Variable Metric Forward-Backward Splitting with Applications to Monotone Inclusions in Duality*

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Abstract

We propose a variable metric forward-backward splitting algorithm and prove its convergence in real Hilbert spaces. We then use this framework to derive primal-dual splitting algorithms for solving various classes of monotone inclusions in duality. Some of these algorithms are new even when specialized to the fixed metric case. Various applications are discussed.

Keywords: cocoercive operator, composite operator, demiregularity, duality, forward-backward splitting algorithm, monotone inclusion, monotone operator, primal-dual algorithm, quasi-Fejér sequence, variable metric.

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1 Introduction

The forward-backward algorithm has a long history going back to the projected gradient method (see [1, 12] for historical background). It addresses the problem of finding a zero of the sum of two operators acting on a real Hilbert space $\mathcal{H}$, namely,

$$\text{find} \quad x \in \mathcal{H} \quad \text{such that} \quad 0 \in Ax + Bx, \tag{1.1}$$

under the assumption that $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone and that $B\colon \mathcal{H} \to \mathcal{H}$ is β-cocoercive for some $\beta \in \left]0,+\infty\right[$, i.e. [4],

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \langle x - y \mid Bx - By\rangle \geqslant \beta\|Bx - By\|^2. \tag{1.2}$$

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This framework is quite central due to the large class of problems it encompasses in areas such as partial differential equations, mechanics, evolution inclusions, signal and image processing, best approximation, convex optimization, learning theory, inverse problems, statistics, game theory, and variational inequalities [1, 4, 7, 10, 12, 15, 18, 20, 21, 23, 24, 29, 30, 39, 40, 42]. The forward-backward algorithm operates according to the routine

$$x_0 \in \mathcal{H} \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad x_{n+1} = (\mathrm{Id} + \gamma_n A)^{-1}(x_n - \gamma_n B x_n), \quad \text{where} \quad 0 < \gamma_n < 2\beta. \tag{1.3}$$

In classical optimization methods, the benefits of changing the underlying metric over the course of the iterations to improve convergence profiles has long been recognized [19, 33]. In proximal methods, variable metrics have been investigated mostly when $B = 0$ in (1.1). In such instances (1.3) reduces to the proximal point algorithm

$$x_0 \in \mathcal{H} \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad x_{n+1} = (\mathrm{Id} + \gamma_n A)^{-1} x_n, \quad \text{where} \quad \gamma_n > 0. \tag{1.4}$$

In the case when A is the subdifferential of a real-valued convex function in a finite dimensional setting, variable metric versions of (1.4) have been proposed in [5, 11, 27, 35]. These methods draw heavily on the fact that the proximal point algorithm for minimizing a function corresponds to the gradient descent method applied to its Moreau envelope. In the same spirit, variable metric proximal point algorithms for a general maximally monotone operator A were considered in [8, 36]. In [8], superlinear convergence rates were shown to be achievable under suitable hypotheses (see also [9] for further developments). The finite dimensional variable metric proximal point algorithm proposed in [32] allows for errors in the proximal steps and features a flexible class of exogenous metrics to implement the algorithm. The first variable metric forward-backward algorithm appears to be that introduced in [10, Section 5]. It focuses on linear convergence results in the case when $A + B$ is strongly monotone and $\mathcal{H}$ is finite-dimensional. The variable metric splitting algorithm of [28] provides a framework which can be used to solve (1.1) in instances when $\mathcal{H}$ is finite-dimensional and B is merely Lipschitzian. However, it does not exploit the cocoercivity property (1.2) and it is more cumbersome to implement than the forward-backward iteration. Let us add that, in the important case when B is the gradient of a convex function, the Baillon-Haddad theorem asserts that the notions of cocoercivity and Lipschitz-continuity coincide [4, Corollary 18.16].

The goal of this paper is two-fold. First, we propose a general purpose variable metric forward-backward algorithm to solve (1.1)–(1.2) in Hilbert spaces and analyze its asymptotic behavior, both in terms of weak and strong convergence. Second, we show that this algorithm can be used to solve a broad class of composite monotone inclusion problems in duality by formulating them as instances of (1.1)–(1.2) in alternate Hilbert spaces. Even when restricted to the constant metric case, some of these results are new.

The paper is organized as follows. Section 2 is devoted to notation and background. In Section 3, we provide preliminary results. The variable metric forward-backward algorithm is introduced and analyzed in Section 4. In Section 5, we present a new variable metric primal-dual splitting algorithm for strongly monotone composite inclusions. This algorithm is obtained by applying the forward-backward algorithm of Section 4 to the dual inclusion. In Section 6, we consider a more general class of composite inclusions in duality and show that they can be solved by applying the forward-backward algorithm of Section 4 to a certain inclusion problem posed in the primal-dual product space. Applications to minimization problems, variational inequalities, and best approximation are discussed.

2 Notation and background

We recall some notation and background from convex analysis and monotone operator theory (see [4] for a detailed account).

Throughout, $\mathcal{H}$, $\mathcal{G}$, and $(\mathcal{G}_i)_{1\leqslant i\leqslant m}$ are real Hilbert spaces. We denote the scalar product of a Hilbert space by $\langle \cdot \mid \cdot \rangle$ and the associated norm by $\|\cdot\|$. The symbols $\rightharpoonup$ and $\to$ denote respectively weak and strong convergence, and Id denotes the identity operator. We denote by $\mathcal{B}(\mathcal{H},\mathcal{G})$ the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{G}$, we set $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H},\mathcal{H})$ and $\mathcal{S}(\mathcal{H}) = \big\{L \in \mathcal{B}(\mathcal{H}) \mid L = L^*\big\}$, where L^* denotes the adjoint of L. The Loewner partial ordering on $\mathcal{S}(\mathcal{H})$ is defined by

$$(\forall U \in \mathcal{S}(\mathcal{H}))(\forall V \in \mathcal{S}(\mathcal{H})) \quad U \succcurlyeq V \quad \Leftrightarrow \quad (\forall x \in \mathcal{H}) \quad \langle Ux \mid x\rangle \geqslant \langle Vx \mid x\rangle . \tag{2.1}$$

Now let $\alpha \in [0,+\infty[$. We set

$$\mathcal{P}_\alpha(\mathcal{H}) = \big\{U \in \mathcal{S}(\mathcal{H}) \mid U \succcurlyeq \alpha\,\mathrm{Id}\big\}, \tag{2.2}$$

and we denote by $\sqrt{U}$ the square root of $U \in \mathcal{P}_\alpha(\mathcal{H})$. Moreover, for every $U \in \mathcal{P}_\alpha(\mathcal{H})$, we define a semi-scalar product and a semi-norm (a scalar product and a norm if $\alpha > 0$) by

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \langle x \mid y\rangle_U = \langle Ux \mid y\rangle \quad \text{and} \quad \|x\|_U = \sqrt{\langle Ux \mid x\rangle}. \tag{2.3}$$

Notation 2.1 We denote by $\boldsymbol{\mathcal{G}} = \mathcal{G}_1\oplus\cdots\oplus\mathcal{G}_m$ the Hilbert direct sum of the Hilbert spaces $(\mathcal{G}_i)_{1\leqslant i\leqslant m}$, i.e., their product space equipped with the scalar product and the associated norm respectively defined by

$$\langle\!\langle \cdot \mid \cdot \rangle\!\rangle \colon (\boldsymbol{x},\boldsymbol{y}) \mapsto \sum_{i=1}^m \langle x_i \mid y_i\rangle \quad \text{and} \quad |||\cdot|||\colon \boldsymbol{x} \mapsto \sqrt{\sum_{i=1}^m \|x_i\|^2}, \tag{2.4}$$

where $\boldsymbol{x} = (x_i)_{1\leqslant i\leqslant m}$ and $\boldsymbol{y} = (y_i)_{1\leqslant i\leqslant m}$ denote generic elements in $\boldsymbol{\mathcal{G}}$.

Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued operator. The domain and the graph of A are respectively defined by $\operatorname{dom} A = \big\{x \in \mathcal{H} \mid Ax \neq \varnothing\big\}$ and $\operatorname{gra} A = \big\{(x,u) \in \mathcal{H}\times\mathcal{H} \mid u \in Ax\big\}$. We denote by $\operatorname{zer} A = \big\{x \in \mathcal{H} \mid 0 \in Ax\big\}$ the set of zeros of A and by $\operatorname{ran} A = \big\{u \in \mathcal{H} \mid (\exists\, x \in \mathcal{H})\; u \in Ax\big\}$ the range of A. The inverse of A is $A^{-1}\colon \mathcal{H} \mapsto 2^{\mathcal{H}}\colon u \mapsto \big\{x \in \mathcal{H} \mid u \in Ax\big\}$, and the resolvent of A is

$$J_A = (\mathrm{Id} + A)^{-1}. \tag{2.5}$$

Moreover, A is monotone if

$$(\forall (x,y) \in \mathcal{H}\times\mathcal{H})(\forall (u,v) \in Ax \times Ay) \quad \langle x - y \mid u - v\rangle \geqslant 0, \tag{2.6}$$

and maximally monotone if it is monotone and there exists no monotone operator $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ such that $\operatorname{gra} A \subset \operatorname{gra} B$ and $A \neq B$. The parallel sum of A and $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ is

$$A \,\Box\, B = (A^{-1} + B^{-1})^{-1}. \tag{2.7}$$

The conjugate of $f\colon \mathcal{H} \to \,]-\infty,+\infty]$ is

$$f^*\colon \mathcal{H} \to [-\infty,+\infty] : u \mapsto \sup_{x\in\mathcal{H}} \big(\langle x \mid u\rangle - f(x)\big), \tag{2.8}$$

and the infimal convolution of f with $g\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ is

$$f \,\square\, g\colon \mathcal{H} \to [-\infty,+\infty] : x \mapsto \inf_{y\in\mathcal{H}} \big(f(y) + g(x-y)\big). \tag{2.9}$$

The class of lower semicontinuous convex functions $f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ such that $\operatorname{dom} f = \big\{x \in \mathcal{H} \mid f(x) < +\infty\big\} \neq \varnothing$ is denoted by $\Gamma_0(\mathcal{H})$. If $f \in \Gamma_0(\mathcal{H})$, then $f^* \in \Gamma_0(\mathcal{H})$ and the subdifferential of f is the maximally monotone operator

$$\partial f\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{u \in \mathcal{H} \mid (\forall y \in \mathcal{H})\ \langle y - x \mid u\rangle + f(x) \leqslant f(y)\big\} \tag{2.10}$$

with inverse $(\partial f)^{-1} = \partial f^*$. Let C be a nonempty subset of $\mathcal{H}$. The indicator function and the distance function of C are defined on $\mathcal{H}$ as

$$\iota_C\colon x \mapsto \begin{cases} 0, & \text{if } x \in C;\\ +\infty, & \text{if } x \notin C \end{cases} \qquad \text{and} \qquad d_C = \iota_C \,\square\, \|\cdot\|\colon x \mapsto \inf_{y\in C}\|x-y\|. \tag{2.11}$$

respectively. The interior of C is $\operatorname{int} C$ and the support function of C is $\sigma_C = \iota_C^*$. Now suppose that C is convex. The normal cone operator of C is defined as

$$N_C = \partial\iota_C\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \begin{cases} \big\{u \in \mathcal{H} \mid (\forall y \in C)\ \ \langle y - x \mid u\rangle \leqslant 0\big\}, & \text{if } x \in C;\\ \varnothing, & \text{otherwise.}\end{cases} \tag{2.12}$$

The strong relative interior of C, i.e., the set of points $x \in C$ such that the conical hull of $-x + C$ is a closed vector subspace of $\mathcal{H}$, is denoted by $\operatorname{sri} C$; if $\mathcal{H}$ is finite-dimensional, $\operatorname{sri} C$ coincides with the relative interior of C, denoted by $\operatorname{ri} C$. If C is also closed, its projector is denoted by P_C, i.e., $P_C\colon \mathcal{H} \to C\colon x \mapsto \operatorname{argmin}_{y\in C}\|x-y\|$.

Finally, $\ell_+^1(\mathbb{N})$ denotes the set of summable sequences in $\left[0,+\infty\right[$.

3 Preliminary results

3.1 Technical results

The following properties can be found in [26, Section VI.2.6] (see [17, Lemma 2.1] for an alternate short proof).

Lemma 3.1 *Let $\alpha \in \left]0,+\infty\right[$ and $\mu \in \left]0,+\infty\right[$, and assume that A and B are operators in $\mathcal{S}(\mathcal{H})$ such that $\mu\operatorname{Id} \succcurlyeq A \succcurlyeq B \succcurlyeq \alpha\operatorname{Id}$. Then the following hold.*

(i) $\alpha^{-1}\operatorname{Id} \succcurlyeq B^{-1} \succcurlyeq A^{-1} \succcurlyeq \mu^{-1}\operatorname{Id}$.

(ii) $(\forall x \in \mathcal{H})\ \big\langle A^{-1}x \mid x\big\rangle \geqslant \|A\|^{-1}\|x\|^2$.

(iii) $\|A^{-1}\| \leqslant \alpha^{-1}$.

The next fact concerns sums of composite cocoercive operators.

Proposition 3.2 *Let I be a finite index set. For every $i \in I$, let $0 \neq L_i \in \mathcal{B}(\mathcal{H},\mathcal{G}_i)$, let $\beta_i \in \left]0,+\infty\right[$, and let $T_i\colon \mathcal{G}_i \to \mathcal{G}_i$ be β_i-cocoercive. Set $T = \sum_{i\in I} L_i^* T_i L_i$ and $\beta = 1/\big(\sum_{i\in I}\|L_i\|^2/\beta_i\big)$. Then T is β-cocoercive.*

Proof. Set $(\forall i \in I)$ $\alpha_i = \beta\|L_i\|^2/\beta_i$. Then $\sum_{i\in I}\alpha_i = 1$ and, using the convexity of $\|\cdot\|^2$ and (1.2), we have

$$
\begin{aligned}
(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \langle x - y \mid Tx - Ty\rangle &= \sum_{i\in I} \langle x - y \mid L_i^* T_i L_i x - L_i^* T_i L_i y\rangle \\
&= \sum_{i\in I} \langle L_i x - L_i y \mid T_i L_i x - T_i L_i y\rangle \\
&\geqslant \sum_{i\in I} \beta_i \|T_i L_i x - T_i L_i y\|^2 \\
&\geqslant \sum_{i\in I} \frac{\beta_i}{\|L_i\|^2} \|L_i^* T_i L_i x - L_i^* T_i L_i y\|^2 \\
&= \beta \sum_{i\in I} \alpha_i \left\| \frac{1}{\alpha_i}(L_i^* T_i L_i x - L_i^* T_i L_i y)\right\|^2 \\
&\geqslant \beta \left\| \sum_{i\in I}(L_i^* T_i L_i x - L_i^* T_i L_i y)\right\|^2 \\
&= \beta\|Tx - Ty\|^2, \qquad (3.1)
\end{aligned}
$$

which concludes the proof. ☐

3.2 Variable metric quasi-Fejér sequences

The following results are from [17].

