A Primal-Dual Method of Partial Inverses
for Composite Inclusions[*]

Maryam A. Alghamdi,[♯] Abdullah Alotaibi,[♯] Patrick L. Combettes,[♭] and Naseer Shahzad [♮]

[♯]King Abdulaziz University
Department of Mathematics
Sciences Faculty for Girls, P. O. Box 4087
Jeddah 21491, Saudi Arabia

[♮]King Abdulaziz University
Department of Mathematics, P. O. Box 80203
Jeddah 21859, Saudi Arabia

[♭]Sorbonne Universités – UPMC Univ. Paris 06
UMR 7598, Laboratoire Jacques-Louis Lions
F-75005, Paris, France

Abstract

Spingarn's method of partial inverses has found many applications in nonlinear analysis and in optimization. We show that it can be employed to solve composite monotone inclusions in duality, thus opening a new range of applications for the partial inverse formalism. The versatility of the resulting primal-dual splitting algorithm is illustrated through applications to structured monotone inclusions and optimization.

Keywords convex optimization, duality, method of partial inverses, monotone operator, splitting algorithm

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[*]Contact author: P. L. Combettes, plc@math.jussieu.fr, phone: +33 1 4427 6319, fax: +33 1 4427 7200.

1 Introduction

Let $\mathcal{H}$ be a real Hilbert space, let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued operator, let $\operatorname{gra} A = \{(x,u) \in \mathcal{H} \times \mathcal{H} \mid u \in Ax\}$ denote the graph of A, let V be a closed vector subspace of $\mathcal{H}$, and let P_V and $P_{V^\perp}$ denote respectively the projectors onto V and onto its orthogonal complement. The partial inverse of A with respect to V is defined through

$$\operatorname{gra} A_V = \big\{(P_V x + P_{V^\perp} u, P_V u + P_{V^\perp} x) \mid (x,u) \in \operatorname{gra} A\big\}. \tag{1.1}$$

This operator, which was introduced by Spingarn in [25], can be regarded as an intermediate object between A and A^{-1}. A key result of [25] is that, if A is maximally monotone, problems of the form

$$\text{find } x \in V \text{ and } u \in V^\perp \text{ such that } u \in Ax \tag{1.2}$$

can be solved by applying the proximal point algorithm algorithm [24] to the partial inverse A_V. The resulting algorithm, known as the *method of partial inverses,* has been applied to various problems in nonlinear analysis and optimization; see, e.g., [5, 9, 11, 13, 16, 17, 18, 20, 25, 26, 27]. The goal of the present paper is to propose a new range of applications of the method of partial inverses by showing that it can be applied to solving the following type of monotone inclusions in duality.

Problem 1.1 Let $\mathcal{H}$ and $\mathcal{G}$ be real Hilbert spaces, let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ and $B\colon \mathcal{G} \to 2^{\mathcal{G}}$ be maximally monotone operators, and let $L\colon \mathcal{H} \to \mathcal{G}$ be a bounded linear operator. Solve the primal inclusion

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in Ax + L^*BLx \tag{1.3}$$

together with the dual inclusion

$$\text{find } v \in \mathcal{G} \text{ such that } 0 \in -LA^{-1}(-L^*v) + B^{-1}v. \tag{1.4}$$

The operator duality described in (1.3)–(1.4) is an extension of the classical Fenchel-Rockafellar duality setting for functions [23] which has been studied in particular in [3, 8, 14, 19, 22]. Our main result shows that, through a suitable reformulation, Problem 1.1 can be reduced to a problem of the form (1.2) and that the method of partial inverses applied to the latter leads to a splitting algorithm in which the operators A, B, and L are used separately.

The remainder of the paper is organized as follows. In Section 2, we revisit the method of partial inverses and propose new convergence results. Our method of partial inverses for solving Problem 1.1 is presented in Section 3, together with applications. An alternative implementation of this method is proposed in Section 4, where further applications are provided. Section 5 contains concluding remarks.

Notation. The scalar product of a real Hilbert space $\mathcal{H}$ is denoted by $\langle \cdot \mid \cdot \rangle$ and the associated norm by $\|\cdot\|$; $\rightharpoonup$ and $\to$ denote, respectively, weak and strong convergence, and Id is the identity operator. We denote by $\mathcal{B}(\mathcal{H}, \mathcal{G})$ the class of bounded linear operators from $\mathcal{H}$ to a real Hilbert space $\mathcal{G}$. Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$. The inverse A^{-1} of A is defined via $\operatorname{gra} A^{-1} = \{(u,x) \in \mathcal{H} \times \mathcal{H} \mid (x,u) \in \operatorname{gra} A\}$, $J_A = (\operatorname{Id} + A)^{-1}$ is the resolvent of A, $\operatorname{ran} A = \bigcup_{x \in \mathcal{H}} Ax$ is the range of A, and $\operatorname{zer} A = \{x \in \mathcal{H} \mid 0 \in Ax\}$ is the set of zeros of A. We denote by $\Gamma_0(\mathcal{H})$ the class of lower semicontinuous convex functions $f\colon \mathcal{H} \to \,]{-\infty}, +\infty]$ such that $\operatorname{dom} f = \{x \in \mathcal{H} \mid f(x) < +\infty\} \neq \varnothing$. Let $f \in \Gamma_0(\mathcal{H})$. The conjugate of f is the function $f^* \in \Gamma_0(\mathcal{H})$ defined by $f^*\colon u \mapsto \sup_{x \in \mathcal{H}}(\langle x \mid u \rangle - f(x))$. For every $x \in \mathcal{H}$, $f + \|x - \cdot\|^2/2$ possesses a unique minimizer, which is denoted by $\operatorname{prox}_f x$. We have

$$\operatorname{prox}_f = J_{\partial f}, \quad \text{where} \quad \partial f\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \{u \in \mathcal{H} \mid (\forall y \in \mathcal{H}) \ \langle y - x \mid u \rangle + f(x) \leqslant f(y)\} \tag{1.5}$$

is the subdifferential of f. See [4] for background on monotone operators and convex analysis.

2　A method of partial inverses

Throughout this section, $\mathcal{K}$ denotes a real Hilbert space. We establish the convergence of a relaxed, error-tolerant method of partial inverses. First, we review some results about partial inverses and the proximal point algorithm.

