{ij}, \text{ for every } j \in J(i); \\ \nu_i^\top D\overline{u}_i = -\nu_j^\top D\overline{u}_j, & \text{on } \Upsilon_{ij}, \text{ for every } j \in J(i) \end{cases} \tag{4.27}$$

in the sense of distributions, which, from (3.9), yields

$$(\forall i \in I) \quad \overline{u}_i = Q_i(f, (-\nu_j^\top D\overline{u}_j)_{j\in J(i+)}, (\nu_j^\top D\overline{u}_j)_{j\in J(i-)}). \tag{4.28}$$

Let us observe that, because of the regularity $\overline{u} \in H^2(\Omega)$, the transmission conditions satisfied by $\overline{u}$ can be expressed as equalities in the spaces $L^2(\Upsilon_{ij})$, which fits in our abstract framework. Since, for every $(i,j) \in K$, $\nu_i^\top D\overline{u}_i = -\nu_j^\top D\overline{u}_j$, (4.26) implies that

$$\overline{u}_i = Q_i\big(f, (-\nu_j^\top D\overline{u}_j)_{j\in J(i+)}, (\nu_j^\top D\overline{u}_j)_{j\in J(i-)}\big) = Q_i\big(f, (-\overline{g}_{ij})_{j\in J(i+)}, (-\overline{g}_{ji})_{j\in J(i-)}\big). \tag{4.29}$$

Hence, upon invoking Proposition 4.2(ii) and the linearity of Q_i, we obtain

$$\begin{aligned}\nabla\varphi_i(\overline{u}_i) &= \overline{u}_i - Q_i(f,0,\ldots,0)\\ &= Q_i\big(f,(-\overline{g}_{ij})_{j\in J(i+)},(-\overline{g}_{ji})_{j\in J(i-)}\big) - Q_i(f,0,\ldots,0)\\ &= Q_i\big(0,(-\overline{g}_{ij})_{j\in J(i+)},(-\overline{g}_{ji})_{j\in J(i-)}\big)\\ &= -Q_i\big(0,(\overline{g}_{ij})_{j\in J(i+)},(\overline{g}_{ji})_{j\in J(i-)}\big),\end{aligned} \tag{4.30}$$

which is the second condition in (3.10). On the other hand, it follows from (2.5) and (4.25) that, for every $(i,j)\in K$ and every $\mu\in\,]0,+\infty[$, $\mathrm{prox}_{\mu\psi_{ij}}\equiv 0$. Hence, we deduce from Proposition 4.2(ii) that (4.24) yields

$$(\forall n\in\mathbb{N})\quad \begin{cases}(\forall i\in I) & p_{i,n}=\mathrm{prox}_{\gamma_n\varphi_i}v_{i,n}\\ (\forall (i,j)\in K) & q_{ij,n}=\mathrm{prox}_{\mu_n\psi_{ij}}(l_{ij,n}+\mu_n g_{ij,n}),\end{cases} \tag{4.31}$$

and the result follows from Theorem 3.6 with $(\varphi_i)_{i\in I}$ and $(\psi_{ij})_{(i,j)\in K}$ defined as in (4.25). □

Remark 4.6

(i) Note that $(\overline{g}_{ij})_{(i,j)\in K}$ defined in (4.26) is a solution to the dual problem associated with Problem 4.3. The method proposed in Theorem 4.5 also converge in the dual variables, but for the sake of simplicity we provide only the convergence in primal variables.

(ii) In (3.21) we have

$$(\forall n\in\mathbb{N})(\forall i\in I)\quad v_{i,n}=u_{i,n}-\gamma_n Q_i\big(0,(g_{ij,n})_{j\in J(i+)},(g_{ji,n})_{j\in J(i-)}\big). \tag{4.32}$$

Hence, since the operators $(Q_i)_{i\in I}$ defined in (3.9) are multilinear, the sequences $(p_{i,n})_{i\in I,\,n\in\mathbb{N}}$ can be computed more efficiently via

$$(\forall n\in\mathbb{N})(\forall i\in I)\quad p_{i,n}=\frac{1}{1+\gamma_n}u_{i,n}+\frac{\gamma_n}{1+\gamma_n}Q_i\big(f,(-g_{ij,n})_{j\in J(i+)},(-g_{ji,n})_{j\in J(i-)}\big). \tag{4.33}$$

This allows us to solve only m auxiliary PDE's for updating $(p_{i,n})_{i\in I}$ at each iteration n.