Proposition 3.3 *Let $\alpha \in \left]0, +\infty\right[$, let $(W_n)_{n\in\mathbb{N}}$ be in $\mathcal{P}_\alpha(\mathcal{H})$, let C be a nonempty subset of $\mathcal{H}$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that*

$$
\big(\exists\, (\eta_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})\big)\big(\forall z \in C\big)\big(\exists\, (\varepsilon_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})\big)(\forall n \in \mathbb{N}) \quad \|x_{n+1} - z\|_{W_{n+1}} \leqslant (1 + \eta_n)\|x_n - z\|_{W_n} + \varepsilon_n. \qquad (3.2)
$$

Then $(x_n)_{n\in\mathbb{N}}$ is bounded and, for every $z \in C$, $(\|x_n - z\|_{W_n})_{n\in\mathbb{N}}$ converges.

Proposition 3.4 *Let $\alpha \in \left]0, +\infty\right[$, and let $(W_n)_{n\in\mathbb{N}}$ and W be operators in $\mathcal{P}_\alpha(\mathcal{H})$ such that $W_n \to W$ pointwise as $n \to +\infty$, as is the case when*

$$
\sup_{n\in\mathbb{N}} \|W_n\| < +\infty \quad \textit{and} \quad (\exists\, (\eta_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N}))(\forall n \in \mathbb{N}) \quad (1 + \eta_n)W_n \succcurlyeq W_{n+1}. \qquad (3.3)
$$

Let C be a nonempty subset of $\mathcal{H}$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that (3.2) is satisfied. Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point in C if and only if every weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$ is in C.

Proposition 3.5 *Let $\alpha \in \left]0, +\infty\right[$, let $(W_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{H})$ such that $\sup_{n\in\mathbb{N}}\|W_n\| < +\infty$, let C be a nonempty closed subset of $\mathcal{H}$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that*

$$
\big(\exists\, (\varepsilon_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})\big)\big(\exists\, (\eta_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})\big)(\forall z \in C)(\forall n \in \mathbb{N}) \quad \|x_{n+1} - z\|_{W_{n+1}} \leqslant (1 + \eta_n)\|x_n - z\|_{W_n} + \varepsilon_n. \qquad (3.4)
$$

Then $(x_n)_{n\in\mathbb{N}}$ converges strongly to a point in C if and only if $\varliminf d_C(x_n) = 0$.

Proposition 3.6 *Let $\alpha \in \left]0,+\infty\right[$, let $(\nu_n)_{n\in\mathbb{N}} \in \ell^1_+(\mathbb{N})$, and let $(W_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{H})$ such that $\sup_{n\in\mathbb{N}} \|W_n\| < +\infty$ and $(\forall n \in \mathbb{N})$ $(1+\nu_n)W_{n+1} \succcurlyeq W_n$. Furthermore, let C be a subset of $\mathcal{H}$ such that $\operatorname{int} C \neq \varnothing$, let $z \in C$ and $\rho \in \left]0,+\infty\right[$ be such that $B(z;\rho) \subset C$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that*

$$\big(\exists\,(\varepsilon_n)_{n\in\mathbb{N}} \in \ell^1_+(\mathbb{N})\big)\big(\exists\,(\eta_n)_{n\in\mathbb{N}} \in \ell^1_+(\mathbb{N})\big)(\forall x \in B(z;\rho))(\forall n \in \mathbb{N}) \\ \|x_{n+1} - x\|^2_{W_{n+1}} \leqslant (1+\eta_n)\|x_n - x\|^2_{W_n} + \varepsilon_n. \quad (3.5)$$

Then $(x_n)_{n\in\mathbb{N}}$ converges strongly.

3.3 Monotone operators

We establish some results on monotone operators in a variable metric environment.

Lemma 3.7 *Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone, let $\alpha \in \left]0,+\infty\right[$, let $U \in \mathcal{P}_\alpha(\mathcal{H})$, and let $\mathcal{G}$ be the real Hilbert space obtained by endowing $\mathcal{H}$ with the scalar product $(x,y) \mapsto \langle x \mid y\rangle_{U^{-1}} = \left\langle x \mid U^{-1}y\right\rangle$. Then the following hold.*

(i) *$UA\colon \mathcal{G} \to 2^{\mathcal{G}}$ is maximally monotone.*

(ii) *$J_{UA}\colon \mathcal{G} \to \mathcal{G}$ is 1-cocoercive, i.e., firmly nonexpansive, hence nonexpansive.*

(iii) *$J_{UA} = (U^{-1} + A)^{-1} \circ U^{-1}$.*

Proof. (i): Set $B = UA$ and $V = U^{-1}$. For every $(x,u) \in \operatorname{gra} B$ and every $(y,v) \in \operatorname{gra} B$, $Vu \in VBx = Ax$ and $Vv \in VBy = Ay$, so that

$$\langle x - y \mid u - v\rangle_V = \langle x - y \mid Vu - Vv\rangle \geqslant 0 \quad (3.6)$$

by monotonicity of A on $\mathcal{H}$. This shows that B is monotone on $\mathcal{G}$. Now let $(y,v) \in \mathcal{H}^2$ be such that

$$(\forall (x,u) \in \operatorname{gra} B) \quad \langle x - y \mid u - v\rangle_V \geqslant 0. \quad (3.7)$$

Then, for every $(x,u) \in \operatorname{gra} A$, $(x,Uu) \in \operatorname{gra} B$ and we derive from (3.7) that

$$\langle x - y \mid u - Vv\rangle = \langle x - y \mid Uu - v\rangle_V \geqslant 0. \quad (3.8)$$

Since A is maximally monotone on $\mathcal{H}$, (3.8) gives $(y,Vv) \in \operatorname{gra} A$, which implies that $(y,v) \in \operatorname{gra} B$. Hence, B is maximally monotone on $\mathcal{G}$.

(ii): This follows from (i) and [4, Corollary 23.8].

(iii): Let x and p be in $\mathcal{G}$. Then $p = J_{UA}x \Leftrightarrow x \in p + UAp \Leftrightarrow U^{-1}x \in (U^{-1} + A)p \Leftrightarrow p = (U^{-1} + A)^{-1}(U^{-1}x)$. ☐

Remark 3.8 let $\alpha \in \left]0,+\infty\right[$, let $U \in \mathcal{P}_\alpha(\mathcal{H})$, set $f\colon \mathcal{H} \to \mathbb{R}\colon x \mapsto \left\langle U^{-1}x \mid x\right\rangle/2$, and let $D\colon (x,y) \mapsto f(x) - f(y) - \langle x - y \mid \nabla f(y)\rangle$ be the associated Bregman distance. Then Lemma 3.7(iii) asserts that $J_{UA} = (\nabla f + A)^{-1} \circ \nabla f$. In other words, J_{UA} is the D-resolvent of A introduced in [3, Definition 3.7].

Let $U \in \mathcal{P}_\alpha(\mathcal{H})$ for some $\alpha \in \,]0,+\infty[$. The proximity operator of $f \in \Gamma_0(\mathcal{H})$ relative to the metric induced by U is [25, Section XV.4]

$$\operatorname{prox}_f^U \colon \mathcal{H} \to \mathcal{H} \colon x \mapsto \underset{y\in\mathcal{H}}{\operatorname{argmin}}\Big(f(y) + \frac{1}{2}\|x-y\|_U^2\Big), \tag{3.9}$$

and the projector onto a nonempty closed convex subset C of $\mathcal{H}$ relative to the norm $\|\cdot\|_U$ is denoted by P_C^U. We have

$$\operatorname{prox}_f^U = J_{U^{-1}\partial f} \quad\text{and}\quad P_C^U = \operatorname{prox}_{\iota_C}^U, \tag{3.10}$$

and we write $\operatorname{prox}_f^{\mathrm{Id}} = \operatorname{prox}_f$.

In the case when $U = \mathrm{Id}$ in Lemma 3.7, examples of closed form expressions for J_{UA} and basic resolvent calculus rules can be found in [4, 15, 18]. A few examples illustrating the case when $U \neq \mathrm{Id}$ are provided below. The first result is an extension of the well-known resolvent identity $J_A + J_{A^{-1}} = \mathrm{Id}$.

Example 3.9 Let $\alpha \in \,]0,+\infty[$, let $\gamma \in \,]0,+\infty[$, and let $U \in \mathcal{P}_\alpha(\mathcal{H})$. Then the following hold.

(i) Let $A \colon \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone. Then

$$J_{\gamma UA} = \sqrt{U} J_{\gamma\sqrt{U}A\sqrt{U}} \sqrt{U}^{\,-1} = \mathrm{Id} - \gamma U J_{\gamma^{-1}U^{-1}A^{-1}}(\gamma^{-1}U^{-1}). \tag{3.11}$$

(ii) Let $f \in \Gamma_0(\mathcal{H})$. Then $\operatorname{prox}_{\gamma f}^U = \sqrt{U}^{\,-1} \operatorname{prox}_{\gamma f\circ\sqrt{U}^{-1}} \sqrt{U} = \mathrm{Id} - \gamma U^{-1} \operatorname{prox}_{\gamma^{-1}f^*}^{U^{-1}}(\gamma^{-1}U)$.

(iii) Let C be a nonempty closed convex subset of $\mathcal{H}$. Then $\operatorname{prox}_{\gamma\sigma_C}^U = \sqrt{U}^{\,-1} \operatorname{prox}_{\gamma\sigma_C\circ\sqrt{U}^{-1}} \sqrt{U} = \mathrm{Id} - \gamma U^{-1} P_C^{U^{-1}}(\gamma^{-1}U)$.

Proof. (i): Let x and p be in $\mathcal{H}$. Then

$$\begin{aligned} p = J_{\gamma UA}x \quad &\Leftrightarrow \quad x - p \in \gamma UAp \\ &\Leftrightarrow \quad \sqrt{U}^{-1}x - \sqrt{U}^{-1}p \in \gamma\sqrt{U}A\sqrt{U}\sqrt{U}^{-1}p \\ &\Leftrightarrow \quad \sqrt{U}^{-1}p = J_{\gamma\sqrt{U}A\sqrt{U}}\big(\sqrt{U}^{-1}x\big) \\ &\Leftrightarrow \quad p = \sqrt{U}J_{\gamma\sqrt{U}A\sqrt{U}}\big(\sqrt{U}^{-1}x\big). \end{aligned} \tag{3.12}$$

Furthermore, by [4, Proposition 23.23(ii)], $J_{\sqrt{U}(\gamma A)\sqrt{U}} = \mathrm{Id} - \sqrt{U}\big(U + (\gamma A)^{-1}\big)^{-1}\sqrt{U}$. Hence, (3.12) yields

$$J_{\gamma UA} = \mathrm{Id} - U\big(U + (\gamma A)^{-1}\big)^{-1}. \tag{3.13}$$

However

$$\begin{aligned} p = \big(U + (\gamma A)^{-1}\big)^{-1}x \quad &\Leftrightarrow \quad x \in Up + (\gamma A)^{-1}p \\ &\Leftrightarrow \quad \gamma^{-1}p \in A(x - Up) \\ &\Leftrightarrow \quad x - Up \in A^{-1}(\gamma^{-1}p) \\ &\Leftrightarrow \quad \gamma^{-1}U^{-1}x \in \big(\mathrm{Id} + \gamma^{-1}U^{-1}A^{-1}\big)(\gamma^{-1}p) \\ &\Leftrightarrow \quad \gamma^{-1}p = J_{\gamma^{-1}U^{-1}A^{-1}}(\gamma^{-1}U^{-1}x). \end{aligned} \tag{3.14}$$

Hence, $(U+(\gamma A)^{-1})^{-1} = \gamma J_{\gamma^{-1}U^{-1}A^{-1}}(\gamma^{-1}U^{-1})$ and, using (3.13), we obtain the rightmost identity in (i).

(ii): Apply (i) to $A = \partial f$, and use (3.10) and the fact that $\partial(f \circ \sqrt{U}^{-1}) = (\sqrt{U}^{-1})^* \circ (\partial f) \circ \sqrt{U}^{-1} = \sqrt{U}^{-1} \circ (\partial f) \circ \sqrt{U}^{-1}$ [4, Corollary 16.42(i)].

(iii): Apply (ii) to $f = \sigma_C$, and use (3.10). ☐

Example 3.10 Define $\boldsymbol{\mathcal{G}}$ as in Notation 2.1, let $\alpha \in \mathbb{R}$, and, for every $i \in \{1, \ldots, m\}$, let $A_i \colon \mathcal{G}_i \to 2^{\mathcal{G}_i}$ be maximally monotone and let $U_i \in \mathcal{P}_\alpha(\mathcal{G}_i)$. Set $\boldsymbol{A} \colon \boldsymbol{\mathcal{G}} \to 2^{\boldsymbol{\mathcal{G}}} \colon (x_i)_{1 \leqslant i \leqslant m} \mapsto \times_{i=1}^m A_i x_i$ and $\boldsymbol{U} \colon \boldsymbol{\mathcal{G}} \to \boldsymbol{\mathcal{G}} \colon (x_i)_{1 \leqslant i \leqslant m} \mapsto (U_i x_i)_{1 \leqslant i \leqslant m}$. Then $\boldsymbol{U}\boldsymbol{A}$ is maximally monotone and

$$(\forall (x_i)_{1 \leqslant i \leqslant m} \in \boldsymbol{\mathcal{G}}) \quad J_{\boldsymbol{U}\boldsymbol{A}}(x_i)_{1 \leqslant i \leqslant m} = (J_{U_i A_i} x_i)_{1 \leqslant i \leqslant m}. \tag{3.15}$$

Proof. This follows from Lemma 3.7(i) and [4, Proposition 23.16]. ☐

Example 3.11 Let $\alpha \in \,]0, +\infty[$, let $\xi \in \mathbb{R}$, let $U \in \mathcal{P}_\alpha(\mathcal{H})$, let $\phi \in \Gamma_0(\mathbb{R})$, suppose that $0 \neq u \in \mathcal{H}$, and set $H = \big\{x \in \mathcal{H} \mid \langle x \mid u \rangle \leqslant \xi\big\}$ and $g = \phi(\langle \cdot \mid u \rangle)$. Then $g \in \Gamma_0(\mathcal{H})$ and

$$(\forall x \in \mathcal{H}) \quad \operatorname{prox}_g^U x = x + \frac{\operatorname{prox}_{\|\sqrt{U^{-1}}u\|^2 \phi} \langle x \mid u \rangle - \langle x \mid u \rangle}{\|\sqrt{U^{-1}}u\|^2} U^{-1}u \tag{3.16}$$

and

$$P_H^U x = \begin{cases} x, & \text{if } \langle x \mid u \rangle \leqslant \xi; \\ x + \dfrac{\xi - \langle x \mid u \rangle}{\langle u \mid U^{-1}u \rangle} U^{-1}u, & \text{if } \langle x \mid u \rangle > \xi. \end{cases} \tag{3.17}$$