Lemma 2.1 [25, Section 2] *Let $A\colon \mathcal{K} \to 2^{\mathcal{K}}$ be maximally monotone, let V be a closed vector subspace of $\mathcal{K}$, and let $z \in \mathcal{K}$. Then A_V is maximally monotone and $z \in \operatorname{zer} A_V \Leftrightarrow (P_V z, P_{V^\perp} z) \in \operatorname{gra} A$.*

Lemma 2.2 *Let $A\colon \mathcal{K} \to 2^{\mathcal{K}}$ be maximally monotone, let V be a closed vector subspace of $\mathcal{K}$, let $z \in \mathcal{K}$, and let $p \in \mathcal{K}$. Then $p = J_{A_V} z \Leftrightarrow P_V p + P_{V^\perp}(z - p) = J_A z$.*

Proof. This result, which appears implicitly in [25, Section 4], follows from the equivalences

$$
\begin{aligned}
P_V p + P_{V^\perp}(z - p) = J_A z \quad &\Leftrightarrow \quad \big(P_V p + P_{V^\perp}(z - p),\, z - P_V p - P_{V^\perp}(z - p)\big) \in \operatorname{gra} A \\
&\Leftrightarrow \quad \big(P_V p + P_{V^\perp}(z - p),\, P_V(z - p) + P_{V^\perp} p\big) \in \operatorname{gra} A \\
&\Leftrightarrow \quad (p,\, z - p) \in \operatorname{gra} A_V \\
&\Leftrightarrow \quad p = J_{A_V} z,
\end{aligned}
\tag{2.1}
$$

where we have used (1.1). □

Lemma 2.3 [11, Remark 2.2(vi)] *Let $B\colon \mathcal{K} \to 2^{\mathcal{K}}$ be a maximally monotone operator such that $\operatorname{zer} B \neq \varnothing$, let $z_0 \in \mathcal{K}$, let $(c_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{K}$, and let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence in $]0, 2[$. Suppose that $\sum_{n \in \mathbb{N}} \lambda_n(2 - \lambda_n) = +\infty$ and $\sum_{n \in \mathbb{N}} \lambda_n \|c_n\| < +\infty$, and set*

$$
(\forall n \in \mathbb{N}) \quad z_{n+1} = z_n + \lambda_n\big(J_B z_n + c_n - z_n\big).
\tag{2.2}
$$

Then $J_B z_n - z_n \to 0$ and $(z_n)_{n \in \mathbb{N}}$ converges weakly to a point in $\operatorname{zer} B$.

The next theorem analyzes a method of partial inverses and extends the results of [13] and [25].

Theorem 2.4 *Let $A\colon \mathcal{K} \to 2^{\mathcal{K}}$ be a maximally monotone operator, let V be a closed vector subspace of $\mathcal{K}$, let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence in $]0, 2[$, and let $(e_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{K}$ such that $\sum_{n \in \mathbb{N}} \lambda_n(2 - \lambda_n) = +\infty$ and $\sum_{n \in \mathbb{N}} \lambda_n \|e_n\| < +\infty$. Suppose that the problem*

$$
\text{find } x \in V \text{ and } u \in V^\perp \text{ such that } u \in Ax
\tag{2.3}
$$

has at least one solution, let $x_0 \in V$, let $u_0 \in V^\perp$, and set

$$
(\forall n \in \mathbb{N}) \quad
\left|
\begin{aligned}
p_n &= J_A(x_n + u_n) + e_n \\
r_n &= x_n + u_n - p_n \\
x_{n+1} &= x_n - \lambda_n P_V r_n \\
u_{n+1} &= u_n - \lambda_n P_{V^\perp} p_n.
\end{aligned}
\right.
\tag{2.4}
$$

Then the following hold:

  (i) *$P_V(p_n - e_n) - x_n \to 0$ and $P_{V^\perp}(r_n + e_n) - u_n \to 0$.*

  (ii) *There exists a solution $(\overline{x}, \overline{u})$ to (2.3) such that $x_n \rightharpoonup \overline{x}$ and $u_n \rightharpoonup \overline{u}$.*

Proof. Set

$$(\forall n \in \mathbb{N}) \quad \boldsymbol{z}_n = \boldsymbol{x}_n + \boldsymbol{u}_n \quad \text{and} \quad \boldsymbol{c}_n = P_{\boldsymbol{V}}\boldsymbol{e}_n - P_{\boldsymbol{V}^\perp}\boldsymbol{e}_n. \tag{2.5}$$

Then $\sum_{n \in \mathbb{N}} \lambda_n \|\boldsymbol{c}_n\| \leqslant \sum_{n \in \mathbb{N}} \lambda_n (\|P_{\boldsymbol{V}}\boldsymbol{e}_n\| + \|P_{\boldsymbol{V}^\perp}\boldsymbol{e}_n\|) \leqslant 2 \sum_{n \in \mathbb{N}} \lambda_n \|\boldsymbol{e}_n\| < +\infty$. Furthermore, since $(\boldsymbol{x}_n)_{n \in \mathbb{N}}$ lies in $\boldsymbol{V}$ and $(\boldsymbol{u}_n)_{n \in \mathbb{N}}$ lies in $\boldsymbol{V}^\perp$, (2.4) can be rewritten as

$$(\forall n \in \mathbb{N}) \quad \left| \begin{array}{l} \boldsymbol{p}_n = J_{\boldsymbol{A}}(\boldsymbol{x}_n + \boldsymbol{u}_n) + \boldsymbol{e}_n \\ \boldsymbol{r}_n = \boldsymbol{x}_n + \boldsymbol{u}_n - \boldsymbol{p}_n \\ \boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \lambda_n(P_{\boldsymbol{V}}\boldsymbol{p}_n - \boldsymbol{x}_n) \\ \boldsymbol{u}_{n+1} = \boldsymbol{u}_n + \lambda_n(P_{\boldsymbol{V}^\perp}\boldsymbol{r}_n - \boldsymbol{u}_n), \end{array} \right. \tag{2.6}$$

which yields

$$\begin{aligned}
(\forall n \in \mathbb{N}) \quad P_{\boldsymbol{V}}&\left(\frac{\boldsymbol{z}_{n+1} - \boldsymbol{z}_n}{\lambda_n} + \boldsymbol{z}_n - \boldsymbol{c}_n \right) + P_{\boldsymbol{V}^\perp}\left(\boldsymbol{z}_n - \left(\frac{\boldsymbol{z}_{n+1} - \boldsymbol{z}_n}{\lambda_n} + \boldsymbol{z}_n - \boldsymbol{c}_n \right) \right) \\
&= P_{\boldsymbol{V}}\left(\frac{\boldsymbol{z}_{n+1} - \boldsymbol{z}_n}{\lambda_n} + \boldsymbol{z}_n - \boldsymbol{c}_n \right) + P_{\boldsymbol{V}^\perp}\left(\frac{\boldsymbol{z}_n - \boldsymbol{z}_{n+1}}{\lambda_n} + \boldsymbol{c}_n \right) \\
&= P_{\boldsymbol{V}}\left(\frac{\boldsymbol{x}_{n+1} - \boldsymbol{x}_n}{\lambda_n} + \boldsymbol{x}_n \right) + P_{\boldsymbol{V}^\perp}\left(\frac{\boldsymbol{u}_n - \boldsymbol{u}_{n+1}}{\lambda_n} \right) - \boldsymbol{e}_n \\
&= P_{\boldsymbol{V}}\boldsymbol{p}_n + P_{\boldsymbol{V}^\perp}(\boldsymbol{u}_n - \boldsymbol{r}_n) - \boldsymbol{e}_n \\
&= P_{\boldsymbol{V}}\boldsymbol{p}_n + P_{\boldsymbol{V}^\perp}(\boldsymbol{p}_n - \boldsymbol{x}_n) - \boldsymbol{e}_n \\
&= \boldsymbol{p}_n - \boldsymbol{e}_n \\
&= J_{\boldsymbol{A}}\boldsymbol{z}_n. \tag{2.7}
\end{aligned}$$