Remark 4.7 The analysis of Theorem 4.5 can be adapted to the case of the linear elasticity system by using Korn's inequality instead of Poincaré's inequality. A key ingredient (and possible limitation) of our approach is the H^2 regularity property of the solution to the problem in the case of the linear elasticity system. Likewise fluid-solid interactions can be handled via our framework.

4.2 *p*-Laplacian

It has long been observed that semi-linear and quasi-linear monotone problems can be efficiently analyzed using modern convex-analytical tools [6, 15, 43]. We follow a similar approach in applying our variational decomposition method to the p-Laplacian operator Δ_p.

Let $p\in\,]1,+\infty[$, let $f\in L^\infty(\Omega)$, and consider the partial differential equation governed by the p-Laplacian operator with Dirichlet boundary conditions

$$\begin{cases}-\mathrm{div}\,\big(|Du|^{p-2}Du\big)=f, & \text{on }\Omega;\\ u=0, & \text{on }\mathrm{bdry}\,\Omega.\end{cases} \tag{4.34}$$

Note that, if $p = 2$, (4.34) reduces to (4.21). This problem possesses a unique weak solution $\overline{u} \in W_0^{1,p}(\Omega)$, which can be obtained by solving the strictly convex minimization problem [24, Section IV.2.2]

$$\underset{u \in W_0^{1,p}(\Omega)}{\text{minimize}} \ \frac{1}{p}\int_\Omega |Du|^p - \int_\Omega fu. \tag{4.35}$$

As another example of our framework, we are interested to solve (4.35) by decomposing the domain Ω in subdomains satisfying the hypotheses in Problem 3.1, and considering continuity conditions on the interfaces. More precisely, we are interested in the following problem.

Problem 4.8 Consider the setting of Problem 3.1. Let $p \in \,]1,+\infty[$, let $\alpha \in \,]0,1]$, and let $f \in L^\infty(\Omega)$. Suppose that the unique solution to (4.35) is in $\mathcal{C}^{1,\alpha}(\overline{\Omega})$. The problem is to

$$\underset{\substack{(u_i)_{i\in I} \in \times_{i\in I} E_i^p \\ (\forall (i,j)\in K)\ \mathsf{T}_{ij}u_i = \mathsf{T}_{ji}u_j}}{\text{minimize}} \ \sum_{i=1}^m \frac{1}{p}\int_{\Omega_i} |Du_i|^p - \int_{\Omega_i} fu_i. \tag{4.36}$$

Proposition 4.9 *Problem 4.8 has a unique solution $(\overline{u}_i)_{i\in I}$. Moreover, the function $\overline{u}$ defined in (4.5) is the unique solution to (4.35).*

Proof. This is a consequence of Proposition 4.1 where, for every $i \in I$, $\phi_i \colon u \mapsto \frac{1}{p}\int_{\Omega_i}|Du|^p - \int_{\Omega_i} fu$, which is strictly convex and coercive. In this case $\phi \colon u \mapsto \frac{1}{p}\int_\Omega |Du|^p - \int_\Omega fu$. ☐

We now present our method for solving Problem 4.8.

Theorem 4.10 *In algorithm (3.21) of Theorem 3.6, replace the steps defining $p_{i,n}$ and $q_{ij,n}$ by*

$$p_{i,n} = \underset{w \in \mathcal{H}_i \cap E_i^p}{\text{argmin}} \ \gamma_n\left(\frac{1}{p}\int_{\Omega_i}|Dw|^p - \int_{\Omega_i} fw\right) + \frac{1}{2}\int_{\Omega_i}|Dw - Dv_{i,n}|^2 \quad \textit{and} \quad q_{ij,n} = 0, \tag{4.37}$$

respectively. Then, for every $i \in I$, the sequence $(u_{i,n})_{n\in\mathbb{N}}$ generated by (3.21) converges strongly to $\overline{u}_i$ in $\mathcal{H}_i$.

Proof. We consider two cases.