Proof. It follows from Example 3.9(ii) that

$$(\forall x \in \mathcal{H}) \quad \operatorname{prox}_g^U x = \sqrt{U^{-1}} \operatorname{prox}_{g \circ \sqrt{U^{-1}}} \sqrt{U} x. \tag{3.18}$$

Moreover, $g \circ \sqrt{U^{-1}} = \phi(\langle \cdot \mid \sqrt{U^{-1}}u \rangle)$. Hence, using (3.18) and [4, Corollary 23.33], we obtain

$$\begin{aligned} (\forall x \in \mathcal{H}) \quad \operatorname{prox}_g^U x &= \sqrt{U^{-1}} \operatorname{prox}_{\phi(\langle \cdot \mid \sqrt{U^{-1}}u \rangle)} \sqrt{U} x \\ &= x + \frac{\operatorname{prox}_{\|\sqrt{U^{-1}}u\|^2 \phi} \langle x \mid u \rangle - \langle x \mid u \rangle}{\|\sqrt{U^{-1}}u\|^2} U^{-1}u. \end{aligned} \tag{3.19}$$

Finally, upon setting $\phi = \iota_{]-\infty, \xi]}$, we obtain (3.17) from (3.16). ☐

Example 3.12 Let $\alpha \in \,]0, +\infty[$, let $\gamma \in \mathbb{R}$, let $A \in \mathcal{P}_0(\mathcal{H})$, let $u \in \mathcal{H}$, let $U \in \mathcal{P}_\alpha(\mathcal{H})$, and set $\varphi \colon \mathcal{H} \to \mathbb{R} \colon x \mapsto \langle Ax \mid x \rangle /2 + \langle x \mid u \rangle + \gamma$. Then $\varphi \in \Gamma_0(\mathcal{H})$ and

$$(\forall x \in \mathcal{H}) \quad \operatorname{prox}_\varphi^U x = (\operatorname{Id} + U^{-1}A)^{-1}(x - U^{-1}u). \tag{3.20}$$

Proof. Let $x \in \mathcal{H}$. Then $p = \operatorname{prox}_\varphi^U x \Leftrightarrow x - p = U^{-1}\nabla\varphi(p) \Leftrightarrow x - p = U^{-1}(Ap + u) \Leftrightarrow x - U^{-1}u = (\operatorname{Id} + U^{-1}A)p \Leftrightarrow p = (\operatorname{Id} + U^{-1}A)^{-1}(x - U^{-1}u)$. ☐

Example 3.13 Let $\alpha \in\,]0,+\infty[$ and let $U \in \mathcal{P}_\alpha(\mathcal{H})$. For every $i \in \{1,\ldots,m\}$, let $r_i \in \mathcal{G}_i$, let $\omega_i \in\,]0,+\infty[$, and let $L_i \in \mathcal{B}(\mathcal{H},\mathcal{G}_i)$. Set $\varphi\colon x \mapsto (1/2)\sum_{i=1}^m \omega_i \|L_i x - r_i\|^2$. Then $\varphi \in \Gamma_0(\mathcal{H})$ and

$$(\forall x \in \mathcal{H}) \quad \operatorname{prox}_\varphi^U x = \bigg(\operatorname{Id} + U^{-1}\sum_{i=1}^m \omega_i L_i^* L_i\bigg)^{-1}\bigg(x + U^{-1}\sum_{i=1}^m \omega_i L_i^* r_i\bigg). \tag{3.21}$$

Proof. We have $\varphi\colon x \mapsto \langle Ax \mid x\rangle/2 + \langle x \mid u\rangle + \gamma$, where $A = \sum_{i=1}^m \omega_i L_i^* L_i$, $u = -\sum_{i=1}^m \omega_i L_i^* r_i$, and $\gamma = \sum_{i=1}^m \omega_i \|r_i\|^2/2$. Hence, (3.21) follows from (3.20). ◻

3.4 Demiregularity

Definition 3.14 [1, Definition 2.3] *An operator $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is* demiregular *at $x \in \operatorname{dom} A$ if, for every sequence $((x_n,u_n))_{n\in\mathbb{N}}$ in* $\operatorname{gra} A$ *and every $u \in Ax$ such that $x_n \rightharpoonup x$ and $u_n \to u$ as $n \to +\infty$, we have $x_n \to x$ as $n \to +\infty$.*

Lemma 3.15 [1, Proposition 2.4] *Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be monotone and suppose that $x \in \operatorname{dom} A$. Then A is demiregular at x in each of the following cases.*

(i) *A is uniformly monotone at x, i.e., there exists an increasing function $\phi\colon [0,+\infty[\, \to [0,+\infty]$ that vanishes only at 0 such that $(\forall u \in Ax)(\forall (y,v) \in \operatorname{gra} A)$ $\langle x - y \mid u - v\rangle \geqslant \phi(\|x-y\|)$.*

(ii) *A is strongly monotone, i.e., there exists $\alpha \in\,]0,+\infty[$ such that $A - \alpha\operatorname{Id}$ is monotone.*

(iii) *J_A is compact, i.e., for every bounded set $C \subset \mathcal{H}$, the closure of $J_A(C)$ is compact. In particular,* $\operatorname{dom} A$ *is boundedly relatively compact, i.e., the intersection of its closure with every closed ball is compact.*

(iv) *$A\colon \mathcal{H} \to \mathcal{H}$ is single-valued with a single-valued continuous inverse.*

(v) *A is single-valued on* $\operatorname{dom} A$ *and* $\operatorname{Id} - A$ *is demicompact, i.e., for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in* $\operatorname{dom} A$ *such that $(Ax_n)_{n\in\mathbb{N}}$ converges strongly, $(x_n)_{n\in\mathbb{N}}$ admits a strong cluster point.*

(vi) *$A = \partial f$, where $f \in \Gamma_0(\mathcal{H})$ is uniformly convex at x, i.e., there exists an increasing function $\phi\colon [0,+\infty[\, \to [0,+\infty]$ that vanishes only at 0 such that $(\forall \alpha \in\,]0,1[)(\forall y \in \operatorname{dom} f)$ $f\big(\alpha x + (1-\alpha)y\big) + \alpha(1-\alpha)\phi(\|x-y\|) \leqslant \alpha f(x) + (1-\alpha) f(y)$.*

(vii) *$A = \partial f$, where $f \in \Gamma_0(\mathcal{H})$ and, for every $\xi \in \mathbb{R}$, $\big\{x \in \mathcal{H} \mid f(x) \leqslant \xi\big\}$ is boundedly compact.*

4 Algorithm and convergence

Our main result is stated in the following theorem.

Theorem 4.1 *Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone, let $\alpha \in\,]0,+\infty[$, let $\beta \in\,]0,+\infty[$, let $B\colon \mathcal{H} \to \mathcal{H}$ be β-cocoercive, let $(\eta_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})$, and let $(U_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{H})$ such that*

$$\mu = \sup_{n\in\mathbb{N}} \|U_n\| < +\infty \quad \textit{and} \quad (\forall n \in \mathbb{N}) \quad (1+\eta_n)U_{n+1} \succcurlyeq U_n. \tag{4.1}$$

Let $\varepsilon \in \left]0, \min\{1, 2\beta/(\mu+1)\}\right[$*, let* $(\lambda_n)_{n\in\mathbb{N}}$ *be a sequence in* $[\varepsilon, 1]$*, let* $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence in* $[\varepsilon, (2\beta-\varepsilon)/\mu]$*, let* $x_0 \in \mathcal{H}$*, and let* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be absolutely summable sequences in* $\mathcal{H}$*. Suppose that*

$$Z = \operatorname{zer}(A+B) \neq \varnothing, \tag{4.2}$$

and set

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_n = x_n - \gamma_n U_n (Bx_n + b_n) \\ x_{n+1} = x_n + \lambda_n \big(J_{\gamma_n U_n A}(y_n) + a_n - x_n\big). \end{array} \right. \tag{4.3}$$

Then the following hold for some $\overline{x} \in Z$*.*

(i) $x_n \rightharpoonup \overline{x}$ *as* $n \to +\infty$*.*

(ii) $\sum_{n\in\mathbb{N}} \|Bx_n - B\overline{x}\|^2 < +\infty$*.*

(iii) *Suppose that one of the following holds.*

  (a) $\underline{\lim}\, d_Z(x_n) = 0$*.*

  (b) *At every point in* Z*,* A *or* B *is demiregular (see Lemma 3.15 for special cases).*

  (c) $\operatorname{int} Z \neq \varnothing$ *and there exists* $(\nu_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})$ *such that* $(\forall n \in \mathbb{N})\ (1+\nu_n)U_n \succcurlyeq U_{n+1}$*.*

  Then $x_n \to \overline{x}$ *as* $n \to +\infty$*.*

Proof. Set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} A_n = \gamma_n U_n A \\ B_n = \gamma_n U_n B \end{cases} \quad \text{and} \quad \begin{cases} p_n = J_{A_n} y_n \\ q_n = J_{A_n}(x_n - B_n x_n) \\ s_n = x_n + \lambda_n (q_n - x_n). \end{cases} \tag{4.4}$$

Then (4.3) can be written as

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = x_n + \lambda_n (p_n + a_n - x_n). \tag{4.5}$$

On the other hand, (4.1) and Lemma 3.1(i)&(iii) yield

$$(\forall n \in \mathbb{N}) \quad \|U_n^{-1}\| \leqslant \frac{1}{\alpha}, \quad U_n^{-1} \in \mathcal{P}_{1/\mu}(\mathcal{H}), \quad \text{and} \quad (1+\eta_n)U_n^{-1} \succcurlyeq U_{n+1}^{-1} \tag{4.6}$$

and, therefore,

$$(\forall n \in \mathbb{N})(\forall x \in \mathcal{H}) \quad (1+\eta_n)\|x\|^2_{U_n^{-1}} \geqslant \|x\|^2_{U_{n+1}^{-1}}. \tag{4.7}$$

Hence, we derive from (4.5), (4.4), Lemma 3.7(ii), (4.6) and (4.1) that

$$\begin{aligned} (\forall n \in \mathbb{N}) \quad \|x_{n+1} - s_n\|_{U_n^{-1}} &\leqslant \lambda_n \Big(\|a_n\|_{U_n^{-1}} + \|p_n - q_n\|_{U_n^{-1}}\Big) \\ &\leqslant \|a_n\|_{U_n^{-1}} + \|y_n - x_n + B_n x_n\|_{U_n^{-1}} \\ &\leqslant \|a_n\|_{U_n^{-1}} + \gamma_n \|U_n b_n\|_{U_n^{-1}} \\ &\leqslant \sqrt{\|U_n^{-1}\|}\, \|a_n\| + \gamma_n \sqrt{\|U_n\|}\, \|b_n\| \\ &\leqslant \frac{1}{\sqrt{\alpha}}\|a_n\| + \frac{2\beta-\varepsilon}{\sqrt{\mu}}\|b_n\|. \end{aligned} \tag{4.8}$$

Now let $z \in Z$. Since B is β-cocoercive,

$$(\forall n \in \mathbb{N}) \quad \langle x_n - z \mid Bx_n - Bz\rangle \geqslant \beta\|Bx_n - Bz\|^2. \tag{4.9}$$

On the other hand, it follows from (4.1) that

$$\big(\forall n \in \mathbb{N}\big) \quad \|B_n x_n - B_n z\|^2_{U_n^{-1}} \leqslant \gamma_n^2\|U_n\|\,\|Bx_n - Bz\|^2 \leqslant \gamma_n^2\mu\|Bx_n - Bz\|^2. \tag{4.10}$$

We also note that, since $-Bz \in Az$, (4.4) yields

$$\big(\forall n \in \mathbb{N}\big) \quad z = J_{A_n}(z - B_n z). \tag{4.11}$$

Altogether, it follows from (4.4), (4.11), Lemma 3.7(ii), (4.9), and (4.10) that

$$\begin{aligned}(\forall n \in \mathbb{N}) \quad \|q_n - z\|^2_{U_n^{-1}} &\leqslant \|(x_n - z) - (B_n x_n - B_n z)\|^2_{U_n^{-1}}\\ &\quad - \|(x_n - q_n) - (B_n x_n - B_n z)\|^2_{U_n^{-1}}\\ &= \|x_n - z\|^2_{U_n^{-1}} - 2\left\langle x_n - z \mid B_n x_n - B_n z\right\rangle_{U_n^{-1}} + \|B_n x_n - B_n z\|^2_{U_n^{-1}}\\ &\quad - \|(x_n - q_n) - (B_n x_n - B_n z)\|^2_{U_n^{-1}}\\ &= \|x_n - z\|^2_{U_n^{-1}} - 2\gamma_n\left\langle x_n - z \mid Bx_n - Bz\right\rangle + \|B_n x_n - B_n z\|^2_{U_n^{-1}}\\ &\quad - \|(x_n - q_n) - (B_n x_n - B_n z)\|^2_{U_n^{-1}}\\ &\leqslant \|x_n - z\|^2_{U_n^{-1}} - \gamma_n(2\beta - \mu\gamma_n)\|Bx_n - Bz\|^2\\ &\quad - \|(x_n - q_n) - (B_n x_n - B_n z)\|^2_{U_n^{-1}}\\ &\leqslant \|x_n - z\|^2_{U_n^{-1}} - \varepsilon^2\|Bx_n - Bz\|^2\\ &\quad - \|(x_n - q_n) - (B_n x_n - B_n z)\|^2_{U_n^{-1}}.\end{aligned} \tag{4.12}$$

In turn, we derive from (4.7) and (4.4) that

$$\begin{aligned}(\forall n \in \mathbb{N}) \quad (1 + \eta_n)^{-1}\|s_n - z\|^2_{U_{n+1}^{-1}} &\leqslant \|s_n - z\|^2_{U_n^{-1}}\\ &\leqslant (1 - \lambda_n)\|x_n - z\|^2_{U_n^{-1}} + \lambda_n\|q_n - z\|^2_{U_n^{-1}}\\ &\leqslant \|x_n - z\|^2_{U_n^{-1}} - \varepsilon^3\|Bx_n - Bz\|^2\\ &\qquad - \varepsilon\|(x_n - q_n) - (B_n x_n - B_n z)\|^2_{U_n^{-1}},\end{aligned} \tag{4.13}$$

which implies that

$$\begin{aligned}(\forall n \in \mathbb{N}) \quad \|s_n - z\|^2_{U_{n+1}^{-1}} &\leqslant (1 + \eta_n)\|x_n - z\|^2_{U_n^{-1}} - \varepsilon^3\|Bx_n - Bz\|^2\\ &\quad - \varepsilon\|(x_n - q_n) - (B_n x_n - B_n z)\|^2_{U_n^{-1}} && (4.14)\\ &\leqslant \delta^2\|x_n - z\|^2_{U_n^{-1}}, && (4.15)\end{aligned}$$

where

$$\delta = \sup_{n\in\mathbb{N}} \sqrt{1 + \eta_n}. \tag{4.16}$$

Next, we set

$$(\forall n \in \mathbb{N}) \quad \varepsilon_n = \delta\bigg(\frac{1}{\sqrt{\alpha}}\|a_n\| + \frac{2\beta - \varepsilon}{\sqrt{\mu}}\|b_n\|\bigg). \tag{4.17}$$