Hence, it follows from (2.5), (2.6), and Lemma 2.2 that

$$(\forall n \in \mathbb{N}) \quad \boldsymbol{z}_{n+1} = \boldsymbol{z}_n + \lambda_n\big(J_{\boldsymbol{A_V}}\boldsymbol{z}_n + \boldsymbol{c}_n - \boldsymbol{z}_n\big). \tag{2.8}$$

Altogether, we derive from Lemmas 2.1 and 2.3 that

$$J_{\boldsymbol{A_V}}\boldsymbol{z}_n - \boldsymbol{z}_n \to \boldsymbol{0} \tag{2.9}$$

and that there exists $\boldsymbol{z} \in \operatorname{zer} \boldsymbol{A_V}$ such that

$$\boldsymbol{z}_n \rightharpoonup \boldsymbol{z}. \tag{2.10}$$

(i): In view of (2.4), (2.5), Lemma 2.2, and (2.9), we have

$$\boldsymbol{x}_n - P_{\boldsymbol{V}}(\boldsymbol{p}_n - \boldsymbol{e}_n) = \boldsymbol{x}_n - P_{\boldsymbol{V}}J_{\boldsymbol{A}}\boldsymbol{z}_n = \boldsymbol{x}_n - P_{\boldsymbol{V}}J_{\boldsymbol{A_V}}\boldsymbol{z}_n = P_{\boldsymbol{V}}\big(\boldsymbol{z}_n - J_{\boldsymbol{A_V}}\boldsymbol{z}_n\big) \to \boldsymbol{0} \tag{2.11}$$

and

$$\boldsymbol{u}_n - P_{\boldsymbol{V}^\perp}(\boldsymbol{r}_n + \boldsymbol{e}_n) = P_{\boldsymbol{V}^\perp}\boldsymbol{z}_n - P_{\boldsymbol{V}^\perp}\big(\boldsymbol{z}_n - J_{\boldsymbol{A}}\boldsymbol{z}_n\big) = P_{\boldsymbol{V}^\perp}J_{\boldsymbol{A}}\boldsymbol{z}_n = P_{\boldsymbol{V}^\perp}\big(\boldsymbol{z}_n - J_{\boldsymbol{A_V}}\boldsymbol{z}_n\big) \to \boldsymbol{0}. \tag{2.12}$$

(ii): Since $\boldsymbol{z} \in \operatorname{zer}(\boldsymbol{A_V})$, it follows from Lemma 2.1 that $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}}) = (P_{\boldsymbol{V}}\boldsymbol{z}, P_{\boldsymbol{V}^\perp}\boldsymbol{z})$ solves (2.3). Furthermore, using (2.5), (2.10), and the weak continuity of $P_{\boldsymbol{V}}$ and $P_{\boldsymbol{V}^\perp}$, we get $\boldsymbol{x}_n = P_{\boldsymbol{V}}\boldsymbol{z}_n \rightharpoonup P_{\boldsymbol{V}}\boldsymbol{z} = \overline{\boldsymbol{x}}$ and $\boldsymbol{u}_n = P_{\boldsymbol{V}^\perp}\boldsymbol{z}_n \rightharpoonup P_{\boldsymbol{V}^\perp}\boldsymbol{z} = \overline{\boldsymbol{u}}$. $\square$

Remark 2.5 Theorem 2.4(ii) was obtained in [13, Section 5] under the additional assumptions that $\sum_{n \in \mathbb{N}} \|\boldsymbol{e}_n\| < +\infty$, $\inf_{n \in \mathbb{N}} \lambda_n > 0$, and $\sup_{n \in \mathbb{N}} \lambda_n < 2$; the original weak convergence result of [25] corresponds to the case when $(\forall n \in \mathbb{N})$ $\boldsymbol{e}_n = \boldsymbol{0}$ and $\lambda_n = 1$. As noted in [11, Remark 2.2(iii)], our assumptions do not require that $\sum_{n \in \mathbb{N}} \|\boldsymbol{e}_n\| < +\infty$ and they even allow for situations in which $(\boldsymbol{e}_n)_{n \in \mathbb{N}}$ does not converge weakly to $\boldsymbol{0}$.

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Remark 2.6 It follows from Lemma 2.1 that any algorithm that constructs a zero of $\boldsymbol{A_V}$ can be used to solve (2.3). We have chosen to employ the proximal point algorithm of Lemma 2.3 because it features a flexible error model and it allows for under- and over-relaxations which can prove very useful in improving the convergence pattern of the algorithm. In addition, (2.8) leads to the simple implementation (2.4) in terms of the resolvent $J_{\boldsymbol{A}}$. An alternative proximal point method is

$$(\forall n \in \mathbb{N}) \quad \boldsymbol{z}_{n+1} = \boldsymbol{z}_n + \lambda_n\big(J_{\gamma_n \boldsymbol{A_V}} \boldsymbol{z}_n + \boldsymbol{c}_n - \boldsymbol{z}_n\big), \tag{2.13}$$

where $(\gamma_n)_{n\in\mathbb{N}}$ lies $]0, +\infty[$ (weak convergence conditions under various hypotheses on $(\lambda_n)_{n\in\mathbb{N}}$, $(\gamma_n)_{n\in\mathbb{N}}$, and $(\boldsymbol{e}_n)_{n\in\mathbb{N}}$ exist; see for instance [10, Corollary 4.5] and [13, Theorem 3]). However, due to the presence of the parameters $(\gamma_n)_{n\in\mathbb{N}}$, (2.13) results in an algorithm which is much less straightforward to execute than (2.4). This issue is also discussed in [13, 18, 25].

3 Primal-dual composite method of partial inverses

The following technical facts will be needed subsequently.