(a) $p \geqslant 2$: Since Ω is bounded, we have $W^{1,p}(\Omega) \subset H^1(\Omega)$, and hence it follows from (4.1) that $E_i^p \subset \mathcal{H}_i$. Thus, Problem 4.8 corresponds to the special case of Problem 3.1 in which

$$\begin{cases} (\forall i \in I) \quad \varphi_i \colon \mathcal{H}_i \to \,]-\infty,+\infty] : u_i \mapsto \begin{cases} \dfrac{1}{p}\displaystyle\int_{\Omega_i}|Du_i|^p - \int_{\Omega_i} fu_i, & \text{if } u_i \in E_i^p; \\ +\infty, & \text{otherwise} \end{cases} \\ (\forall (i,j) \in K) \quad \psi_{ij} = \iota_{\{0\}}. \end{cases} \tag{4.38}$$

It is clear that the functions $(\psi_{ij})_{(i,j)\in K}$ are proper, lower semicontinuous, and convex. Since the convexity of functions $(\varphi_i)_{i\in I}$ is clear, let us show that they are lower semicontinuous. To this end, fix $i \in I$, take $\lambda \in \mathbb{R}$, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}_i$ such that $u_n \xrightarrow{\mathcal{H}_i} u \in \mathcal{H}_i$ and $(\forall n \in \mathbb{N})\ \varphi_i(u_n) \leqslant \lambda$. We deduce from [4, Theorem 5.4.3] that the norm in $W^{1,p}(\Omega_i)$ and the norm

$$u \mapsto \left(\int_{\Omega_i}|Du|^p\right)^{1/p} = \|Du\|_{L^p(\Omega_i)} \tag{4.39}$$

are equivalent in E_i^p, which yields the coercivity of φ_i in E_i^p. Therefore, $(u_n)_{n\in\mathbb{N}}$ is bounded in E_i^p and, hence, it converges weakly to u in E_i^p. Moreover, the function φ_i is convex and continuous on E_i^p, and hence weakly lower semicontinuous, which yields

$$\varphi_i(u) \leqslant \underline{\lim}\, \varphi_i(u_n) \leqslant \lambda. \tag{4.40}$$

Let us show that condition (3.10) holds. Let $(\overline{u}_i)_{i\in I} \in \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$ be the solution to Problem 4.8, and let $\overline{u} \in H_0^1(\Omega)$ be as in (4.5). Since $\psi_{ij} = \iota_{\{0\}}$, we have $\partial\psi_{ij}(0) = L^2(\Upsilon_{ij})$, and the first condition in (3.10) is therefore satisfied. Now since $\overline{u} \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$, for every $(i,j) \in K$, $\nu_i^\top |D\overline{u}_i|^{p-2} D\overline{u}_i \in L^2(\Upsilon_{ij})$ and $\nu_j^\top |D\overline{u}_j|^{p-2} D\overline{u}_j \in L^2(\Upsilon_{ij})$. Let us show that the second condition in (3.10) holds with

$$(\forall (i,j) \in K) \quad \overline{g}_{ij} = |D\overline{u}_j|^{p-2} \nu_j^\top D\overline{u}_j \in L^2(\Upsilon_{ij}). \tag{4.41}$$

The Euler equation associated with Problem 4.8 yields

$$(\forall i \in I) \quad \begin{cases} -\text{div}\left(|D\overline{u}_i|^{p-2} D\overline{u}_i\right) = f, & \text{on } \Omega_i;\\ \overline{u}_i = 0, & \text{on } \Upsilon_{ii};\\ \mathsf{T}_{ij}\,\overline{u}_i = \mathsf{T}_{ji}\overline{u}_j, & \text{on } \Upsilon_{ij},\ \text{ for every } j \in J(i);\\ |D\overline{u}_i|^{p-2}\nu_i^\top D\overline{u}_i = -|D\overline{u}_j|^{p-2}\nu_j^\top D\overline{u}_j, & \text{on } \Upsilon_{ij},\ \text{ for every } j \in J(i). \end{cases} \tag{4.42}$$

Now, for every $i \in I$, let us compute an element $v_i \in \partial\varphi_i(\overline{u}_i)$. By a classical directional differentiation argument (see [4, Theorem 6.6.1] for a detailed proof) we obtain

$$(\forall u \in \mathcal{H}_i) \quad \int_{\Omega_i} (|D\overline{u}_i|^{p-2} D\overline{u}_i - Dv_i)^\top Du = \int_{\Omega_i} fu, \tag{4.43}$$

from which we deduce that v_i satisfies, in sense of distributions, the boundary value problem

$$\begin{cases} -\Delta v_i = -f - \text{div}\left(|D\overline{u}_i|^{p-2} D\overline{u}_i\right), & \text{on } \Omega_i;\\ v_i = 0, & \text{on } \Upsilon_{ii};\\ \nu_i^\top Dv_i = \nu_i^\top |D\overline{u}_i|^{p-2} D\overline{u}_i, & \text{on } \Upsilon_{ij},\ \text{ for every } j \in J(i), \end{cases} \tag{4.44}$$

which, using (4.42) and (4.41), reduces to

$$\begin{cases} \Delta v_i = 0, & \text{on } \Omega_i;\\ v_i = 0, & \text{on } \Upsilon_{ii};\\ \nu_i^\top Dv_i = -\overline{g}_{ij}, & \text{on } \Upsilon_{ij},\ \text{ for every } j \in J(i+);\\ \nu_i^\top Dv_i = \overline{g}_{ji}, & \text{on } \Upsilon_{ij},\ \text{ for every } j \in J(i-). \end{cases} \tag{4.45}$$