Then our assumptions yield

$$\sum_{n\in\mathbb{N}} \varepsilon_n < +\infty. \tag{4.18}$$

Moreover, using (4.7), (4.14), and (4.8), we obtain

$$\begin{aligned}(\forall n\in\mathbb{N})\quad \|x_{n+1}-z\|_{U_{n+1}^{-1}} &\leqslant \|x_{n+1}-s_n\|_{U_{n+1}^{-1}} + \|s_n-z\|_{U_{n+1}^{-1}}\\ &\leqslant \sqrt{1+\eta_n}\|x_{n+1}-s_n\|_{U_n^{-1}} + \sqrt{1+\eta_n}\|x_n-z\|_{U_n^{-1}}\\ &\leqslant \delta\|x_{n+1}-s_n\|_{U_n^{-1}} + \sqrt{1+\eta_n}\|x_n-z\|_{U_n^{-1}}\\ &\leqslant \sqrt{1+\eta_n}\|x_n-z\|_{U_n^{-1}} + \varepsilon_n\\ &\leqslant (1+\eta_n)\|x_n-z\|_{U_n^{-1}} + \varepsilon_n. \end{aligned} \tag{4.19}$$

In view of (4.6), (4.18), and (4.19), we can apply Proposition 3.3 to assert that $(\|x_n - z\|_{U_n^{-1}})_{n\in\mathbb{N}}$ converges and, therefore, that

$$\zeta = \sup_{n\in\mathbb{N}} \|x_n-z\|_{U_n^{-1}} < +\infty. \tag{4.20}$$

On the other hand, (4.7), (4.8), and (4.17) yield

$$(\forall n\in\mathbb{N})\quad \|x_{n+1}-s_n\|^2_{U_{n+1}^{-1}} \leqslant (1+\eta_n)\|x_{n+1}-s_n\|^2_{U_n^{-1}} \leqslant \varepsilon_n^2. \tag{4.21}$$

Hence, using (4.14), (4.15), (4.16), and (4.20), we get

$$\begin{aligned}(\forall n\in\mathbb{N})\quad \|x_{n+1}-z\|^2_{U_{n+1}^{-1}} &\leqslant \|s_n-z\|^2_{U_{n+1}^{-1}} + 2\|s_n-z\|_{U_{n+1}^{-1}}\,\|x_{n+1}-s_n\|_{U_{n+1}^{-1}} + \|x_{n+1}-s_n\|^2_{U_{n+1}^{-1}}\\ &\leqslant (1+\eta_n)\|x_n-z\|^2_{U_n^{-1}} - \varepsilon^3\|Bx_n-Bz\|^2\\ &\quad - \varepsilon\|x_n-q_n-B_nx_n+B_nz\|^2_{U_n^{-1}} + 2\delta\zeta\varepsilon_n + \varepsilon_n^2\\ &\leqslant \|x_n-z\|^2_{U_n^{-1}} - \varepsilon^3\|Bx_n-Bz\|^2 - \varepsilon\|x_n-q_n-B_nx_n+B_nz\|^2_{U_n^{-1}}\\ &\quad + \zeta^2\eta_n + 2\delta\zeta\varepsilon_n + \varepsilon_n^2. \end{aligned} \tag{4.22}$$

Consequently, for every $N\in\mathbb{N}$,

$$\begin{aligned}\varepsilon^3\sum_{n=0}^{N}\|Bx_n-Bz\|^2 &\leqslant \|x_0-z\|^2_{U_0^{-1}} - \|x_{N+1}-z\|^2_{U_{N+1}^{-1}} + \sum_{n=0}^{N}\big(\zeta^2\eta_n + 2\delta\zeta\varepsilon_n + \varepsilon_n^2\big)\\ &\leqslant \zeta^2 + \sum_{n=0}^{N}\big(\zeta^2\eta_n + 2\delta\zeta\varepsilon_n + \varepsilon_n^2\big). \end{aligned} \tag{4.23}$$

Appealing to (4.18) and the summability of $(\eta_n)_{n\in\mathbb{N}}$, taking the limit as $N\to+\infty$, yields

$$\sum_{n\in\mathbb{N}}\|Bx_n-Bz\|^2 \leqslant \frac{1}{\varepsilon^3}\Bigg(\zeta^2 + \sum_{n\in\mathbb{N}}\big(\zeta^2\eta_n + 2\delta\zeta\varepsilon_n + \varepsilon_n^2\big)\Bigg) < +\infty. \tag{4.24}$$

We likewise derive from (4.22) that

$$\sum_{n\in\mathbb{N}}\|x_n-q_n-B_nx_n+B_nz\|^2_{U_n^{-1}} < +\infty. \tag{4.25}$$

(i): Let x be a weak sequential cluster point of $(x_n)_{n\in\mathbb{N}}$, say $x_{k_n} \rightharpoonup x$ as $n \to +\infty$. In view of (4.19), (4.6), and Proposition 3.4, it is enough to show that $x \in Z$. On the one hand, (4.24) yields $Bx_{k_n} \to Bz$ as $n \to +\infty$. On the other hand, since B is cocoercive, it is maximally monotone [4, Example 20.28] and its graph is therefore sequentially closed in $\mathcal{H}^{\text{weak}} \times \mathcal{H}^{\text{strong}}$ [4, Proposition 20.33(ii)]. This implies that $Bx = Bz$ and hence that $Bx_{k_n} \to Bx$ as $n \to +\infty$. Thus, in view of (4.24),

$$\sum_{n\in\mathbb{N}} \|Bx_n - Bx\|^2 < +\infty. \tag{4.26}$$

Now set

$$(\forall n \in \mathbb{N}) \quad u_n = \frac{1}{\gamma_n} U_n^{-1}(x_n - q_n) - Bx_n. \tag{4.27}$$

Then it follows from (4.4) that

$$(\forall n \in \mathbb{N}) \quad u_n \in Aq_n. \tag{4.28}$$

In addition, (4.4), (4.6), and (4.25) yield

$$\begin{aligned}
\|u_n + Bx\| &= \frac{1}{\gamma_n}\|U_n^{-1}(x_n - q_n - B_n x_n + B_n x)\| \\
&\leqslant \frac{1}{\varepsilon\alpha}\|x_n - q_n - B_n x_n + B_n x\| \\
&\leqslant \frac{\sqrt{\mu}}{\varepsilon\alpha}\|x_n - q_n - B_n x_n + B_n x\|_{U_n^{-1}} \\
&\to 0 \quad \text{as} \quad n \to +\infty.
\end{aligned} \tag{4.29}$$

Moreover, it follows from (4.4), (4.1), and (4.26) that

$$\begin{aligned}
\|x_n - q_n\| &\leqslant \|x_n - q_n - B_n x_n + B_n x\| + \|B_n x_n - B_n x\| \\
&\leqslant \|x_n - q_n - B_n x_n + B_n x\| + \gamma_n \|U_n\|\, \|Bx_n - Bx\| \\
&\leqslant \|x_n - q_n - B_n x_n + B_n x\| + (2\beta - \varepsilon)\|Bx_n - Bx\| \\
&\to 0 \quad \text{as} \quad n \to +\infty.
\end{aligned} \tag{4.30}$$

and, therefore, since $x_{k_n} \rightharpoonup x$ as $n \to +\infty$, that $q_{k_n} \rightharpoonup x$ as $n \to +\infty$. To sum up,

$$\begin{cases} q_{k_n} \rightharpoonup x \\ u_{k_n} \to -Bx \end{cases} \quad \text{as } n \to +\infty, \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad (q_{k_n}, u_{k_n}) \in \operatorname{gra} A. \tag{4.31}$$

Hence, using the sequential closedness of $\operatorname{gra} A$ in $\mathcal{H}^{\text{weak}} \times \mathcal{H}^{\text{strong}}$ [4, Proposition 20.33(ii)], we conclude that $-Bx \in Ax$, i.e., $x \in Z$.

(ii): Since $\overline{x} \in Z$, the claim follows from (4.24).

(iii): We now prove strong convergence.

(iii)(a): Since A and B are maximally monotone and $\operatorname{dom} B = \mathcal{H}$, $A + B$ is maximally monotone [4, Corollary 24.4(i)] and Z is therefore closed [4, Proposition 23.39]. Hence, the claim follows from (i), (4.19), and Proposition 3.5.

(iii)(b): It follows from (i) and (4.30) that $q_n \rightharpoonup \overline{x} \in Z$ as $n \to +\infty$ and from (4.29) that $u_n \to -B\overline{x} \in A\overline{x}$ as $n \to +\infty$. Hence, if A is demiregular at $\overline{x}$, (4.28) yields $q_n \to \overline{x}$ as $n \to +\infty$. In view of (4.30), we conclude that $x_n \to \overline{x}$ as $n \to +\infty$. Now suppose that B is demiregular at $\overline{x}$. Then since $x_n \rightharpoonup \overline{x} \in Z$ as $n \to +\infty$ by (i) and $Bx_n \to B\overline{x}$ as $n \to +\infty$ by (ii), we conclude that $x_n \to \overline{x}$ as $n \to +\infty$.

(iii)(c): Suppose that $z \in \operatorname{int} Z$ and fix $\rho \in \left]0,+\infty\right[$ such that $B(z;\rho) \subset Z$. It follows from (4.20) that $\theta = \sup_{x\in B(z;\rho)} \sup_{n\in\mathbb{N}} \|x_n - x\|_{U_n^{-1}} \leqslant (1/\sqrt{\alpha})(\sup_{n\in\mathbb{N}} \|x_n - z\| + \sup_{x\in B(z;\rho)} \|x - z\|) < +\infty$ and from (4.22) that

$$(\forall n \in \mathbb{N})(\forall x \in B(z;\rho)) \quad \|x_{n+1} - x\|^2_{U_{n+1}^{-1}} \leqslant \|x_n - x\|^2_{U_n^{-1}} + \theta^2\eta_n + 2\delta\theta\varepsilon_n + \varepsilon_n^2. \tag{4.32}$$

Hence, the claim follows from (i), Lemma 3.1, and Proposition 3.6. ☐

Remark 4.2 Here are some observations on Theorem 4.1.

(i) Suppose that $(\forall n \in \mathbb{N})\ U_n = \mathrm{Id}$. Then (4.3) relapses to the forward-backward algorithm studied in [1, 12], which itself captures those of [27, 29, 40]. Theorem 4.1 extends the convergence results of these papers.

(ii) As shown in [18, Remark 5.12], the convergence of the forward-backward iterates to a solution may be only weak and not strong, hence the necessity of the additional conditions in Theorem 4.1(iii).

(iii) In Euclidean spaces, condition (4.1) was used in [32] in a variable metric proximal point algorithm and then in [28] in a more general splitting algorithm.

Next, we describe direct applications of Theorem 4.1, which yield new variable metric splitting schemes. We start with minimization problems, an area in which the forward-backward algorithm has found numerous applications, e.g., [15, 18, 21, 39, 40].

Example 4.3 Let $f \in \Gamma_0(\mathcal{H})$, let $\alpha \in \left]0,+\infty\right[$, let $\beta \in \left]0,+\infty\right[$, let $g\colon \mathcal{H} \to \mathbb{R}$ be convex and differentiable with a $1/\beta$-Lipschitzian gradient, let $(\eta_n)_{n\in\mathbb{N}} \in \ell^1_+(\mathbb{N})$, and let $(U_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{H})$ such that (4.1) holds. Furthermore, let $\varepsilon \in \left]0, \min\{1, 2\beta/(\mu+1)\}\right[$ where μ is given by (4.1), let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, 1]$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (2\beta-\varepsilon)/\mu]$, let $x_0 \in \mathcal{H}$, and let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{H}$. Suppose that $\operatorname{Argmin}(f+g) \neq \varnothing$ and set

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_n = x_n - \gamma_n U_n(\nabla g(x_n) + b_n) \\ x_{n+1} = x_n + \lambda_n \Big(\operatorname{prox}^{U_n^{-1}}_{\gamma_n f} y_n + a_n - x_n \Big). \end{array} \right. \tag{4.33}$$

Then the following hold for some $\overline{x} \in \operatorname{Argmin}(f+g)$.

(i) $x_n \rightharpoonup \overline{x}$ as $n \to +\infty$.

(ii) $\sum_{n\in\mathbb{N}} \|\nabla g(x_n) - \nabla g(\overline{x})\|^2 < +\infty$.

(iii) Suppose that one of the following holds.

(a) $\varliminf d_{\operatorname{Argmin}(f+g)}(x_n) = 0$.

(b) At every point in $\operatorname{Argmin}(f+g)$, f or g is uniformly convex (see Lemma 3.15(vi)).

(c) $\operatorname{int}\operatorname{Argmin}(f+g) \neq \varnothing$ and there exists $(\nu_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})$ such that $(\forall n \in \mathbb{N})$ $(1+\nu_n)U_n \succcurlyeq U_{n+1}$.

Then $x_n \to \overline{x}$ as $n \to +\infty$.

Proof. An application of Theorem 4.1 with $A = \partial f$ and $B = \nabla g$, since the Baillon-Haddad theorem [4, Corollary 18.16] ensures that ∇g is β-cocoercive and since, by [4, Corollary 26.3], $\operatorname{Argmin}(f+g) = \operatorname{zer}(A+B)$. ☐

The next example addresses variational inequalities, another area of application of forward-backward splitting [4, 23, 39, 40].

Example 4.4 Let $f \in \Gamma_0(\mathcal{H})$, let $\alpha \in \left]0,+\infty\right[$, let $\beta \in \left]0,+\infty\right[$, let $B\colon \mathcal{H} \to \mathcal{H}$ be β-cocoercive, let $(\eta_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})$, and let $(U_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{H})$ that satisfies (4.1). Furthermore, let $\varepsilon \in \left]0, \min\{1, 2\beta/(\mu+1)\}\right[$ where μ is given by (4.1), let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, 1]$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (2\beta-\varepsilon)/\mu]$, let $x_0 \in \mathcal{H}$, and let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{H}$. Suppose that the variational inequality

$$\text{find} \quad x \in \mathcal{H} \quad \text{such that} \quad (\forall y \in \mathcal{H}) \quad \langle x-y \mid Bx\rangle + f(x) \leqslant f(y) \tag{4.34}$$

admits at least one solution and set

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_n = x_n - \gamma_n U_n(Bx_n + b_n) \\ x_{n+1} = x_n + \lambda_n\big(\operatorname{prox}_{\gamma_n f}^{U_n^{-1}} y_n + a_n - x_n\big). \end{array} \right. \tag{4.35}$$

Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a solution $\overline{x}$ to (4.34).