Lemma 3.1 *Let $\mathcal{H}$ and $\mathcal{G}$ be real Hilbert spaces, and let $L \in \mathcal{B}(\mathcal{H}, \mathcal{G})$. Set $\mathcal{K} = \mathcal{H} \oplus \mathcal{G}$ and $\boldsymbol{V} = \big\{(x, y) \in \mathcal{K} \mid Lx = y\big\}$. Then, for every $(x, y) \in \mathcal{K}$, the following hold:*

(i) $P_{\boldsymbol{V}}(x, y) = \big(x - L^*(\mathrm{Id} + LL^*)^{-1}(Lx - y), y + (\mathrm{Id} + LL^*)^{-1}(Lx - y)\big).$

(ii) $P_{\boldsymbol{V}}(x, y) = \big((\mathrm{Id} + L^*L)^{-1}(x + L^*y), L(\mathrm{Id} + L^*L)^{-1}(x + L^*y)\big).$

(iii) $P_{\boldsymbol{V}^\perp}(x, y) = \big(L^*(\mathrm{Id} + LL^*)^{-1}(Lx - y), -(\mathrm{Id} + LL^*)^{-1}(Lx - y)\big).$

(iv) $P_{\boldsymbol{V}^\perp}(x, y) = \big(x - (\mathrm{Id} + L^*L)^{-1}(x + L^*y), y - L(\mathrm{Id} + L^*L)^{-1}(x + L^*y)\big).$

Proof. Set $\boldsymbol{T} \colon \mathcal{K} \to \mathcal{G} \colon (x, y) \mapsto Lx - y$. Then $\boldsymbol{T} \in \mathcal{B}(\mathcal{K}, \mathcal{G})$, $\boldsymbol{T}^* \colon \mathcal{G} \to \mathcal{K} \colon v \mapsto (L^*v, -v)$, $\boldsymbol{TT}^* = (\mathrm{Id} + LL^*)$ is invertible, and $\boldsymbol{V} = \ker \boldsymbol{T}$ is a closed vector subspace of $\mathcal{K}$. Let us fix $(x, y) \in \mathcal{K}$.

(i): Since $\boldsymbol{V} = \ker \boldsymbol{T}$, it follows from [4, Example 28.14(iii)] that

$$\begin{aligned} P_{\boldsymbol{V}}(x, y) &= (x, y) - \boldsymbol{T}^*(\boldsymbol{TT}^*)^{-1}\boldsymbol{T}(x, y) \\ &= \big(x - L^*(\mathrm{Id} + LL^*)^{-1}(Lx - y), y + (\mathrm{Id} + LL^*)^{-1}(Lx - y)\big). \end{aligned} \tag{3.1}$$

(i)$\Rightarrow$(ii): We have

$$\begin{aligned} (\mathrm{Id} + L^*L)\big(x - L^*(\mathrm{Id} &+ LL^*)^{-1}(Lx - y) - (\mathrm{Id} + L^*L)^{-1}(x + L^*y)\big) \\ &= x + L^*Lx - L^*(\mathrm{Id} + LL^*)^{-1}(Lx - y) - L^*LL^*(\mathrm{Id} + LL^*)^{-1}(Lx - y) - x - L^*y \\ &= L^*(Lx - y) - L^*(\mathrm{Id} + LL^*)^{-1}(Lx - y) - L^*LL^*(\mathrm{Id} + LL^*)^{-1}(Lx - y) \\ &= L^*\big(\mathrm{Id} + LL^* - \mathrm{Id} - LL^*\big)(\mathrm{Id} + LL^*)^{-1}(Lx - y) \\ &= 0. \end{aligned} \tag{3.2}$$

Therefore, $x - L^*(\mathrm{Id} + LL^*)^{-1}(Lx - y) = (\mathrm{Id} + L^*L)^{-1}(x + L^*y)$. Likewise,

$$\begin{aligned} (\mathrm{Id} + LL^*)\big(y + (\mathrm{Id} &+ LL^*)^{-1}(Lx - y) - L(\mathrm{Id} + L^*L)^{-1}(x + L^*y)\big) \\ &= y + LL^*y + Lx - y - L(\mathrm{Id} + L^*L)^{-1}(x + L^*y) - LL^*L(\mathrm{Id} + L^*L)^{-1}(x + L^*y) \\ &= L(x + L^*y) - L(\mathrm{Id} + L^*L)^{-1}(x + L^*y) - LL^*L(\mathrm{Id} + L^*L)^{-1}(x + L^*y) \\ &= \big(L(\mathrm{Id} + L^*L) - L - LL^*L\big)(\mathrm{Id} + L^*L)^{-1}(x + L^*y) \\ &= 0 \end{aligned} \tag{3.3}$$

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and hence $y + (\mathrm{Id} + LL^*)^{-1}(Lx - y) = L(\mathrm{Id} + L^*L)^{-1}(x + L^*y)$.

(i)$\Rightarrow$(iii) and (ii)$\Rightarrow$(iv): These follow from the identity $P_{V^\perp} = \mathrm{Id} - P_V$. $\square$

Our main algorithm is introduced and analyzed in the next theorem. It consists of applying (2.4) to the operator $A\colon (x,y) \mapsto Ax \times By$ and the subspace $V = \{(x,y) \in \mathcal{H} \oplus \mathcal{G} \mid Lx = y\}$.

Theorem 3.2 *In Problem 1.1, set $Q = (\mathrm{Id} + L^*L)^{-1}$ and assume that $\mathrm{zer}\,(A + L^*BL) \neq \varnothing$. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $]0, 2[$, let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$, and let $(b_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{G}$ such that $\sum_{n\in\mathbb{N}} \lambda_n(2 - \lambda_n) = +\infty$ and $\sum_{n\in\mathbb{N}} \lambda_n \sqrt{\|a_n\|^2 + \|b_n\|^2} < +\infty$. Let $x_0 \in \mathcal{H}$ and $v_0 \in \mathcal{G}$, and set $y_0 = Lx_0$, $u_0 = -L^*v_0$, and*

$$(\forall n \in \mathbb{N}) \quad \left|
\begin{aligned}
p_n &= J_A(x_n + u_n) + a_n \\
q_n &= J_B(y_n + v_n) + b_n \\
r_n &= x_n + u_n - p_n \\
s_n &= y_n + v_n - q_n \\
t_n &= Q(r_n + L^*s_n) \\
w_n &= Q(p_n + L^*q_n) \\
x_{n+1} &= x_n - \lambda_n t_n \\
y_{n+1} &= y_n - \lambda_n L t_n \\
u_{n+1} &= u_n + \lambda_n(w_n - p_n) \\
v_{n+1} &= v_n + \lambda_n(Lw_n - q_n).
\end{aligned}
\right. \tag{3.4}$$

Then the following hold:

(i) $x_n - w_n + Q(a_n + L^*b_n) \to 0$ *and* $y_n - Lw_n + LQ(a_n + L^*b_n) \to 0$.