Hence, we derive from (3.9) that $v_i = Q_i(0, (-\overline{g}_{ij})_{j\in J(i+)}, (-\overline{g}_{ji})_{j\in J(i-)}) \in \partial\varphi_i(\overline{u}_i)$ which yields (3.10). On the other hand, it follows from (2.5) and (4.25) that, for every $(i,j) \in K$ and every $\mu \in \left]0,+\infty\right[$, $\text{prox}_{\mu\psi_{ij}} \equiv 0$. Hence, we deduce from (2.5) that (4.37) yields

$$(\forall n \in \mathbb{N}) \quad \begin{cases} (\forall i \in I) \quad p_{i,n} = \text{prox}_{\gamma_n\varphi_i} v_{i,n}\\ (\forall (i,j) \in K) \quad q_{ij,n} = \text{prox}_{\mu_n\psi_{ij}}(l_{ij,n} + \mu_n g_{ij,n}). \end{cases} \tag{4.46}$$

Therefore, when $(\varphi_i)_{i\in I}$ and $(\psi_{ij})_{(i,j)\in K}$ are defined by (4.38), we deduce from Theorem 3.6 that $u_{i,n} \xrightarrow{\mathcal{H}_i} \overline{u}_i$.

(b) $1 < p < 2$: In this case, for every $i \in I$, $\mathcal{H}_i \subset W^{1,p}(\Omega_i)$, with continuous embedding. Let us assume that the solution $\overline{u}$ of problem (4.35) belongs to $H_0^1(\Omega)$ (indeed we shall further state regularity properties of $\overline{u}$ which make this property satisfied). Combining this property with the density of $H_0^1(\Omega)$ in $W_0^{1,p}(\Omega)$ (for the norm topology of $W_0^{1,p}(\Omega)$), the variational problem (4.35) equivalently writes

$$\underset{u\in H_0^1(\Omega)}{\text{minimize}} \ \frac{1}{p}\int_\Omega |Du|^p - \int_\Omega fu. \tag{4.47}$$

Using the same argument as in Proposition 3.5, this is equivalent to solving

$$\underset{\substack{(u_i)_{i\in I}\in\bigoplus_{i\in I}\mathcal{H}_i \\ (\forall (i,j)\in K)\ \mathsf{T}_{ij}u_i=\mathsf{T}_{ji}u_j}}{\text{minimize}} \quad \sum_{i=1}^m \frac{1}{p}\int_{\Omega_i} |Du_i|^p - \int_{\Omega_i} fu_i. \tag{4.48}$$

Thus we are led to set

$$(\forall i \in I) \quad \varphi_i\colon \mathcal{H}_i \to \mathbb{R}\colon u_i \mapsto \frac{1}{p}\int_{\Omega_i} |Du_i|^p - \int_{\Omega_i} fu_i, \tag{4.49}$$

which is continuous on $\mathcal{H}_i$. The remainder of the proof is identical to the case $p \geqslant 2$. Just notice that, when $p < 2$, the p-Laplacian becomes a singular elliptic operator. The global regularity of the solution $\overline{u}$ to problem (4.35), with a globally continuous gradient, is well established [12, 35]. ☐

Remark 4.11

(i) A recent account of regularity properties for the solution to the p-Laplacian equation can be found in [12, 33, 39]. Note that, in contrast with the case $p = 2$, the degeneracy of the elliptic operator $-\Delta_p$ for $p > 2$ makes the regularity study more involved. In [12], global $H^2(\Omega)$ regularity is obtained for the regularized operator $-\varepsilon\Delta - \Delta_p$ $(\varepsilon > 0)$. In general, for smooth data, the local regularity $C^{1,\alpha}_{loc}(\Omega)$ holds $(\alpha \in \left]0,+\infty\right[)$.