Proof. Set $A = \partial f$ in Theorem 4.1(i). ☐

5 Strongly monotone inclusions in duality

In [13], strongly convex composite minimization problems of the form

$$\underset{x\in\mathcal{H}}{\text{minimize}} \;\; f(x) + g(Lx - r) + \frac{1}{2}\|x - z\|^2, \tag{5.1}$$

where $z \in \mathcal{H}$, $r \in \mathcal{G}$, $f \in \Gamma_0(\mathcal{H})$, $g \in \Gamma_0(\mathcal{G})$, and $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$, were solved by applying the forward-backward algorithm to the Fenchel-Rockafellar dual problem

$$\underset{v\in\mathcal{G}}{\text{minimize}} \;\; \widetilde{f^*}(z - L^*v) + g^*(v) + \langle v \mid r\rangle\,, \tag{5.2}$$

where $\widetilde{f^*} = f^* \,\square\, (\|\cdot\|^2/2)$ denotes the Moreau envelope of f^*. This framework was shown to capture and extend various formulations in areas such as sparse signal recovery, best approximation theory, and inverse problems. In this section, we use the results of Section 4 to generalize this framework in several directions simultaneously. First, we consider general monotone inclusions, not just minimization problems. Second, we incorporate parallel sum components (see (2.7)) in the model. Third, our algorithm allows for a variable metric. The following problem is formulated using the duality framework of [16], which itself extends those of [2, 22, 31, 34, 37, 38].

Problem 5.1 Let $z \in \mathcal{H}$, let $\rho \in\,]0,+\infty[$, let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone, and let m be a strictly positive integer. For every $i \in \{1,\ldots,m\}$, let $r_i \in \mathcal{G}_i$, let $B_i\colon \mathcal{G}_i \to 2^{\mathcal{G}_i}$ be maximally monotone, let $\nu_i \in\,]0,+\infty[$, let $D_i\colon \mathcal{G}_i \to 2^{\mathcal{G}_i}$ be maximally monotone and ν_i-strongly monotone, and suppose that $0 \neq L_i \in \mathcal{B}(\mathcal{H},\mathcal{G}_i)$. Furthermore, suppose that

$$z \in \operatorname{ran}\Bigg(A + \sum_{i=1}^m L_i^*\big((B_i \,\square\, D_i)(L_i \cdot - r_i)\big) + \rho\,\mathrm{Id}\Bigg). \tag{5.3}$$

The problem is to solve the primal inclusion

$$\text{find } \overline{x} \in \mathcal{H} \text{ such that } z \in A\overline{x} + \sum_{i=1}^m L_i^*\big((B_i \,\square\, D_i)(L_i\overline{x} - r_i)\big) + \rho\overline{x}, \tag{5.4}$$

together with the dual inclusion

$$\text{find } \overline{v_1} \in \mathcal{G}_1,\ \ldots,\ \overline{v_m} \in \mathcal{G}_m \text{ such that}$$
$$(\forall i \in \{1,\ldots,m\})\quad r_i \in L_i\Bigg(J_{\rho^{-1}A}\Bigg(\rho^{-1}\Bigg(z - \sum_{j=1}^m L_j^*\overline{v_j}\Bigg)\Bigg)\Bigg) - B_i^{-1}\overline{v_i} - D_i^{-1}\overline{v_i}. \tag{5.5}$$

Let us start with some properties of Problem 5.1.

Proposition 5.2 *In Problem 5.1, set*

$$\overline{x} = J_{\rho^{-1}M}\big(\rho^{-1}z\big), \quad \textit{where} \quad M = A + \sum_{i=1}^m L_i^* \circ (B_i \,\square\, D_i) \circ (L_i \cdot - r_i). \tag{5.6}$$

Then the following hold.

(i) $\overline{x}$ *is the unique solution to the primal problem* (5.4).

(ii) *The dual problem* (5.5) *admits at least one solution.*

(iii) *Let* $(\overline{v_1},\ldots,\overline{v_m})$ *be a solution to* (5.5). *Then* $\overline{x} = J_{\rho^{-1}A}\big(\rho^{-1}\big(z - \sum_{i=1}^m L_i^*\overline{v_i}\big)\big)$.

(iv) *Condition* (5.3) *is satisfied for every* z *in* $\mathcal{H}$ *if and only if* M *is maximally monotone. This is true when one of the following holds.*

(a) *The conical hull of*

$$E = \Bigg\{\big(L_ix - r_i - v_i\big)_{1\leqslant i\leqslant m} \;\Bigg|\; x \in \operatorname{dom} A \textit{ and } (v_i)_{1\leqslant i\leqslant m} \in \bigtimes_{i=1}^m \operatorname{ran}\big(B_i^{-1} + D_i^{-1}\big)\Bigg\} \tag{5.7}$$

is a closed vector subspace.

(b) $A = \partial f$ *for some* $f \in \Gamma_0(\mathcal{H})$, *for every* $i \in \{1,\ldots,m\}$, $B_i = \partial g_i$ *for some* $g_i \in \Gamma_0(\mathcal{G}_i)$ *and* $D_i = \partial \ell_i$ *for some strongly convex function* $\ell_i \in \Gamma_0(\mathcal{G}_i)$, *and one of the following holds.*

1/ $(r_1,\ldots,r_m) \in \operatorname{sri}\big\{(L_ix - y_i)_{1\leqslant i\leqslant m} \mid x \in \operatorname{dom} f$ *and* $(\forall i \in \{1,\ldots,m\})\ y_i \in \operatorname{dom} g_i + \operatorname{dom} \ell_i\big\}$.

2/ *For every* $i \in \{1,\ldots,m\}$, g_i *or* ℓ_i *is real-valued.*

3/ $\mathcal{H}$ *and* $(\mathcal{G}_i)_{1\leqslant i\leqslant m}$ *are finite-dimensional, and there exists* $x \in \operatorname{ri}\operatorname{dom} f$ *such that*

$$(\forall i \in \{1,\ldots,m\}) \quad L_i x - r_i \in \operatorname{ri}\operatorname{dom} g_i + \operatorname{ri}\operatorname{dom}\ell_i. \tag{5.8}$$

Proof. (i): It follows from our assumptions and [4, Proposition 20.10] that $\rho^{-1}M$ is a monotone operator. Hence, $J_{\rho^{-1}M}$ is a single-valued operator with domain $\operatorname{ran}(\operatorname{Id}+\rho^{-1}M)$ [4, Proposition 23.9(ii)]. Moreover, (5.3) $\Leftrightarrow$ $\rho^{-1}z \in \operatorname{ran}(\operatorname{Id}+\rho^{-1}M) = \operatorname{dom} J_{\rho^{-1}M}$, and, in view of (2.5), the inclusion in (5.4) is equivalent to $\overline{x} = J_{\rho^{-1}M}(\rho^{-1}z)$.

(ii)&(iii): It follows from (2.5) and (2.7) that

$$\begin{aligned}
\text{(i)} \quad &\Leftrightarrow \quad (\exists\, \overline{v_1} \in \mathcal{G}_1)\cdots(\exists\, \overline{v_m} \in \mathcal{G}_m) \quad \begin{cases}(\forall i \in \{1,\ldots,m\}) \quad \overline{v_i} \in (B_i \,\square\, D_i)(L_i\overline{x} - r_i)\\ z - \sum_{i=1}^m L_i^*\overline{v_i} \in A\overline{x} + \rho\overline{x}\end{cases}\\
&\Leftrightarrow \quad (\exists\, \overline{v_1} \in \mathcal{G}_1)\cdots(\exists\, \overline{v_m} \in \mathcal{G}_m) \quad \begin{cases}(\forall i \in \{1,\ldots,m\}) \quad r_i \in L_i\overline{x} - B_i^{-1}\overline{v_i} - D_i^{-1}\overline{v_i}\\ \overline{x} = J_{\rho^{-1}A}\big(\rho^{-1}\big(z - \sum_{j=1}^m L_j^*\overline{v_j}\big)\big)\end{cases}\\
&\Leftrightarrow \quad \begin{cases}(\overline{v_1},\ldots,\overline{v_m}) \text{ solves } (5.5)\\ \overline{x} = J_{\rho^{-1}A}\big(\rho^{-1}\big(z - \sum_{j=1}^m L_j^*\overline{v_j}\big)\big).\end{cases}
\end{aligned} \tag{5.9}$$

(iv): It follows from Minty's theorem [4, Theorem 21.1], that $M + \rho\operatorname{Id}$ is surjective if and only if M is maximally monotone.

(iv)(a): Using Notation 2.1, let us set

$$\boldsymbol{L}\colon \mathcal{H} \to \boldsymbol{\mathcal{G}}\colon x \mapsto \big(L_i x\big)_{1\leqslant i\leqslant m} \quad \text{and} \quad \boldsymbol{B}\colon \boldsymbol{\mathcal{G}} \to 2^{\boldsymbol{\mathcal{G}}}\colon \boldsymbol{y} \mapsto \big((B_i \,\square\, D_i)(y_i - r_i)\big)_{1\leqslant i\leqslant m}. \tag{5.10}$$

Then it follows from (5.6) that $M = A + \boldsymbol{L}^* \circ \boldsymbol{B} \circ \boldsymbol{L}$ and from (5.7) that $E = \boldsymbol{L}(\operatorname{dom} A) - \operatorname{dom}\boldsymbol{B}$. Hence, since $\operatorname{cone}(E) = \overline{\operatorname{span}}\,(E)$, in view of [6, Section 24], to conclude that M is maximally monotone, it is enough to show that $\boldsymbol{B}$ is. For every $i \in \{1,\ldots,m\}$, since D_i is maximally monotone and strongly monotone, $\operatorname{dom} D_i^{-1} = \operatorname{ran} D_i = \mathcal{G}_i$ [4, Proposition 22.8(ii)] and it follows from [4, Proposition 20.22 and Corollary 24.4(i)] that $B_i \,\square\, D_i$ is maximally monotone. This shows that $\boldsymbol{B}$ is maximally monotone.

(iv)(b): This follows from [16, Proposition 4.3]. ☐

Remark 5.3 In connection with Proposition 5.2(iv), let us note that even in the simple setting of normal cone operators in finite dimension, some constraint qualification is required to ensure the existence of a primal solution for every $z \in \mathcal{H}$. To see this, suppose that, in Problem 5.1, $\mathcal{H}$ is the Euclidean plane, $m = 1$, $\rho = 1$, $\mathcal{G}_1 = \mathcal{H}$, $L_1 = \operatorname{Id}$, $z = (\zeta_1, \zeta_2)$, $r_1 = 0$, $D_1 = \{0\}^{-1}$, $A = N_C$, and $B_1 = N_K$, where $C = \big\{(\xi_1,\xi_2) \in \mathbb{R}^2 \mid (\xi_1 - 1)^2 + \xi_2^2 \leqslant 1\big\}$ and $K = \big\{(\xi_1,\xi_2) \in \mathbb{R}^2 \mid \xi_1 \leqslant 0\big\}$. Then $\operatorname{dom}(A + B_1 + \operatorname{Id}) = \operatorname{dom} A \cap \operatorname{dom} B_1 = C \cap K = \{0\}$ and the primal inclusion $z \in A\overline{x} + B_1\overline{x} + \overline{x}$ reduces to $(\zeta_1,\zeta_2) \in N_C 0 + N_K 0 = \left]-\infty, 0\right] \times \{0\} + \left[0, +\infty\right[\times \{0\} = \mathbb{R} \times \{0\}$, which has no solution if $\zeta_2 \neq 0$. Here $\operatorname{cone}(\operatorname{dom} A - \operatorname{dom} B_1) = \operatorname{cone}(C - K) = -K$ is not a vector subspace.

In the following result we derive from Theorem 4.1 a parallel primal-dual algorithm for solving Problem 5.1.

Corollary 5.4 *In Problem 5.1, set*

$$\beta = \frac{1}{\displaystyle\max_{1\leqslant i\leqslant m} \frac{1}{\nu_i} + \frac{1}{\rho}\sum_{1\leqslant i\leqslant m} \|L_i\|^2}. \tag{5.11}$$

Let $(a_n)_{n\in\mathbb{N}}$ *be an absolutely summable sequence in* $\mathcal{H}$*, let* $\alpha \in \left]0,+\infty\right[$*, and let* $(\eta_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})$*. For every* $i \in \{1,\ldots,m\}$*, let* $v_{i,0} \in \mathcal{G}_i$*, let* $(b_{i,n})_{n\in\mathbb{N}}$ *and* $(d_{i,n})_{n\in\mathbb{N}}$ *be absolutely summable sequences in* $\mathcal{G}_i$*, and let* $(U_{i,n})_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{P}_\alpha(\mathcal{G}_i)$*. Suppose that*

$$\mu = \max_{1\leqslant i\leqslant m} \sup_{n\in\mathbb{N}} \|U_{i,n}\| < +\infty \quad \text{and} \quad (\forall i \in \{1,\ldots,m\})(\forall n \in \mathbb{N}) \quad (1+\eta_n)U_{i,n+1} \succcurlyeq U_{i,n}. \tag{5.12}$$

Let $\varepsilon \in \left]0, \min\{1, 2\beta/(\mu+1)\}\right[$*, let* $(\lambda_n)_{n\in\mathbb{N}}$ *be a sequence in* $[\varepsilon, 1]$*, and let* $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence in* $[\varepsilon, (2\beta-\varepsilon)/\mu]$*. Set*

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} s_n = z - \sum_{i=1}^m L_i^* v_{i,n} \\ x_n = J_{\rho^{-1}A}(\rho^{-1}s_n) + a_n \\ \text{For } i = 1,\ldots,m \\ \left\lfloor \begin{array}{l} w_{i,n} = v_{i,n} + \gamma_n U_{i,n}\big(L_i x_n - r_i - D_i^{-1}v_{i,n} - d_{i,n}\big) \\ v_{i,n+1} = v_{i,n} + \lambda_n\Big(J_{\gamma_n U_{i,n} B_i^{-1}}(w_{i,n}) + b_{i,n} - v_{i,n}\Big). \end{array} \right. \end{array} \right. \tag{5.13}$$

Then the following hold for the solution $\overline{x}$ *to (5.4) and for some solution* $(\overline{v_1},\ldots,\overline{v_m})$ *to (5.5).*

(i) $(\forall i \in \{1,\ldots,m\})$ $v_{i,n} \rightharpoonup \overline{v_i}$ *as* $n \to +\infty$*. In addition,* $\overline{x} = J_{\rho^{-1}A}\big(\rho^{-1}\big(z - \sum_{i=1}^m L_i^*\overline{v_i}\big)\big)$*.*