(ii) $u_n - r_n + t_n - a_n + Q(a_n + L^*b_n) \to 0$ *and* $v_n - s_n + Lt_n - b_n + LQ(a_n + L^*b_n) \to 0$.

Moreover, there exist a solution $\overline{x}$ to (1.3) and a solution $\overline{v}$ to (1.4) such that the following hold:

(iii) $-L^*\overline{v} \in A\overline{x}$ *and* $\overline{v} \in BL\overline{x}$.

(iv) $x_n \rightharpoonup \overline{x}$ *and* $v_n \rightharpoonup \overline{v}$.

Proof. Set

$$\mathcal{K} = \mathcal{H} \oplus \mathcal{G} \quad \text{and} \quad V = \{(x,y) \in \mathcal{K} \mid Lx = y\} \tag{3.5}$$

and note that

$$V^\perp = \{(u,v) \in \mathcal{K} \mid u = -L^*v\}. \tag{3.6}$$

In addition, set

$$Z = \{(x,v) \in \mathcal{K} \mid -L^*v \in Ax \text{ and } v \in BLx\} \tag{3.7}$$

and

$$A\colon \mathcal{K} \to 2^{\mathcal{K}}\colon (x,y) \mapsto Ax \times By. \tag{3.8}$$

We also introduce the set

$$S = \{(x, u) \in V \times V^\perp \mid u \in Ax\}. \tag{3.9}$$

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Observe that

$$\boldsymbol{S} = \big\{ \big((x, Lx), (-L^*v, v)\big) \in \mathcal{K} \times \mathcal{K} \mid (x, v) \in \boldsymbol{Z} \big\}. \tag{3.10}$$

Thus (see [8, 19] for the first two equivalences),

$$\operatorname{zer}\big(A + L^*BL\big) \neq \varnothing \;\Leftrightarrow\; \operatorname{zer}\big(-LA^{-1}(-L^*) + B^{-1}\big) \neq \varnothing \;\Leftrightarrow\; \boldsymbol{Z} \neq \varnothing \;\Leftrightarrow\; \boldsymbol{S} \neq \varnothing. \tag{3.11}$$

Now define $(\forall n \in \mathbb{N})$ $\boldsymbol{e}_n = (a_n, b_n)$, $\boldsymbol{p}_n = (p_n, q_n)$, $\boldsymbol{r}_n = (r_n, s_n)$, $\boldsymbol{u}_n = (u_n, v_n)$, and $\boldsymbol{x}_n = (x_n, y_n)$. Then $\boldsymbol{x}_0 \in \boldsymbol{V}$ and $\boldsymbol{u}_0 \in \boldsymbol{V}^\perp$. Moreover, by [4, Proposition 23.16], $\boldsymbol{A}$ is maximally monotone and

$$(\forall n \in \mathbb{N}) \quad J_{\boldsymbol{A}}(\boldsymbol{x}_n + \boldsymbol{u}_n) = \big(J_A(x_n + u_n), J_B(y_n + v_n)\big). \tag{3.12}$$

Furthermore, it follows from (3.5) and Lemma 3.1(ii) that

$$(\forall n \in \mathbb{N}) \quad P_{\boldsymbol{V}}\boldsymbol{r}_n = \big(Q(r_n + L^*s_n), LQ(r_n + L^*s_n)\big), \tag{3.13}$$

and from Lemma 3.1(iv) that

$$(\forall n \in \mathbb{N}) \quad P_{\boldsymbol{V}^\perp}\boldsymbol{p}_n = \big(p_n - Q(p_n + L^*q_n), q_n - LQ(p_n + L^*q_n)\big). \tag{3.14}$$

Thus, we derive from (3.12), (3.13), and (3.14) that (2.4) yields (3.4). Altogether, since $\sum_{n \in \mathbb{N}} \lambda_n \|\boldsymbol{e}_n\| = \sum_{n \in \mathbb{N}} \lambda_n \sqrt{\|a_n\|^2 + \|b_n\|^2} < +\infty$, Theorem 2.4(i) and Lemma 3.1 imply that (i) and (ii) are satisfied, and Theorem 2.4(ii) implies that there exists $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}}) \in \boldsymbol{S}$ such that $\boldsymbol{x}_n \rightharpoonup \overline{\boldsymbol{x}}$ and $\boldsymbol{u}_n \rightharpoonup \overline{\boldsymbol{u}}$. Therefore, by (3.10), there exists $(\overline{x}, \overline{v}) \in \boldsymbol{Z}$ such that $(x_n, v_n) \rightharpoonup (\overline{x}, \overline{v})$. Since $\boldsymbol{Z} \subset (\operatorname{zer}(A + L^*BL)) \times (\operatorname{zer}(-LA^{-1}(-L^*) + B^{-1}))$ [8, Proposition 2.8(i)], the proof is complete. $\square$

Remark 3.3 In the special case when $A = 0$ and L has closed range, an algorithm was proposed in [20] to solve the primal problem (1.3), i.e., to find a point in $\operatorname{zer}(L^*BL)$. It employs the method of partial inverses in $\mathcal{G}$ for finding $y \in V$ and $v \in V^\perp$ such that $v \in By$, where $V = \operatorname{ran} L$, and then solves $Lx = y$. Each iteration of the resulting algorithm requires the computation of the generalized inverse of L, which is numerically demanding.

Remark 3.4 The first algorithm for solving Problem 1.1 appears to be that of [8], which revolves around the application of a two-operator splitting technique in the primal-dual space $\mathcal{H} \oplus \mathcal{G}$. This algorithm, and those devised subsequently on the same principle [6, 12, 28], use the linear operators L and L^* explicitly, without inversion. By contrast, the algorithm described in Theorem 3.2 is not based on a splitting technique but on the proximal point algorithm, and it requires the inversion of the linear operator $\operatorname{Id} + L^*L$. In finite-dimensional applications, this operator is associated with an $N \times N$ symmetric positive-definite matrix and this inversion can be performed off-line with a complexity of $O(N^3)$ [15]. On can then store the matrix $Q = (\operatorname{Id} + L^*L)^{-1}$ and reuse it at each iteration. In some instances, the inversion can be performed more efficiently because of the structure of L, e.g., by means of the fast Fourier transform [1, 2, 21]. The algorithms of [6, 8, 12, 28] and (3.4) should be compared in concrete situations to assess their respective merits in terms of iteration-complexity, overall speed of convergence, and memory requirements.

The following application of Theorem 3.2 concerns multi-operator inclusions.