(ii) Our approach makes it possible to consider the case when p assumes different values on each subdomain Ω_i. In this case, the minimization problem becomes

$$\underset{\substack{(u_i)_{i\in I}\in\bigoplus_{i\in I}\mathcal{H}_i \\ (\forall (i,j)\in K)\ \mathsf{T}_{ij}u_i=\mathsf{T}_{ji}u_j}}{\text{minimize}} \quad \sum_{i=1}^m \varphi_i(u_i) \tag{4.50}$$

where, for every $i \in I$,

$$\varphi_i\colon \mathcal{H}_i \to \left]-\infty,+\infty\right] : u_i \mapsto \begin{cases} \dfrac{1}{p_i}\displaystyle\int_{\Omega_i} |Du_i|^{p_i} - \int_{\Omega_i} fu_i, & \text{if } u_i \in E_i^{p_i} \cap \mathcal{H}_i; \\ +\infty, & \text{otherwise}, \end{cases} \tag{4.51}$$

and $p_i \in \left]1,+\infty\right[$. This modification is motivated by bonding problems in continuum mechanics.

(iii) Note that $(\overline{g}_{ij})_{(i,j)\in K}$ defined in (4.41) is a solution to the dual problem associated with Problem 4.8. The method proposed in Theorem 4.10 also converges in the dual variables, but for the sake of simplicity we provide only the convergence in primal variables.

Remark 4.12 The Plateau problem, i.e., the non parametric zero mean curvature problem, can be treated similar to the p-Laplacian problem (case $1 < p < 2$). The variational problem reads

$$\underset{\substack{u\in W^{1,1}(\Omega)\\ u=\phi \text{ on bdry}\,\Omega}}{\text{minimize}} \quad \int_\Omega \sqrt{1+|Du|^2}dx, \tag{4.52}$$

where $\phi\colon \text{bdry}\,\Omega \to \mathbb{R}$ is a given boundary data. The main issue in that situation is the existence and regularity of the solution of the variational problem. The regularity of the solution to (4.52) has been the object of active research. When bdry Ω is regular with nonnegative mean curvature and $\phi \in C^3(\bar{\Omega})$, there exists a unique solution of problem (4.52) which is regular, and the boundary condition is satisfied in a classical sense (by contrast with the relaxed boundary condition in the general case), see [24, Theorem 2.2, pp. 130]. Then one has to modify the function φ_i by introducing the non homogeneous Dirichlet boundary condition in its domain (i.e., φ_i is set to $+\infty$ when this condition is not satisfied). The function φ_i is still convex and lower semicontinuous on $\mathcal{H}_i = H^1(\Omega_i)$.

4.3 Obstacle problem

We adopt the notation of the Poisson Problem 4.3. Let $h\colon \Omega \to \mathbb{R}$ be an obstacle function of class $C^{1,1}$, and suppose that the constraint set

$$C = \big\{u \in H_0^1(\Omega) \;\big|\; u \geqslant h \;\text{ a.e. in } \Omega\big\} \tag{4.53}$$

is nonempty. This clearly requires that $h \leqslant 0$ on bdry Ω.

We consider the convex minimization problem called obstacle problem

$$\underset{u\in C}{\text{minimize}} \;\; \frac{1}{2}\int_\Omega |Du|^2 - \int_\Omega fu. \tag{4.54}$$

This strongly convex minimization problem admits a unique solution u (see [7, 31] for a general presentation and analysis of this problem). We are interested in solving it using the following equivalent formulation, which fits in our domain decomposition approach.

Problem 4.13 Consider the setting of Problem 3.1. Let $f \in L^2(\Omega)$, let $h \in C^{1,1}(\Omega)$, and, for every $i \in I$, define $C_i = \big\{u \in \mathcal{H}_i \;\big|\; u \geqslant h \;\text{ a.e. in } \Omega_i\big\}$. Suppose that, for every $(i,j) \in K$, Υ_{ij} and bdry Ω are of class $\mathcal{C}^2$. The problem is to

$$\underset{\substack{(u_i)_{i\in I}\in \times_{i\in I} C_i\\ (\forall (i,j)\in K)\; \mathsf{T}_{ij}u_i=\mathsf{T}_{ji}u_j}}{\text{minimize}} \quad \sum_{i=1}^m \frac{1}{2}\int_{\Omega_i} |Du_i|^2 - \int_{\Omega_i} fu_i. \tag{4.55}$$

Proposition 4.14 *Problem 4.13 has a unique solution $(\overline{u}_i)_{i\in I}$. Moreover, the function defined in (4.5) is the unique solution to (4.54).*