(ii) $x_n \to \overline{x}$ *as* $n \to +\infty$*.*

Proof. For every $i \in \{1,\ldots,m\}$, since D_i is maximally monotone and ν_i-strongly monotone, D_i^{-1} is ν_i-cocoercive with $\operatorname{dom} D_i^{-1} = \operatorname{ran} D_i = \mathcal{G}_i$ [4, Proposition 22.8(ii)]. Let us define $\boldsymbol{\mathcal{G}}$ as in Notation 2.1, and let us introduce the operators

$$\begin{cases} T\colon \mathcal{H} \to \mathcal{H}\colon x \mapsto J_{\rho^{-1}A}\big(\rho^{-1}(z-x)\big) \\ \boldsymbol{A}\colon \boldsymbol{\mathcal{G}} \to 2^{\boldsymbol{\mathcal{G}}}\colon \boldsymbol{v} \mapsto \big(B_i^{-1}v_i\big)_{1\leqslant i\leqslant m} \\ \boldsymbol{D}\colon \boldsymbol{\mathcal{G}} \to \boldsymbol{\mathcal{G}}\colon \boldsymbol{v} \mapsto \big(r_i + D_i^{-1}v_i\big)_{1\leqslant i\leqslant m} \\ \boldsymbol{L}\colon \mathcal{H} \to \boldsymbol{\mathcal{G}}\colon x \mapsto \big(L_i x\big)_{1\leqslant i\leqslant m} \end{cases} \tag{5.14}$$

and

$$(\forall n \in \mathbb{N}) \quad \boldsymbol{U}_n\colon \boldsymbol{\mathcal{G}} \to \boldsymbol{\mathcal{G}}\colon \boldsymbol{v} \mapsto \big(U_{i,n}v_i\big)_{1\leqslant i\leqslant m}. \tag{5.15}$$

(i): In view of (2.4) and (5.14),

$$\boldsymbol{A} \text{ is maximally monotone,} \tag{5.16}$$

$\boldsymbol{D}$ is $(\min_{1\leqslant i\leqslant m} \nu_i)$-cocoercive, Lemma 3.7(ii) implies that

$$-T \text{ is } \rho\text{-cocoercive,} \tag{5.17}$$

while $\|\boldsymbol{L}\|^2 \leqslant \sum_{i=1}^m \|L_i\|^2$. Hence, we derive from (5.11) and Proposition 3.2 that

$$\boldsymbol{B} = \boldsymbol{D} - \boldsymbol{L}T\boldsymbol{L}^* \quad \text{is} \quad \beta\text{-cocoercive}. \tag{5.18}$$

Moreover, it follows from (5.12), (5.15), and (2.4) that

$$\sup_{n\in\mathbb{N}} \|\boldsymbol{U}_n\| = \mu \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad (1+\eta_n)\boldsymbol{U}_{n+1} \succcurlyeq \boldsymbol{U}_n \in \mathcal{P}_\alpha(\boldsymbol{\mathcal{G}}). \tag{5.19}$$

Now set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \boldsymbol{a}_n = \big(b_{i,n}\big)_{1\leqslant i\leqslant m} \\ \boldsymbol{b}_n = \big(d_{i,n} - L_i a_n\big)_{1\leqslant i\leqslant m} \\ \boldsymbol{v}_n = \big(v_{i,n}\big)_{1\leqslant i\leqslant m} \\ \boldsymbol{w}_n = \big(w_{i,n}\big)_{1\leqslant i\leqslant m}. \end{cases} \tag{5.20}$$

Then $\sum_{n\in\mathbb{N}} |||\boldsymbol{a}_n||| < +\infty$, $\sum_{n\in\mathbb{N}} |||\boldsymbol{b}_n||| < +\infty$, and (5.13) can be rewritten as

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \boldsymbol{w}_n = \boldsymbol{v}_n - \gamma_n \boldsymbol{U}_n(\boldsymbol{B}\boldsymbol{v}_n + \boldsymbol{b}_n) \\ \boldsymbol{v}_{n+1} = \boldsymbol{v}_n + \lambda_n\big(J_{\gamma_n \boldsymbol{U}_n \boldsymbol{A}}(\boldsymbol{w}_n) + \boldsymbol{a}_n - \boldsymbol{v}_n\big). \end{array} \right. \tag{5.21}$$

Furthermore, the dual problem (5.5) is equivalent to

$$\text{find} \quad \overline{\boldsymbol{v}} \in \boldsymbol{\mathcal{G}} \quad \text{such that} \quad \boldsymbol{0} \in \boldsymbol{A}\overline{\boldsymbol{v}} + \boldsymbol{B}\overline{\boldsymbol{v}} \tag{5.22}$$

which, in view of (5.16), (5.18), and Proposition 5.2(ii), can be solved using (5.21). Altogether, the claims follow from Theorem 4.1(i) and Proposition 5.2(iii).

(ii): Set $(\forall n \in \mathbb{N})$ $z_n = x_n - a_n$. It follows from (i), (5.13) and (5.14) that

$$\overline{x} = T(\boldsymbol{L}^*\overline{\boldsymbol{v}}) \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad z_n = T(\boldsymbol{L}^*\boldsymbol{v}_n). \tag{5.23}$$

In turn, we deduce from (5.17), (i), (5.18), and the monotonicity of $\boldsymbol{D}$ that

$$\begin{aligned} \rho\|z_n - \overline{x}\|^2 &= \rho\|T(\boldsymbol{L}^*\boldsymbol{v}_n) - T(\boldsymbol{L}^*\overline{\boldsymbol{v}})\|^2 \\ &\leqslant \langle \boldsymbol{L}^*(\boldsymbol{v}_n - \overline{\boldsymbol{v}}) \mid T(\boldsymbol{L}^*\overline{\boldsymbol{v}}) - T(\boldsymbol{L}^*\boldsymbol{v}_n)\rangle \\ &\leqslant \langle\langle \boldsymbol{v}_n - \overline{\boldsymbol{v}} \mid \boldsymbol{L}T(\boldsymbol{L}^*\overline{\boldsymbol{v}}) - \boldsymbol{L}T(\boldsymbol{L}^*\boldsymbol{v}_n)\rangle\rangle \\ &\leqslant \langle\langle \boldsymbol{v}_n - \overline{\boldsymbol{v}} \mid \boldsymbol{D}\boldsymbol{v}_n - \boldsymbol{D}\overline{\boldsymbol{v}}\rangle\rangle - \langle\langle \boldsymbol{v}_n - \overline{\boldsymbol{v}} \mid \boldsymbol{L}T(\boldsymbol{L}^*\boldsymbol{v}_n) - \boldsymbol{L}T(\boldsymbol{L}^*\overline{\boldsymbol{v}})\rangle\rangle \\ &= \langle\langle \boldsymbol{v}_n - \overline{\boldsymbol{v}} \mid \boldsymbol{B}\boldsymbol{v}_n - \boldsymbol{B}\overline{\boldsymbol{v}}\rangle\rangle \\ &\leqslant \delta |||\boldsymbol{B}\boldsymbol{v}_n - \boldsymbol{B}\overline{\boldsymbol{v}}|||, \end{aligned} \tag{5.24}$$

where $\delta = \sup_{n\in\mathbb{N}} |||\boldsymbol{v}_n - \overline{\boldsymbol{v}}||| < +\infty$ by (i). Therefore, it follows from (5.21) and Theorem 4.1(ii) that $\|z_n - \overline{x}\| \to 0$. Since $a_n \to 0$ as $n \to +\infty$, we conclude that $x_n \to \overline{x}$ as $n \to +\infty$. ☐

Remark 5.5 Here are some observations on Corollary 5.4.

(i) At iteration n, the vectors a_n, $b_{i,n}$, and $d_{i,n}$ model errors in the implementation of the nonlinear operators. Note also that, thanks to Example 3.9(i), the computation of $v_{i,n+1}$ in (5.13) can be implemented using $J_{\gamma_n^{-1}U_{i,n}^{-1}B_i}$ rather than $J_{\gamma_n U_{i,n} B_i^{-1}}$.

(ii) Corollary 5.4 provides a general algorithm for solving strongly monotone composite inclusions which is new even in the fixed standard metric case, i.e., $(\forall i \in \{1,\ldots,m\})(\forall n \in \mathbb{N})\ U_{i,n} = \mathrm{Id}$.

The following example describes an application of Corollary 5.4 to strongly convex minimization problems which extends the primal-dual formulation (5.1)–(5.2) of [13] and solves it with a variable metric scheme. It also extends the framework of [14], where $f = 0$ and $(\forall i \in \{1,\ldots,m\})\ \ell_i = \iota_{\{0\}}$ and $(\forall n \in \mathbb{N})\ U_{i,n} = \mathrm{Id}$.

Example 5.6 Let $z \in \mathcal{H}$, let $f \in \Gamma_0(\mathcal{H})$, let $\alpha \in \left]0,+\infty\right[$, let $(\eta_n)_{n\in\mathbb{N}} \in \ell^1_+(\mathbb{N})$, let $(a_n)_{n\in\mathbb{N}}$ be an absolutely summable sequence in $\mathcal{H}$, and let m be a strictly positive integer. For every $i \in \{1,\ldots,m\}$, let $r_i \in \mathcal{G}_i$, let $g_i \in \Gamma_0(\mathcal{G}_i)$, let $\nu_i \in \left]0,+\infty\right[$, let $\ell_i \in \Gamma_0(\mathcal{G}_i)$ be ν_i-strongly convex, let $v_{i,0} \in \mathcal{G}_i$, let $(b_{i,n})_{n\in\mathbb{N}}$ and $(d_{i,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{G}_i$, let $(U_{i,n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{G}_i)$, and suppose that $0 \neq L_i \in \mathcal{B}(\mathcal{H},\mathcal{G}_i)$. Furthermore, suppose that (see Proposition 5.2(iv)(b) for special cases)

$$z \in \operatorname{ran}\left(\partial f + \sum_{i=1}^m L_i^*(\partial g_i \,\Box\, \partial \ell_i)(L_i \cdot - r_i) + \mathrm{Id}\right). \tag{5.25}$$

The primal problem is

$$\underset{x\in\mathcal{H}}{\text{minimize}}\ \ f(x) + \sum_{i=1}^m (g_i \,\Box\, \ell_i)(L_i x - r_i) + \frac{1}{2}\|x - z\|^2, \tag{5.26}$$

and the dual problem is

$$\underset{v_1\in\mathcal{G}_1,\ldots,v_m\in\mathcal{G}_m}{\text{minimize}}\ \ \widetilde{f^*}\left(z - \sum_{i=1}^m L_i^* v_i\right) + \sum_{i=1}^m \big(g_i^*(v_i) + \ell_i^*(v_i) + \langle v_i \mid r_i\rangle\big). \tag{5.27}$$

Suppose that (5.12) holds, let $\varepsilon \in \left]0, \min\{1, 2\beta/(\mu+1)\}\right[$, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, 1]$, and let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (2\beta-\varepsilon)/\mu]$, where β is defined in (5.11) and μ in (5.12). Set

$$(\forall n \in \mathbb{N})\quad \left\lfloor \begin{array}{l} s_n = z - \sum_{i=1}^m L_i^* v_{i,n} \\ x_n = \operatorname{prox}_f s_n + a_n \\ \text{For } i = 1,\ldots,m \\ \left\lfloor \begin{array}{l} w_{i,n} = v_{i,n} + \gamma_n U_{i,n}\big(L_i x_n - r_i - \nabla \ell_i^*(v_{i,n}) - d_{i,n}\big) \\ v_{i,n+1} = v_{i,n} + \lambda_n\Big(\operatorname{prox}^{U_{i,n}^{-1}}_{\gamma_n g_i^*} w_{i,n} + b_{i,n} - v_{i,n}\Big). \end{array}\right. \end{array}\right. \tag{5.28}$$

Then (5.26) admits a unique solution $\overline{x}$ and the following hold for some solution $(\overline{v_1},\ldots,\overline{v_m})$ to (5.27).

(i) $(\forall i \in \{1,\ldots,m\})\ v_{i,n} \rightharpoonup \overline{v_i}$ as $n \to +\infty$. In addition, $\overline{x} = \operatorname{prox}_f(z - \sum_{i=1}^m L_i^* \overline{v_i})$.

(ii) $x_n \to \overline{x}$ as $n \to +\infty$.

Proof. Set

$$\rho = 1, \quad A = \partial f, \quad \text{and} \quad (\forall i \in \{1,\ldots,m\})\quad B_i = \partial g_i \quad \text{and} \quad D_i = \partial \ell_i. \tag{5.29}$$

It follows from [4, Theorem 20.40] that the operators A, $(B_i)_{1\leqslant i\leqslant m}$, and $(D_i)_{1\leqslant i\leqslant m}$ are maximally monotone. We also observe that (5.25) implies that (5.3) is satisfied. Moreover, for every $i \in \{1,\ldots,m\}$, D_i is ν_i-strongly monotone [4, Example 22.3(iv)], ℓ_i^* is Fréchet differentiable on $\mathcal{G}_i$ [4, Corollary 13.33 and Theorem 18.15], and $D_i^{-1} = (\partial\ell_i)^{-1} = \partial\ell_i^* = \{\nabla\ell_i^*\}$ [4, Corollary 16.24 and Proposition 17.26(i)]. Since, for every $i \in \{1,\ldots,m\}$, $\operatorname{dom}\ell_i^* = \mathcal{G}_i$, [4, Proposition 24.27] yields

$$(\forall i \in \{1,\ldots,m\}) \quad B_i \,\square\, D_i = \partial g_i \,\square\, \partial\ell_i = \partial(g_i \,\square\, \ell_i), \tag{5.30}$$

while [4, Corollaries 16.24 and 16.38(iii)] yield

$$(\forall i \in \{1,\ldots,m\}) \quad B_i^{-1} + D_i^{-1} = \partial g_i^* + \{\nabla\ell_i^*\} = \partial\big(g_i^* + \ell_i^*\big). \tag{5.31}$$

Moreover, (3.10) implies that (5.28) is a special case of (5.13). Hence, in view of Corollary 5.4, it remains to show that (5.4) and (5.5) yield (5.26) and (5.27), respectively. Let us set $q = \|\cdot\|^2/2$. We derive from [4, Example 16.33] that

$$\partial\big(f + q(\cdot - z)\big) = \partial f + \operatorname{Id} - z. \tag{5.32}$$