Problem 3.5 Let m be a strictly positive integer, and let $\mathcal{H}$ and $(\mathcal{G}_i)_{1 \leqslant i \leqslant m}$ be real Hilbert spaces. Let $z \in \mathcal{H}$, let $C \colon \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone, and, for every $i \in \{1, \ldots, m\}$, let $B_i \colon \mathcal{G}_i \to 2^{\mathcal{G}_i}$ be maximally monotone, let $o_i \in \mathcal{G}_i$, and let $L_i \in \mathcal{B}(\mathcal{H}, \mathcal{G}_i)$. Solve

$$\text{find } \overline{x} \in \mathcal{H} \text{ such that } z \in C\overline{x} + \sum_{i=1}^{m} L_i^* B_i(L_i \overline{x} - o_i) \tag{3.15}$$

together with the dual problem

find $\overline{v}_1 \in \mathcal{G}_1, \ldots, \overline{v}_m \in \mathcal{G}_m$ such that

$$(\forall i \in \{1, \ldots, m\}) \quad - o_i \in -L_i C^{-1}\left(z - \sum_{j=1}^m L_j^* \overline{v}_j\right) + B_i^{-1}\overline{v}_i. \quad (3.16)$$

Corollary 3.6 *In Problem 3.5, set $Q = (\mathrm{Id} + \sum_{i=1}^m L_i^* L_i)^{-1}$ and assume that $z \in \mathrm{ran}\,(C + \sum_{i=1}^m L_i^* B_i(L_i \cdot -o_i))$. Let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence in $]0, 2[$ such that $\sum_{n \in \mathbb{N}} \lambda_n(2 - \lambda_n) = +\infty$, let $(a_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{H}$, and let $x_0 \in \mathcal{H}$. For every $i \in \{1, \ldots, m\}$, let $(b_{i,n})_{n \in \mathbb{N}}$ be a sequence in $\mathcal{G}_i$, let $v_{i,0} \in \mathcal{G}_i$, and set $y_{i,0} = L_i x_0$. Suppose that $\sum_{n \in \mathbb{N}} \lambda_n \sqrt{\|a_n\|^2 + \sum_{i=1}^m \|b_{i,n}\|^2} < +\infty$, and set $u_0 = -\sum_{i=1}^m L_i^* v_{i,0}$ and*

$$(\forall n \in \mathbb{N}) \quad \left| \begin{array}{l} p_n = J_C(x_n + u_n + z) + a_n \\ r_n = x_n + u_n - p_n \\ \text{For } i = 1, \ldots, m \\ \quad \left\lfloor \begin{array}{l} q_{i,n} = o_i + J_{B_i}(y_{i,n} + v_{i,n} - o_i) + b_{i,n} \\ s_{i,n} = y_{i,n} + v_{i,n} - q_{i,n} \end{array} \right. \\ t_n = Q(r_n + \sum_{i=1}^m L_i^* s_{i,n}) \\ w_n = Q(p_n + \sum_{i=1}^m L_i^* q_{i,n}) \\ x_{n+1} = x_n - \lambda_n t_n \\ u_{n+1} = u_n + \lambda_n(w_n - p_n) \\ \text{For } i = 1, \ldots, m \\ \quad \left\lfloor \begin{array}{l} y_{i,n+1} = y_{i,n} - \lambda_n L_i t_n \\ v_{i,n+1} = v_{i,n} + \lambda_n(L_i w_n - q_{i,n}). \end{array} \right. \end{array} \right. \quad (3.17)$$

Then there exist a solution $\overline{x}$ to (3.15) and a solution $(\overline{v}_i)_{1 \leqslant i \leqslant m}$ to (3.16) such that the following hold:

(i) $z - \sum_{i=1}^m L_i^* \overline{v}_i \in C\overline{x}$ *and* $(\forall i \in \{1, \ldots, m\})$ $\overline{v}_i \in B_i(L_i \overline{x} - o_i)$.

(ii) $x_n \rightharpoonup \overline{x}$ *and* $(\forall i \in \{1, \ldots, m\})$ $v_{i,n} \rightharpoonup \overline{v}_i$.

Proof. Set $A \colon \mathcal{H} \to 2^{\mathcal{H}} \colon x \mapsto -z + Cx$, $\mathcal{G} = \bigoplus_{i=1}^m \mathcal{G}_i$, $L \colon \mathcal{H} \to \mathcal{G} \colon x \mapsto (L_i x)_{1 \leqslant i \leqslant m}$, and $B \colon \mathcal{G} \to 2^{\mathcal{G}} \colon (y_i)_{1 \leqslant i \leqslant m} \mapsto \bigtimes_{i=1}^m B_i(y_i - o_i)$. Then $L^* \colon \mathcal{G} \to \mathcal{H} \colon (y_i)_{1 \leqslant i \leqslant m} \mapsto \sum_{i=1}^m L_i^* y_i$ and Problem 3.5 is therefore an instantiation of Problem 1.1. Moreover, [4, Propositions 23.15 and 23.16] yield

$$J_A \colon x \mapsto J_C(x + z) \quad \text{and} \quad J_B \colon (y_i)_{1 \leqslant i \leqslant m} \mapsto \left(o_i + J_{B_i}(y_i - o_i)\right)_{1 \leqslant i \leqslant m}. \quad (3.18)$$

Now set $(\forall n \in \mathbb{N})$ $b_n = (b_{i,n})_{1 \leqslant i \leqslant m}$, $q_n = (q_{i,n})_{1 \leqslant i \leqslant m}$, $s_n = (s_{i,n})_{1 \leqslant i \leqslant m}$, $v_n = (v_{i,n})_{1 \leqslant i \leqslant m}$, and $y_n = (y_{i,n})_{1 \leqslant i \leqslant m}$. In this setting, (3.4) coincides with (3.17) and the claims therefore follow from Theorem 3.2(iii)&(iv). $\square$

The next application addresses a primal-dual structured minimization problem.

Problem 3.7 Let m be a strictly positive integer, and let $\mathcal{H}$ and $(\mathcal{G}_i)_{1 \leqslant i \leqslant m}$ be real Hilbert spaces. Let $z \in \mathcal{H}$, let $f \in \Gamma_0(\mathcal{H})$, and, for every $i \in \{1, \ldots, m\}$, let $g_i \in \Gamma_0(\mathcal{G}_i)$, $o_i \in \mathcal{G}_i$, and $L_i \in \mathcal{B}(\mathcal{H}, \mathcal{G}_i)$. Solve the primal problem

$$\underset{x \in \mathcal{H}}{\text{minimize}} \quad f(x) + \sum_{i=1}^m g_i(L_i x - o_i) - \langle x \mid z \rangle \quad (3.19)$$

together with the dual problem

$$\underset{v_1 \in \mathcal{G}_1, \ldots, v_m \in \mathcal{G}_m}{\text{minimize}} \quad f^*\left(z - \sum_{i=1}^m L_i^* v_i\right) + \sum_{i=1}^m \left(g_i^*(v_i) + \langle v_i \mid o_i \rangle\right). \quad (3.20)$$