Proof. This is a consequence of Proposition 4.1 with, for every $i \in I$, $\phi_i \colon u \mapsto \iota_{C_i}(u)+\frac{1}{2}\int_{\Omega_i}|Du|^2-\int_{\Omega_i} fu$, which are strongly convex. In this case, $\phi\colon u \mapsto \iota_C(u)+\frac{1}{2}\int_\Omega |Du|^2-\int_\Omega fu$. □

Theorem 4.15 *In algorithm* (3.21) *of Theorem 3.6, replace the steps defining $p_{i,n}$ and $q_{ij,n}$ by*

$$p_{i,n} = P_{C_i}\left(\frac{1}{1+\gamma_n}v_{i,n}+\frac{\gamma_n}{1+\gamma_n}Q_i(f,0,\ldots,0)\right) \quad \text{and} \quad q_{ij,n}=0, \tag{4.56}$$

respectively. Then, for every $i \in I$, the sequence $(u_{i,n})_{n\in\mathbb{N}}$ generated by (3.21) *converges strongly to $\overline{u}_i$ in $\mathcal{H}_i$.*

Proof. Set

$$\begin{cases}(\forall i \in I) \quad \varphi_i \colon \mathcal{H}_i \to \left]-\infty,+\infty\right] : u_i \mapsto \iota_{C_i}(u_i)+\dfrac{1}{2}\displaystyle\int_{\Omega_i}|Du_i|^2-\int_{\Omega_i} fu_i\\ (\forall (i,j)\in K) \quad \psi_{ij}=\iota_{\{0\}}.\end{cases} \tag{4.57}$$

Since the sets $(C_i)_{i\in I}$ are closed and convex in $\mathcal{H}_i$, the convex functions $(\varphi_i)_{i\in I}$ are lower semicontinuous, and hence, for every $i \in I$, $\varphi_i \in \Gamma_0(\mathcal{H}_i)$. Moreover, for every $(i,j)\in K$, $\psi_{ij}\in\Gamma_0(L^2(\Upsilon_{ij}))$. Altogether, Problem 4.13 is a particular case of Problem 3.1. Let us verify that condition (3.10) holds. Let $(\overline{u}_i)_{i\in I}\in C_1\times\cdots\times C_m$ be the solution to Problem 4.13, and let $\overline{u}\in C$ defined by (4.5) be the unique solution to (4.54) guaranteed by Proposition 4.14. Since $\psi_{ij}=\iota_{\{0\}}$, we have $\partial\psi_{ij}(0)=L^2(\Upsilon_{ij})$, and hence the first condition in (3.10) is satisfied. Since $\operatorname{bdry}\Omega$ and $(\Upsilon_{ij})_{(i,j)\in K}$ are of class $\mathcal{C}^2$ and $h \in C^{1,1}$, we have $\overline{u}\in C^{1,1}$ and, for every $i\in I$ and $j \in J(i)$, $\nu_i^\top D\overline{u}_i \in L^2(\Upsilon_{ij})$ and $\nu_j^\top D\overline{u}_j\in L^2(\Upsilon_{ij})$ [31, Theorem 8.2] (see also [27]). Now let us show that the second condition in (3.10) holds with

$$(\forall (i,j)\in K) \quad \overline{g}_{ij}=\nu_j^\top D\overline{u}_j \in L^2(\Upsilon_{ij}). \tag{4.58}$$

The optimality condition for the solution $\overline{u}$ to (4.54) and Proposition 4.14 yield $\overline{u}\in C$ and

$$(\forall v \in C) \quad \int_\Omega D\overline{u}^\top D(v-\overline{u})-\int_\Omega f(v-\overline{u})\geqslant 0 \tag{4.59}$$

or, equivalently,

$$(\forall i \in I)(\forall v_i \in C_i) \text{ such that } (\forall (i,j)\in K)\ \mathsf{T}_{ij}v_i=\mathsf{T}_{ji}v_j$$
$$\sum_{i\in I}\left(\int_{\Omega_i}D\overline{u}_i^\top D(v_i-\overline{u}_i)-\int_{\Omega_i}f(v_i-\overline{u}_i)\right)\geqslant 0. \tag{4.60}$$