On the other hand, it follows from (5.25) and [4, Proposition 16.5(ii)] that

$$\partial\big(f + q(\cdot - z)\big) + \sum_{i=1}^m L_i^*\big(\partial(g_i \,\square\, \ell_i)\big)(L_i \cdot - r_i) \subset \partial\left(f + q(\cdot - z) + \sum_{i=1}^m (g_i \,\square\, \ell_i)\circ(L_i \cdot - r_i)\right) \tag{5.33}$$

and that $x \mapsto f(x) + \sum_{i=1}^m (g_i \,\square\, \ell_i)(L_i x - r_i) + \|x - z\|^2/2$ is a strongly convex function in $\Gamma_0(\mathcal{H})$. Therefore [4, Corollary 11.16] asserts that (5.26) possesses a unique solution $\overline{x}$. Next, we deduce from (5.32), (5.29), (5.30), and Fermat's rule [4, Theorem 16.2] that, for every $x \in \mathcal{H}$,

$$\begin{aligned}
x \text{ solves (5.4)} \quad &\Leftrightarrow \quad z \in \partial f(x) + \sum_{i=1}^m L_i^*\big((\partial g_i \,\square\, \partial\ell_i)(L_i x - r_i)\big) + x \\
&\Leftrightarrow \quad 0 \in \partial\big(f + q(\cdot - z)\big)(x) + \left(\sum_{i=1}^m L_i^* \circ \partial(g_i \,\square\, \ell_i)\circ(L_i \cdot - r_i)\right)(x) \\
&\Rightarrow \quad 0 \in \partial\left(f + q(\cdot - z) + \sum_{i=1}^m (g_i \,\square\, \ell_i)\circ(L_i \cdot - r_i)\right)(x) \\
&\Leftrightarrow \quad x \text{ solves (5.26)}.
\end{aligned} \tag{5.34}$$

Finally, set $\boldsymbol{L}\colon \mathcal{H} \to \boldsymbol{\mathcal{G}}\colon x \mapsto \big(L_i x\big)_{1\leqslant i\leqslant m}$ and $\boldsymbol{h}\colon \boldsymbol{\mathcal{G}} \to \,]-\infty,+\infty]\colon \boldsymbol{v} \mapsto \sum_{i=1}^m (g_i^*(v_i) + \ell_i^*(v_i) + \langle v_i \mid r_i\rangle)$. We recall that $\widetilde{f^*} = f^* \,\square\, q$ is Fréchet differentiable on $\mathcal{H}$ with $\nabla\widetilde{f^*} = \operatorname{prox}_f$ [4, Remark 14.4]. Hence, it follows from (5.29), (5.31), [4, Proposition 16.8 and Theorem 16.37(i)], and Fermat's rule [4, Theorem 16.2] that, for every $\boldsymbol{v} = (v_i)_{1\leqslant i\leqslant m} \in \boldsymbol{\mathcal{G}}$,

$$\begin{aligned}
\boldsymbol{v} \text{ solves (5.5)} &\Leftrightarrow (\forall i \in \{1,\ldots,m\})\ r_i \in L_i\left(J_A\left(z - \sum_{j=1}^m L_j^* v_j\right)\right) - B_i^{-1} v_i - D_i^{-1} v_i \\
&\Leftrightarrow (\forall i \in \{1,\ldots,m\})\ r_i \in L_i\left(\operatorname{prox}_f\left(z - \sum_{j=1}^m L_j^* v_j\right)\right) - \partial(g_i^* + \ell_i^*)(v_i) \\
&\Leftrightarrow (0,\ldots,0) \in -\boldsymbol{L}\Big(\nabla\widetilde{f^*}(z - \boldsymbol{L}^*\boldsymbol{v})\Big) + \mathop{\times}_{i=1}^m \partial\big(g_i^* + \ell_i^* + \langle \cdot \mid r_i\rangle\big)(v_i) \\
&\qquad\qquad = \big(-\boldsymbol{L}^*\big)^*\Big(\nabla\widetilde{f^*}(z - \boldsymbol{L}^*\boldsymbol{v})\Big) + \partial\boldsymbol{h}(\boldsymbol{v}) = \partial\Big(\widetilde{f^*}(z - \boldsymbol{L}^*\cdot) + \boldsymbol{h}\Big)(\boldsymbol{v}) \\
&\Leftrightarrow \boldsymbol{v} \text{ solves (5.27)},
\end{aligned} \tag{5.35}$$

which completes the proof. ☐

We conclude this section with an application to a composite best approximation problem.

Example 5.7 Let $z \in \mathcal{H}$, let C be a closed convex subset of $\mathcal{H}$, let $\alpha \in \,]0,+\infty[$, let $(\eta_n)_{n\in\mathbb{N}} \in \ell_+^1(\mathbb{N})$, let $(a_n)_{n\in\mathbb{N}}$ be an absolutely summable sequence in $\mathcal{H}$, and let m be a strictly positive integer. For every $i \in \{1,\ldots,m\}$, let $r_i \in \mathcal{G}_i$, let D_i be a closed convex subset of $\mathcal{G}_i$, let $v_{i,0} \in \mathcal{G}_i$, let $(b_{i,n})_{n\in\mathbb{N}}$ be an absolutely summable sequence in $\mathcal{G}_i$, let $(U_{i,n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{G}_i)$, and suppose that $0 \neq L_i \in \mathcal{B}(\mathcal{H},\mathcal{G}_i)$. The problem is

$$\underset{\substack{x\in C\\ L_1x\in r_1+D_1\\ \vdots\\ L_mx\in r_m+D_m}}{\text{minimize}} \quad \|x-z\|. \tag{5.36}$$

Suppose that (5.12) holds, that $(\max_{1\leqslant i\leqslant m}\sup_{n\in\mathbb{N}}\|U_{i,n}\|)\sum_{i=1}^m\|L_i\|^2 < 2$, and that

$$(r_1,\ldots,r_m) \in \operatorname{sri}\big\{(L_ix-y_i)_{1\leqslant i\leqslant m} \mid x\in C \text{ and } (\forall i\in\{1,\ldots,m\})\ \ y_i\in D_i\big\}. \tag{5.37}$$

Set

$$(\forall n\in\mathbb{N})\quad \left\lfloor\begin{array}{l} s_n = z-\sum_{i=1}^m L_i^*v_{i,n}\\ x_n = P_Cs_n+a_n\\ \text{For } i=1,\ldots,m\\ \left\lfloor\begin{array}{l} w_{i,n} = v_{i,n}+U_{i,n}\big(L_ix_n-r_i\big)\\ v_{i,n+1} = w_{i,n}-U_{i,n}\Big(P_{D_i}^{U_{i,n}}\big(U_{i,n}^{-1}w_{i,n}\big)+b_{i,n}\Big). \end{array}\right. \end{array}\right. \tag{5.38}$$

Then $(x_n)_{n\in\mathbb{N}}$ converges strongly to the unique solution $\overline{x}$ to (5.36).

Proof. Set $f=\iota_C$ and $(\forall i\in\{1,\ldots,m\})$ $g_i=\iota_{D_i}$, $\ell_i=\iota_{\{0\}}$, and $(\forall n\in\mathbb{N})$ $\gamma_n=\lambda_n=1$ and $d_{i,n}=0$. Then (5.37) and Proposition 5.2(iv)((b))1/ imply that (5.25) is satisfied. Moreover, in view of Example 3.9(iii), (5.38) is a special case of (5.28). Hence, the claim follows from Example 5.6(ii). ☐

6 Inclusions involving cocoercive operators

We revisit a primal-dual problem investigated first in [16], and then in [41] with the scenario described below.

Problem 6.1 Let $z\in\mathcal{H}$, let $A\colon\mathcal{H}\to 2^{\mathcal{H}}$ be maximally monotone, let $\mu\in\,]0,+\infty[$, let $C\colon\mathcal{H}\to\mathcal{H}$ be μ-cocoercive, and let m be a strictly positive integer. For every $i\in\{1,\ldots,m\}$, let $r_i\in\mathcal{G}_i$, let $B_i\colon\mathcal{G}_i\to 2^{\mathcal{G}_i}$ be maximally monotone, let $\nu_i\in\,]0,+\infty[$, let $D_i\colon\mathcal{G}_i\to 2^{\mathcal{G}_i}$ be maximally monotone and ν_i-strongly monotone, and suppose that $0\neq L_i\in\mathcal{B}(\mathcal{H},\mathcal{G}_i)$. The problem is to solve the primal inclusion

$$\text{find}\quad \overline{x}\in\mathcal{H}\quad \text{such that}\quad z\in A\overline{x}+\sum_{i=1}^m L_i^*\big((B_i\,\Box\,D_i)(L_i\overline{x}-r_i)\big)+C\overline{x}, \tag{6.1}$$

together with the dual inclusion

$$\text{find } \overline{v_1} \in \mathcal{G}_1,\ \dots,\ \overline{v_m} \in \mathcal{G}_m \text{ such that}$$
$$(\exists\, x \in \mathcal{H}) \quad \begin{cases} z - \sum_{i=1}^m L_i^* \overline{v}_i \in Ax + Cx \\ (\forall i \in \{1,\dots,m\})\ \overline{v}_i \in (B_i \,\square\, D_i)(L_i x - r_i). \end{cases} \tag{6.2}$$

Corollary 6.2 *In Problem 6.1, suppose that*

$$z \in \operatorname{ran}\Bigg(A + \sum_{i=1}^m L_i^*\big((B_i \,\square\, D_i)(L_i \cdot - r_i)\big) + C\Bigg), \tag{6.3}$$

and set

$$\beta = \min\{\mu, \nu_1, \dots, \nu_m\}. \tag{6.4}$$

Let $\varepsilon \in \left]0, \min\{1,\beta\}\right[$, let $\alpha \in \left]0,+\infty\right[$, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, 1]$, let $x_0 \in \mathcal{H}$, let $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{H}$, and let $(U_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{H})$ such that $(\forall n \in \mathbb{N})$ $U_{n+1} \succcurlyeq U_n$. For every $i \in \{1,\dots,m\}$, let $v_{i,0} \in \mathcal{G}_i$, and let $(b_{i,n})_{n\in\mathbb{N}}$ and $(d_{i,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{G}_i$, and let $(U_{i,n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{G}_i)$ such that $(\forall n \in \mathbb{N})$ $U_{i,n+1} \succcurlyeq U_{i,n}$. For every $n \in \mathbb{N}$, set

$$\delta_n = \left(\sqrt{\sum_{i=1}^m \|\sqrt{U_{i,n}} L_i \sqrt{U_n}\|^2}\right)^{-1} - 1, \tag{6.5}$$

and suppose that

$$\zeta_n = \frac{\delta_n}{(1+\delta_n)\max\{\|U_n\|, \|U_{1,n}\|, \dots, \|U_{m,n}\|\}} \geqslant \frac{1}{2\beta - \varepsilon}. \tag{6.6}$$

Set

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} p_n = J_{U_n A}\Big(x_n - U_n\big(\sum_{i=1}^m L_i^* v_{i,n} + Cx_n + c_n - z\big)\Big) + a_n \\ y_n = 2p_n - x_n \\ x_{n+1} = x_n + \lambda_n(p_n - x_n) \\ \text{For } i = 1,\dots,m \\ \left\lfloor \begin{array}{l} q_{i,n} = J_{U_{i,n} B_i^{-1}}\Big(v_{i,n} + U_{i,n}\big(L_i y_n - D_i^{-1} v_{i,n} - d_{i,n} - r_i\big)\Big) + b_{i,n} \\ v_{i,n+1} = v_{i,n} + \lambda_n(q_{i,n} - v_{i,n}). \end{array} \right. \end{array} \right. \tag{6.7}$$

Then the following hold for some solution $\overline{x}$ to (6.1) and some solution $(\overline{v_1}, \dots, \overline{v_m})$ to (6.2).

(i) *$x_n \rightharpoonup \overline{x}$ as $n \to +\infty$.*

(ii) *$(\forall i \in \{1,\dots,m\})$ $v_{i,n} \rightharpoonup \overline{v_i}$ as $n \to +\infty$.*

(iii) *Suppose that C is demiregular at $\overline{x}$. Then $x_n \to \overline{x}$ as $n \to +\infty$.*

(iv) *Suppose that, for some $j \in \{1,\dots,m\}$, D_j^{-1} is demiregular at $\overline{v_j}$. Then $v_{j,n} \to \overline{v_j}$ as $n \to +\infty$.*

Proof. Define $\boldsymbol{\mathcal{G}}$ as in Notation 2.1 and set $\boldsymbol{\mathcal{K}} = \mathcal{H} \oplus \boldsymbol{\mathcal{G}}$. We denote the scalar product and the norm of $\boldsymbol{\mathcal{K}}$ by $\langle\langle\langle \cdot \mid \cdot \rangle\rangle\rangle$ and $|||\cdot|||$, respectively. As shown in [16, 41], the operators

$$\begin{cases} \boldsymbol{A} : \boldsymbol{\mathcal{K}} \to 2^{\boldsymbol{\mathcal{K}}} \colon (x, v_1, \ldots, v_m) \mapsto (\sum_{i=1}^m L_i^* v_i - z + Ax) \times (r_1 - L_1 x + B_1^{-1} v_1) \times \ldots \times \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (r_m - L_m x + B_m^{-1} v_m) \\ \boldsymbol{B} : \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}} \colon (x, v_1, \ldots, v_m) \mapsto \big(Cx, D_1^{-1} v_1, \ldots, D_m^{-1} v_m\big) \\ \boldsymbol{S} \ : \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}} \colon (x, v_1, \ldots, v_m) \mapsto \Big(\sum_{i=1}^m L_i^* v_i, -L_1 x, \ldots, -L_m x \Big) \end{cases} \tag{6.8}$$

are maximally monotone and, moreover, $\boldsymbol{B}$ is β-cocoercive [41, Eq. (3.12)]. Furthermore, as shown in [16, Section 3], under condition (6.3), $\operatorname{zer}(\boldsymbol{A} + \boldsymbol{B}) \neq \varnothing$ and

$$(\overline{x}, \overline{\boldsymbol{v}}) \in \operatorname{zer}(\boldsymbol{A} + \boldsymbol{B}) \quad \Rightarrow \quad \overline{x} \text{ solves (6.1) and } \overline{\boldsymbol{v}} \text{ solves (6.2).} \tag{6.9}$$

Next, for every $n \in \mathbb{N}$, define

$$\begin{cases} \boldsymbol{U}_n \colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}} \colon (x, v_1, \ldots, v_m) \mapsto \Big(U_n x, U_{1,n} v_1, \ldots, U_{m,n} v_m\Big) \\ \boldsymbol{V}_n \colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}} \colon (x, v_1, \ldots, v_m) \mapsto \Big(U_n^{-1} x - \sum_{i=1}^m L_i^* v_i, \big(-L_i x + U_{i,n}^{-1} v_i\big)_{1 \leqslant i \leqslant m}\Big) \\ \boldsymbol{T}_n \colon \mathcal{H} \to \boldsymbol{\mathcal{G}} \colon x \mapsto \Big(\sqrt{U_{1,n}} L_1 x, \ldots, \sqrt{U_{m,n}} L_m x\Big). \end{cases} \tag{6.10}$$

It follows from our assumptions and Lemma 3.1(iii) that

$$(\forall n \in \mathbb{N}) \quad \boldsymbol{U}_{n+1} \succcurlyeq \boldsymbol{U}_n \in \mathcal{P}_\alpha(\boldsymbol{\mathcal{K}}) \quad \text{and} \quad ||\boldsymbol{U}_n^{-1}|| \leqslant \frac{1}{\alpha}. \tag{6.11}$$