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Corollary 3.8 *In Problem 3.7, set $Q = (\mathrm{Id} + \sum_{i=1}^{m} L_i^* L_i)^{-1}$ and assume that*

$$z \in \mathrm{ran}\left(\partial f + \sum_{i=1}^{m} L_i^*(\partial g_i)(L_i \cdot - o_i)\right), \tag{3.21}$$

Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $]0,2[$ such that $\sum_{n\in\mathbb{N}} \lambda_n(2-\lambda_n) = +\infty$, let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$, and let $x_0 \in \mathcal{H}$. For every $i \in \{1,\dots,m\}$, let $(b_{i,n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{G}_i$, let $v_{i,0} \in \mathcal{G}_i$, and set $y_{i,0} = L_i x_0$. Suppose that $\sum_{n\in\mathbb{N}} \lambda_n \sqrt{\|a_n\|^2 + \sum_{i=1}^{m}\|b_{i,n}\|^2} < +\infty$, and set $u_0 = -\sum_{i=1}^{m} L_i^* v_{i,0}$ and

$$(\forall n \in \mathbb{N}) \quad \begin{vmatrix} p_n = \mathrm{prox}_f(x_n + u_n + z) + a_n \\ r_n = x_n + u_n - p_n \\ \textit{For } i = 1,\dots,m \\ \quad \begin{vmatrix} q_{i,n} = o_i + \mathrm{prox}_{g_i}(y_{i,n} + v_{i,n} - o_i) + b_{i,n} \\ s_{i,n} = y_{i,n} + v_{i,n} - q_{i,n} \end{vmatrix} \\ t_n = Q(r_n + \sum_{i=1}^{m} L_i^* s_{i,n}) \\ w_n = Q(p_n + \sum_{i=1}^{m} L_i^* q_{i,n}) \\ x_{n+1} = x_n - \lambda_n t_n \\ u_{n+1} = u_n + \lambda_n(w_n - p_n) \\ \textit{For } i = 1,\dots,m \\ \quad \begin{vmatrix} y_{i,n+1} = y_{i,n} - \lambda_n L_i t_n \\ v_{i,n+1} = v_{i,n} + \lambda_n(L_i w_n - q_{i,n}). \end{vmatrix} \end{vmatrix} \tag{3.22}$$

Then there exist a solution $\overline{x}$ to (3.19) and a solution $(\overline{v}_i)_{1\leqslant i\leqslant m}$ to (3.20) such that $z - \sum_{i=1}^{m} L_i^*\overline{v}_i \in \partial f(\overline{x})$, $x_n \rightharpoonup \overline{x}$, and $(\forall i \in \{1,\dots,m\})\ v_{i,n} \rightharpoonup \overline{v}_i \in \partial g_i(L_i\overline{x} - o_i)$.

Proof. Set $C = \partial f$ and $(\forall i \in \{1,\dots,m\}\ B_i = \partial g_i$. Then, using the same type of argument as in the proof of [12, Theorem 4.2], we derive from (3.21) that Problem 3.5 reduces to Problem 3.7 and that (3.17) reduces to (3.22). Thus, the assertions follow from Corollary 3.6. $\square$

4 Alternative composite primal-dual method of partial inverses

The partial inverse method (3.4) relies on the implicit assumption that the operator $(\mathrm{Id} + L^*L)^{-1}$ is relatively easy to implement. In some instances, it may be advantageous to work with $(\mathrm{Id} + LL^*)^{-1}$ instead. In this section we describe an alternative method tailored to such situations.

Theorem 4.1 *In Problem 1.1, set $R = (\mathrm{Id} + LL^*)^{-1}$ and assume that $\mathrm{zer}(A + L^*BL) \neq \varnothing$. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $]0,2[$, let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$, and let $(b_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{G}$ such that $\sum_{n\in\mathbb{N}} \lambda_n(2-\lambda_n) = +\infty$ and $\sum_{n\in\mathbb{N}} \lambda_n \sqrt{\|a_n\|^2 + \|b_n\|^2} < +\infty$. Let $x_0 \in \mathcal{H}$ and $v_0 \in \mathcal{G}$, and set $y_0 = Lx_0$, $u_0 = -L^*v_0$, and*

$$(\forall n \in \mathbb{N}) \quad \begin{vmatrix} p_n = J_A(x_n + u_n) + a_n \\ q_n = J_B(y_n + v_n) + b_n \\ r_n = x_n + u_n - p_n \\ s_n = y_n + v_n - q_n \\ t_n = R(Lr_n - s_n) \\ w_n = R(Lp_n - q_n) \\ x_{n+1} = x_n + \lambda_n(L^*t_n - r_n) \\ y_{n+1} = y_n - \lambda_n(t_n + s_n) \\ u_{n+1} = u_n - \lambda_n L^*w_n \\ v_{n+1} = v_n + \lambda_n w_n. \end{vmatrix} \tag{4.1}$$

Then the conclusions of Theorem 3.2 are true.

Proof. The proof is analogous to that of Theorem 3.2 except that we replace (3.13) by

$$(\forall n \in \mathbb{N}) \quad P_{\boldsymbol{V}}\boldsymbol{r}_n = \big(r_n - L^*R(Lr_n - s_n), s_n + R(Lr_n - s_n)\big), \tag{4.2}$$

and (3.14) by

$$(\forall n \in \mathbb{N}) \quad P_{\boldsymbol{V}^\perp}\boldsymbol{p}_n = \big(L^*R(Lp_n - q_n), -R(Lp_n - q_n)\big) \tag{4.3}$$

by invoking Lemma 3.1(i)&(iii). $\square$

Next, we present an application to a coupled inclusions problem. This problem has essentially the same structure as Problem 3.5, except that primal and dual inclusions are interchanged.

Problem 4.2 Let m be a strictly positive integer, and let $(\mathcal{H}_i)_{1 \leqslant i \leqslant m}$ and $\mathcal{G}$ be real Hilbert spaces. Let $o \in \mathcal{G}$, let $D \colon \mathcal{G} \to 2^{\mathcal{G}}$ be maximally monotone, and, for every $i \in \{1, \ldots, m\}$, let $z_i \in \mathcal{H}_i$, let $A_i \colon \mathcal{H}_i \to 2^{\mathcal{H}_i}$ be maximally monotone, and let $L_i \in \mathcal{B}(\mathcal{H}_i, \mathcal{G})$. Solve the primal problem

$$\text{find } \overline{x}_1 \in \mathcal{H}_1, \ldots, \overline{x}_m \in \mathcal{H}_m \text{ such that } (\forall i \in \{1, \ldots, m\}) \; z_i \in A_i \overline{x}_i + L_i^* D\bigg(\sum_{j=1}^m L_j \overline{x}_j - o\bigg) \tag{4.4}$$

together with the dual problem

$$\text{find } \overline{v} \in \mathcal{G} \text{ such that } -o \in -\sum_{i=1}^m L_i A_i^{-1}\big(z_i - L_i^* \overline{v}\big) + D^{-1}\overline{v}. \tag{4.5}$$