We deduce the system of Kuhn-Tucker conditions: for every $i\in I$ and every $j \in J(i)$, there exist positive Borel measures $\overline{\mu}_i$ on Ω_i and $\overline{\eta}_{ij}$ on Υ_{ij} such that

$$(\forall i \in I) \quad \begin{cases}-\Delta\overline{u}_i-f=\overline{\mu}_i, & \text{on } \Omega_i;\\ \overline{u}_i=0, & \text{on } \Upsilon_{ii};\\ \int_{\Omega_i}(\overline{u}_i-h)d\overline{\mu}_i=0; & \\ \mathsf{T}_{ij}\overline{u}_i=\mathsf{T}_{ji}\overline{u}_j, & \text{on } \Upsilon_{ij},\ \text{for every } j\in J(i);\\ \nu_i^\top D\overline{u}_i+\nu_j^\top D\overline{u}_j=\overline{\eta}_{ij} & \text{on } \Upsilon_{ij},\ \text{for every } j\in J(i);\\ \int_{\Upsilon_{ij}}(\overline{u}_i-h)d\overline{\eta}_{ij}=0 & \text{for every } j \in J(i).\end{cases} \tag{4.61}$$

Analogously, for every $i \in I$, the inclusion $v_i \in \partial\varphi_i(\overline{u}_i)$ is equivalent to the existence, for every $j \in J(i)$, of positive Borel measures μ_i on Ω_i and η_{ij} on Υ_{ij} such that

$$\begin{cases} -\Delta(\overline{u}_i - v_i) - f = \mu_i, & \text{on } \Omega_i; \\ \overline{u}_i - v_i = 0, & \text{on } \Upsilon_{ii}; \\ \int_{\Omega_i}(\overline{u}_i - h)d\mu_i = 0; & \\ \mathsf{T}_{ij}\,\overline{u}_i = \mathsf{T}_{ji}\overline{u}_j, & \text{on } \Upsilon_{ij}, \text{ for every } j \in J(i); \\ \nu_i^\top D(\overline{u}_i - v_i) = \eta_{ij} & \text{on } \Upsilon_{ij}, \text{ for every } j \in J(i); \\ \int_{\Upsilon_{ij}}(\overline{u}_i - h)d\eta_{ij} = 0 & \text{for every } j \in J(i). \end{cases} \tag{4.62}$$

Hence, by taking, for every $i \in I$, v_i to be the solution to the boundary value problem

$$\begin{cases} -\Delta v_i = 0, & \text{on } \Omega_i; \\ v_i = 0, & \text{on } \Upsilon_{ii}; \\ \nu_i^\top D v_i = -\nu_j^\top D\overline{u}_j, & \text{on } \Upsilon_{ij}, \text{ for every } j \in J(i) \end{cases} \tag{4.63}$$

and, for every $j \in J(i)$, $g_{ij} = \nu_j^\top D\overline{u}_j$ on Υ_{ij}, we deduce from (3.9) and (4.63) that $v_i = -Q_i\big(0, (g_{ij})_{j\in J(i+)}, (g_{ji})_{j\in J(i-)}\big) \in \partial\varphi_i(u_i)$ where $g_{ij} = \nu_j^\top D\overline{u}_j$ on Υ_{ij}. Hence, condition (3.10) holds.

On the other hand, it follows from (2.5) and (4.57) that, for every $(i,j) \in K$ and $\mu \in \left]0,+\infty\right[$, $\text{prox}_{\mu\psi_{ij}} \equiv 0$. Hence, we deduce from Proposition 4.2(i) that (4.56) yields

$$(\forall n \in \mathbb{N}) \quad \begin{cases} (\forall i \in I) \quad p_{i,n} = \text{prox}_{\gamma_n\varphi_i} v_{i,n} \\ (\forall (i,j) \in K) \quad q_{ij,n} = \text{prox}_{\mu_n\psi_{ij}}(l_{ij,n} + \mu_n g_{ij,n}). \end{cases} \tag{4.64}$$

Therefore, the result follows from Theorem 3.6, where $(\varphi_i)_{i\in I}$ and $(\psi_{ij})_{(i,j)\in K}$ are defined by (4.57). ◻

Remark 4.16

(i) Note that $(\overline{g}_{ij})_{(i,j)\in K}$ defined in (4.58) is a solution to the dual problem associated with Problem 4.13. The method proposed in Theorem 4.15 also guarantees the convergence of the dual variables, but for the sake of simplicity we provide only the primal convergence statement.