Moreover, for every $n \in \mathbb{N}$, $\boldsymbol{V}_n \in \mathcal{S}(\boldsymbol{\mathcal{K}})$ since $\boldsymbol{U}_n \in \mathcal{S}(\boldsymbol{\mathcal{K}})$. In addition, (6.10) and (6.11) yield

$$\big(\forall n \in \mathbb{N}\big) \quad \|\boldsymbol{V}_n\| \leqslant \|\boldsymbol{U}_n^{-1}\| + \|\boldsymbol{S}\| \leqslant \rho, \quad \text{where} \quad \rho = \frac{1}{\alpha} + \sqrt{\sum_{i=1}^m \|L_i\|^2}. \tag{6.12}$$

On the other hand,

$$\begin{aligned} (\forall n \in \mathbb{N})(\forall x \in \mathcal{H}) \quad |||\boldsymbol{T}_n x|||^2 &= \sum_{i=1}^m \big\| \sqrt{U_{i,n}} L_i \sqrt{U_n} \sqrt{U_n}^{\,-1} x \big\|^2 \\ &\leqslant \|x\|^2_{U_n^{-1}} \sum_{i=1}^m \big\| \sqrt{U_{i,n}} L_i \sqrt{U_n} \big\|^2 \\ &= \beta_n \|x\|^2_{U_n^{-1}}, \end{aligned} \tag{6.13}$$

where $(\forall n \in \mathbb{N})$ $\beta_n = \sum_{i=1}^m \big\|\sqrt{U_{i,n}} L_i \sqrt{U_n}\big\|^2$. Hence, (6.5) yields

$$(\forall n \in \mathbb{N}) \quad (1 + \delta_n)\beta_n = \frac{1}{1 + \delta_n}. \tag{6.14}$$

Therefore, for every $n \in \mathbb{N}$ and every $\boldsymbol{x} = (x, v_1, \ldots, v_m) \in \boldsymbol{\mathcal{K}}$, using (6.10), (6.13), (6.14), Lemma 3.1(ii), and (6.6), we obtain

$$\begin{aligned}
\langle\langle\langle \boldsymbol{x} \mid \boldsymbol{V}_n\boldsymbol{x}\rangle\rangle\rangle &= \big\langle x \mid U_n^{-1}x\big\rangle + \sum_{i=1}^m \Big\langle v_i \mid U_{i,n}^{-1}v_i\Big\rangle - 2\sum_{i=1}^m \langle L_i x \mid v_i\rangle \\
&= \|x\|^2_{U_n^{-1}} + \sum_{i=1}^m \|v_i\|^2_{U_{i,n}^{-1}} - 2\sum_{i=1}^m \Big\langle \sqrt{U_{i,n}}L_i x \mid \sqrt{U_{i,n}}^{\,-1}v_i\Big\rangle \\
&= \|x\|^2_{U_n^{-1}} + \sum_{i=1}^m \|v_i\|^2_{U_{i,n}^{-1}} \\
&\quad - 2\langle\langle \sqrt{(1+\delta_n)\beta_n}^{\,-1}\boldsymbol{T}_n x \mid \sqrt{(1+\delta_n)\beta_n}\big(\sqrt{U_{1,n}}^{\,-1}v_1, \ldots, \sqrt{U_{m,n}}^{\,-1}v_m\big)\rangle\rangle \\
&\geqslant \|x\|^2_{U_n^{-1}} + \sum_{i=1}^m \|v_i\|^2_{U_{i,n}^{-1}} - \left(\frac{|||\boldsymbol{T}_n x|||^2}{(1+\delta_n)\beta_n} + (1+\delta_n)\beta_n\sum_{i=1}^m \|v_i\|^2_{U_{i,n}^{-1}}\right) \\
&\geqslant \|x\|^2_{U_n^{-1}} + \sum_{i=1}^m \|v_i\|^2_{U_{i,n}^{-1}} - \left(\frac{\|x\|^2_{U_n^{-1}}}{(1+\delta_n)} + (1+\delta_n)\beta_n\sum_{i=1}^m \|v_i\|^2_{U_{i,n}^{-1}}\right) \\
&= \frac{\delta_n}{1+\delta_n}\left(\|x\|^2_{U_n^{-1}} + \sum_{i=1}^m \|v_i\|^2_{U_{i,n}^{-1}}\right) \\
&\geqslant \frac{\delta_n}{1+\delta_n}\left(\|U_n\|^{-1}\|x\|^2 + \sum_{i=1}^m \|U_{i,n}\|^{-1}\|v_i\|^2\right) \\
&\geqslant \zeta_n |||\boldsymbol{x}|||^2.
\end{aligned} \tag{6.15}$$

In turn, it follows from Lemma 3.1(iii) and (6.6) that

$$(\forall n \in \mathbb{N}) \quad \|\boldsymbol{V}_n^{-1}\| \leqslant \frac{1}{\zeta_n} \leqslant 2\beta - \varepsilon. \tag{6.16}$$

Moreover, by Lemma 3.1(i), $(\forall n \in \mathbb{N})$ $(\boldsymbol{U}_{n+1} \succcurlyeq \boldsymbol{U}_n \Rightarrow \boldsymbol{U}_n^{-1} \succcurlyeq \boldsymbol{U}_{n+1}^{-1} \Rightarrow \boldsymbol{V}_n \succcurlyeq \boldsymbol{V}_{n+1} \Rightarrow \boldsymbol{V}_{n+1}^{-1} \succcurlyeq \boldsymbol{V}_n^{-1})$. Furthermore, we derive from Lemma 3.1(ii) and (6.12) that

$$(\forall \boldsymbol{x} \in \boldsymbol{\mathcal{K}}) \quad \langle\langle\langle \boldsymbol{V}_n^{-1}\boldsymbol{x} \mid \boldsymbol{x}\rangle\rangle\rangle \geqslant \|\boldsymbol{V}_n\|^{-1}|||\boldsymbol{x}|||^2 \geqslant \frac{1}{\rho}|||\boldsymbol{x}|||^2. \tag{6.17}$$

Altogether,

$$\sup_{n\in\mathbb{N}} \|\boldsymbol{V}_n^{-1}\| \leqslant 2\beta - \varepsilon \qquad \text{and} \quad (\forall n \in \mathbb{N}) \quad \boldsymbol{V}_{n+1}^{-1} \succcurlyeq \boldsymbol{V}_n^{-1} \in \mathcal{P}_{1/\rho}(\boldsymbol{\mathcal{K}}). \tag{6.18}$$

Now set, for every $n \in \mathbb{N}$,

$$\begin{cases}
\boldsymbol{x}_n = (x_n, v_{1,n}, \ldots, v_{m,n}) \\
\boldsymbol{y}_n = (p_n, q_{1,n}, \ldots, q_{m,n}) \\
\boldsymbol{a}_n = (a_n, b_{1,n}, \ldots, b_{m,n}) \\
\boldsymbol{c}_n = (c_n, d_{1,n}, \ldots, d_{m,n}) \\
\boldsymbol{d}_n = (U_n^{-1}a_n, U_{1,n}^{-1}b_{1,n}, \ldots, U_{m,n}^{-1}b_{m,n})
\end{cases} \quad \text{and} \quad \boldsymbol{b}_n = (\boldsymbol{S} + \boldsymbol{V}_n)\boldsymbol{a}_n + \boldsymbol{c}_n - \boldsymbol{d}_n. \tag{6.19}$$

Then $\sum_{n\in\mathbb{N}} |||\boldsymbol{a}_n||| < +\infty$, $\sum_{n\in\mathbb{N}} |||\boldsymbol{c}_n||| < +\infty$, and $\sum_{n\in\mathbb{N}} |||\boldsymbol{d}_n||| < +\infty$. Therefore (6.12) implies that $\sum_{n\in\mathbb{N}} |||\boldsymbol{b}_n||| < +\infty$. Furthermore, using the same arguments as in [41, Eqs. (3.22)–(3.35)], we derive from (6.7) and (6.8) that

$$(\forall n \in \mathbb{N}) \quad \boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \lambda_n\Big(J_{\boldsymbol{V}_n^{-1}\boldsymbol{A}}\big(\boldsymbol{x}_n - \boldsymbol{V}_n^{-1}(\boldsymbol{B}\boldsymbol{x}_n + \boldsymbol{b}_n)\big) + \boldsymbol{a}_n - \boldsymbol{x}_n\Big). \tag{6.20}$$

We observe that (6.20) has the structure of the variable metric forward-backward splitting algorithm (4.3), where $(\forall n \in \mathbb{N})\ \gamma_n = 1$. Finally, (6.16) and (6.18) imply that all the conditions in Theorem 4.1 are satisfied.

(i)&(ii): Theorem 4.1(i) asserts that there exists

$$\overline{\boldsymbol{x}} = (\overline{x}, \overline{v_1}, \ldots, \overline{v_m}) \in \operatorname{zer}(\boldsymbol{A} + \boldsymbol{B}) \tag{6.21}$$

such that $\boldsymbol{x}_n \rightharpoonup \overline{\boldsymbol{x}}$ as $n \to +\infty$. In view of (6.9), the assertions are proved.

(iii)&(iv): It follows from Theorem 4.1(ii) that $\boldsymbol{B}\boldsymbol{x}_n \to \boldsymbol{B}\overline{\boldsymbol{x}}$ as $n \to +\infty$. Hence, (6.8), (6.19), and (6.21) yield

$$Cx_n \to C\overline{x} \quad \text{and} \quad \big(\forall i \in \{1, \ldots, m\}\big) \quad D_i^{-1}v_{i,n} \to D_i^{-1}\overline{v_i} \quad \text{as} \quad n \to +\infty. \tag{6.22}$$

Hence the results follow from (i)&(ii) and Definition 3.14. ☐

Remark 6.3 In the case when $C = \rho\,\mathrm{Id}$ for some $\rho \in\ ]0, +\infty[$, Problem 6.1 reduces to Problem 5.1. However, the algorithm obtained in Corollary 5.2 is quite different from that of Corollary 6.2. Indeed, the former was obtained by applying the forward-backward algorithm (4.3) to the dual inclusion, which was made possible by the strong monotonicity of the primal problem. By contrast, the latter relies on an application of (4.3) in a primal-dual product space.

Example 6.4 Let $z \in \mathcal{H}$, let $f \in \Gamma_0(\mathcal{H})$, let $\mu \in\ ]0, +\infty[$, let $h\colon \mathcal{H} \to \mathbb{R}$ be convex and differentiable with a μ^{-1}-Lipschitzian gradient, let $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{H}$, let $\alpha \in\ ]0, +\infty[$, let m be a strictly positive integer, and let $(U_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{H})$ such that $(\forall n \in \mathbb{N})\ U_{n+1} \succcurlyeq U_n$. For every $i \in \{1, \ldots, m\}$, let $r_i \in \mathcal{G}_i$, let $g_i \in \Gamma_0(\mathcal{G}_i)$, let $\nu_i \in\ ]0, +\infty[$, let $\ell_i \in \Gamma_0(\mathcal{G}_i)$ be ν_i-strongly convex, let $v_{i,0} \in \mathcal{G}_i$, let $(b_{i,n})_{n\in\mathbb{N}}$ and $(d_{i,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{G}_i$, suppose that $0 \neq L_i \in \mathcal{B}(\mathcal{H}, \mathcal{G}_i)$, and let $(U_{i,n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{G}_i)$ such that $(\forall n \in \mathbb{N})\ U_{i,n+1} \succcurlyeq U_{i,n}$. Furthermore, suppose that

$$z \in \operatorname{ran}\left(\partial f + \sum_{i=1}^{m} L_i^*(\partial g_i \,\Box\, \partial \ell_i)(L_i \cdot - r_i) + \nabla h\right). \tag{6.23}$$

The primal problem is

$$\underset{x\in\mathcal{H}}{\text{minimize}}\ f(x) + \sum_{i=1}^{m}(g_i \,\Box\, \ell_i)(L_i x - r_i) + h(x) - \langle x \mid z\rangle\,, \tag{6.24}$$

and the dual problem is

$$\underset{v_1\in\mathcal{G}_1,\ldots,v_m\in\mathcal{G}_m}{\text{minimize}}\ (f^* \,\Box\, h^*)\left(z - \sum_{i=1}^{m} L_i^* v_i\right) + \sum_{i=1}^{m}\big(g_i^*(v_i) + \ell_i^*(v_i) + \langle v_i \mid r_i\rangle\big). \tag{6.25}$$

Let $\beta = \min\{\mu, \nu_1, \ldots, \nu_m\}$, let $\varepsilon \in]0, \min\{1, \beta\}[$, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, 1]$, suppose that (6.6) holds, and set

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} p_n = \operatorname{prox}_f^{U_n^{-1}} \Big(x_n - U_n\big(\sum_{i=1}^m L_i^* v_{i,n} + \nabla h(x_n) + c_n - z\big)\Big) + a_n \\ y_n = 2p_n - x_n \\ x_{n+1} = x_n + \lambda_n(p_n - x_n) \\ \text{For } i = 1, \ldots, m \\ \left\lfloor \begin{array}{l} q_{i,n} = \operatorname{prox}_{g_i^*}^{U_{i,n}^{-1}} \Big(v_{i,n} + U_{i,n}\big(L_i y_n - \nabla \ell_i^*(v_{i,n}) - d_{i,n} - r_i\big)\Big) + b_{i,n} \\ v_{i,n+1} = v_{i,n} + \lambda_n(q_{i,n} - v_{i,n}). \end{array} \right. \end{array} \right. \tag{6.26}$$

Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a solution to (6.24), for every $i \in \{1, \ldots, m\}$ $(v_{i,n})_{n\in\mathbb{N}}$ converges weakly to some $\overline{v_i} \in \mathcal{G}_i$, and $(\overline{v}_1, \ldots, \overline{v}_m)$ is a solution to (6.25).

Proof. Set $A = \partial f$, $C = \nabla h$, and $(\forall i \in \{1, \ldots, m\})$ $B_i = \partial g_i$ and $D_i = \partial \ell_i$. In this setting, it follows from the analysis of [16, Section 4] that (6.24)–(6.25) is a special case of Problem 6.1 and, using (3.10), that (6.26) is a special case of (6.7). Thus, the claims follow from Corollary 6.2(i)&(ii). □

Remark 6.5 Suppose that, in Corollary 6.2 and Example 6.4, there exist τ and $(\sigma_i)_{1\leqslant i\leqslant m}$ in $]0, +\infty[$ such that $(\forall n \in \mathbb{N})$ $U_n = \tau \,\mathrm{Id}$ and $(\forall i \in \{1, \ldots, m\})$ $U_{i,n} = \sigma_i \,\mathrm{Id}$. Then (6.7) and (6.26) reduce to the fixed metric methods appearing in [41, Eq. (3.3)] and [41, Eq. (4.5)], respectively (see [41] for further connections with existing work).

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