Corollary 4.3 *In Problem 4.2, set $R = (\mathrm{Id} + \sum_{i=1}^m L_i L_i^*)^{-1}$ and assume that (4.4) has at least one solution. Let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence in $]0, 2[$ such that $\sum_{n \in \mathbb{N}} \lambda_n (2 - \lambda_n) = +\infty$, let $(b_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{G}$, and let $v_0 \in \mathcal{G}$. For every $i \in \{1, \ldots, m\}$, let $(a_{i,n})_{n \in \mathbb{N}}$ be a sequence in $\mathcal{H}_i$, let $x_{i,0} \in \mathcal{H}_i$, and set $u_{i,0} = -L_i^* v_0$. Suppose that $\sum_{n \in \mathbb{N}} \lambda_n \sqrt{\|b_n\|^2 + \sum_{i=1}^m \|a_{i,n}\|^2} < +\infty$, and set $y_0 = \sum_{i=1}^m L_i x_{i,0}$ and*

$$(\forall n \in \mathbb{N}) \quad \left| \begin{array}{l} \text{For } i = 1, \ldots, m \\ \quad \left| \begin{array}{l} p_{i,n} = J_{A_i}(x_{i,n} + u_{i,n} + z_i) + a_{i,n} \\ r_{i,n} = x_{i,n} + u_{i,n} - p_{i,n} \end{array} \right. \\ q_n = o + J_D(y_n + v_n - o) + b_n \\ s_n = y_n + v_n - q_n \\ t_n = R(\sum_{i=1}^m L_i r_{i,n} - s_n) \\ w_n = R(\sum_{i=1}^m L_i p_{i,n} - q_n) \\ \text{For } i = 1, \ldots, m \\ \quad \left| \begin{array}{l} x_{i,n+1} = x_{i,n} + \lambda_n(L_i^* t_n - r_{i,n}) \\ u_{i,n+1} = u_{i,n} - \lambda_n L_i^* w_n \end{array} \right. \\ y_{n+1} = y_n - \lambda_n(t_n + s_n) \\ v_{n+1} = v_n + \lambda_n w_n. \end{array} \right. \tag{4.6}$$

Then there exist a solution $(\overline{x}_i)_{1 \leqslant i \leqslant m}$ to (4.4) and a solution $\overline{v}$ to (4.5) such that the following hold:

(i) $\overline{v} \in D(\sum_{i=1}^m L_i \overline{x}_i - o)$ *and* $(\forall i \in \{1, \ldots, m\}) \; z_i - L_i^* \overline{v} \in A_i \overline{x}_i$.

(ii) $v_n \rightharpoonup \overline{v}$ *and* $(\forall i \in \{1, \ldots, m\}) \; x_{i,n} \rightharpoonup \overline{x}_i$.

Proof. Set $B\colon \mathcal{G} \to 2^{\mathcal{G}}\colon y \mapsto D(y-o)$, $\mathcal{H} = \bigoplus_{i=1}^{m} \mathcal{H}_i$, $L\colon \mathcal{H} \to \mathcal{G}\colon (x_i)_{1 \leqslant i \leqslant m} \mapsto \sum_{i=1}^{m} L_i x_i$, and $A\colon \mathcal{H} \to 2^{\mathcal{H}}\colon (x_i)_{1 \leqslant i \leqslant m} \mapsto \bigtimes_{i=1}^{m}(-z_i + A_i x_i)$. Then $L^*\colon \mathcal{G} \to \mathcal{H}\colon y \mapsto (L_i^* y)_{1 \leqslant i \leqslant m}$ and hence Problem 4.2 is a special case of Problem 1.1. On the other hand, [4, Propositions 23.15 and 23.16] yield

$$J_A\colon (x_i)_{1 \leqslant i \leqslant m} \mapsto \big(J_{A_i}(x_i + z_i)\big)_{1 \leqslant i \leqslant m} \quad \text{and} \quad J_B\colon y \mapsto o + J_D(y-o). \tag{4.7}$$

Now set $(\forall n \in \mathbb{N})$ $a_n = (a_{i,n})_{1 \leqslant i \leqslant m}$, $p_n = (p_{i,n})_{1 \leqslant i \leqslant m}$, $r_n = (r_{i,n})_{1 \leqslant i \leqslant m}$, $u_n = (u_{i,n})_{1 \leqslant i \leqslant m}$, and $x_n = (x_{i,n})_{1 \leqslant i \leqslant m}$. Then (4.1) reduces to (4.6) and we can appeal to Theorem 4.1 to conclude. $\square$

Remark 4.4 In Problem 4.2, set $D = \partial g$, where $g \in \Gamma_0(\mathcal{G})$, and $(\forall i \in \{1, \ldots, m\}$ $A_i = \partial f_i$, where $f_i \in \Gamma_0(\mathcal{H}_i)$ and $z_i \in \operatorname{ran}\big(\partial f_i + L_i^*(\partial g)(L_i \cdot -o)\big)$. Then, arguing as in the proof of Corollary 3.8, we derive from Corollary 4.3 an algorithm for solving the primal problem

$$\underset{x_1 \in \mathcal{H}_1, \ldots, x_m \in \mathcal{H}_m}{\text{minimize}} \quad \sum_{i=1}^{m} \big(f_i(x_i) - \langle x_i \mid z_i \rangle\big) + g\bigg(\sum_{i=1}^{m} L_i x_i - o\bigg) \tag{4.8}$$

together with the dual problem

$$\underset{v \in \mathcal{G}}{\text{minimize}} \quad \sum_{i=1}^{m} f_i^*\big(z_i - L_i^* v\big) + g^*(v) + \langle v \mid o \rangle \tag{4.9}$$

by replacing J_{A_i} by $\operatorname{prox}_{f_i}$ and J_D by prox_g in (4.6).

5 Concluding remarks

We have shown that the method of partial inverses can be used to solve composite monotone inclusions in duality and have presented a few applications of this new framework. Despite their apparent complexity, all the algorithms developed in this paper are instances of the method of partial inverses, which is itself an instance of the proximal point algorithm. This underlines the fundamental nature of the proximal point algorithm and its far reaching ramifications. Finally, let us note that in [7] the method of partial inverses was coupled to standard splitting methods for the sum of two monotone operators to solve inclusions of the form $0 \in Ax + Bx + N_V x$, where N_V is the normal cone operator of the closed vector subspace V. Combining this approach to our results should lead to new splitting methods for more general problems involving composite operators, such as those studied in [12].

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