(ii) In (3.21) we have

$$(\forall n \in \mathbb{N})(\forall i \in I) \quad v_{i,n} = u_{i,n} - \gamma_n Q_i\big(0, (g_{ij,n})_{j\in J(i+)}, (g_{ji,n})_{j\in J(i-)}\big). \tag{4.65}$$

Hence, since the operators $(Q_i)_{i\in I}$ defined in (3.9) are multilinear, the sequences $(p_{i,n})_{i\in I,\, n\in\mathbb{N}}$ can be computed more efficiently via

$$(\forall n \in \mathbb{N})(\forall i \in I) \quad p_{i,n} = P_{C_i}\Big(\frac{1}{1+\gamma_n}u_{i,n} + \frac{\gamma_n}{1+\gamma_n}Q_i(f, (-g_{ij,n})_{j\in J(i+)}, (-g_{ji,n})_{j\in J(i-)})\Big). \tag{4.66}$$

This allows us to solve only m auxiliary PDE's for updating $(p_{i,n})_{i\in I}$ at each iteration n.

5 Perspectives

In this section we briefly outline possible adaptations and variants of our framework to related problems.

First, in the setting of the Poisson Problem 4.3 let, for every $i \in I$ and $j \in J(i+)$, $\varepsilon_{ij} \in \{-1, 1\}$, and consider the variational problem

$$\underset{\substack{u_1\in\mathcal{H}_1,\ldots,u_m\in\mathcal{H}_m\\(\forall i\in I)(\forall j\in J(i+))\ \varepsilon_{ij}(\mathsf{T}_{ij}u_i-\mathsf{T}_{ji}u_j)\geqslant 0}}{\text{minimize}} \quad \sum_{i=1}^{m} \frac{1}{2}\int_{\Omega_i} |Du_i|^2 - \int_{\Omega_i} f u_i. \tag{5.1}$$

By contrast with the preceding problems in which the bilateral constraint $\mathsf{T}_{ij}u_i - \mathsf{T}_{ji}u_j = 0$ imposes a continuity property at the interfaces, the constraint $\varepsilon_{ij}\,(\mathsf{T}_{ij}u_i - \mathsf{T}_{ji}u_j) \geqslant 0$ models a unilateral transmission condition through the interfaces. This occurs for example in the modelling of fissures and cracks. Depending on the sign of ε_{ij}, we have a nonzero flux from Ω_i towards Ω_j, or in the reverse direction. The main difference with respect to the previous examples is that, instead of using $\psi_{ij} = \iota_{\{0\}}$, in this case we set $\psi_{ij} = \iota_{\{L^2(\Upsilon_{ij})^+\}}$ or $\psi_{ij} = \iota_{\{L^2(\Upsilon_{ij})^-\}}$, depending on the sign of ε_{ij}. Clearly $\psi_{ij} \in \Gamma_0(L^2(\Upsilon_{ij}))$ because $L^2(\Upsilon_{ij})^+$ and $L^2(\Upsilon_{ij})^-$ are closed convex cones in $L^2(\Upsilon_{ij})$.

Modeling semi-permeable membranes gives rise to similar problems, which possibly involve both unilateral transmission conditions and surface energy functions. For example (here $\mu_{ij} > 0$ stands for some permeability coefficients)

$$\underset{\substack{i\in I,\, u_i\in\mathcal{H}_i\\ \varepsilon_{ij}(\mathsf{T}_{ij}u_i-\mathsf{T}_{ji}u_j)\geqslant 0,\\ j\in J(i+)}}{\text{minimize}} \quad \sum_{i=1}^{m} \frac{1}{2}\int_{\Omega_i} |Du_i|^2 - \int_{\Omega_i} f u_i + \sum_{i,j} \frac{\mu_{ij}}{2}\int_{\Upsilon_{ij}} |\mathsf{T}_{ij}u_i - \mathsf{T}_{ji}u_j|^2. \tag{5.2}$$

This problem is within the scope of our study. Depending on the sign of ε_{ij} one can take

$$\psi_{ij}(g) = \iota_{\{L^2(\Upsilon_{ij})^+\}}(g) + \frac{\mu_{ij}}{2}\int_{\Upsilon_{ij}} |g|^2 \tag{5.3}$$

or

$$\psi_{ij}(g) = \iota_{\{L^2(\Upsilon_{ij})^-\}}(g) + \frac{\mu_{ij}}{2}\int_{\Upsilon_{ij}} |g|^2. \tag{5.4}$$

Finally, let us note that in this paper we have considered only Dirichlet boundary conditions. Neumann and mixed boundary conditions can also be considered by working in Sobolev spaces $(\mathcal{H}_i)_{i\in I}$ associated with the corresponding variational formulation (for example, for the Neumann problem, one can take $\mathcal{H}_i = H^1(\Omega_i)$).

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