Interchange Rules for Integral Functions*

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Abstract

We first present an abstract principle for the interchange of infimization and integration over spaces of mappings taking values in topological spaces. New conditions on the underlying space and the integrand are then introduced to convert this principle into concrete scenarios that are shown to capture those of various existing interchange rules. These results are leveraged to improve state-of-the-art interchange rules for evaluating Legendre conjugates, subdifferentials, recessions, Moreau envelopes, and proximity operators of integral functions by bringing the corresponding operations under the integral sign.

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1 Introduction

This paper concerns the interchange of the infimization and integration operations in the context of the following assumption.

Assumption 1.1

[A] X is a real vector space endowed with a Souslin topology $\mathcal{T}_{\mathsf{X}}$ and associated Borel σ-algebra $\mathcal{B}_{\mathsf{X}}$.

[B] The mapping $(\mathsf{X} \times \mathsf{X}, \mathcal{B}_{\mathsf{X}} \otimes \mathcal{B}_{\mathsf{X}}) \to (\mathsf{X}, \mathcal{B}_{\mathsf{X}})\colon (\mathsf{x}, \mathsf{y}) \mapsto \mathsf{x} + \mathsf{y}$ is measurable.

[C] For every $\lambda \in \mathbb{R}$, the mapping $(\mathsf{X}, \mathcal{B}_{\mathsf{X}}) \to (\mathsf{X}, \mathcal{B}_{\mathsf{X}})\colon \mathsf{x} \mapsto \lambda\mathsf{x}$ is measurable.

[D] $(\Omega, \mathcal{F}, \mu)$ is a σ-finite measure space such that $\mu(\Omega) \neq 0$, and $\mathcal{L}(\Omega; \mathsf{X})$ denotes the vector space of measurable mappings from $(\Omega, \mathcal{F})$ to $(\mathsf{X}, \mathcal{B}_{\mathsf{X}})$.

[E] $\mathcal{X}$ is a vector subspace of $\mathcal{L}(\Omega; \mathsf{X})$.

[F] $\varphi\colon (\Omega \times \mathsf{X}, \mathcal{F} \otimes \mathcal{B}_{\mathsf{X}}) \to \overline{\mathbb{R}}$ is an integrand in the sense that it is measurable and, for every $\omega \in \Omega$, $\operatorname{epi} \varphi_\omega \neq \varnothing$, where $\varphi_\omega = \varphi(\omega, \cdot)$.

[G] There exists $\overline{x} \in \mathcal{X}$ such that $\int_\Omega \max\{\varphi(\cdot, \overline{x}(\cdot)), 0\} d\mu < +\infty$.

As is customary, given a measurable function $\varrho\colon (\Omega, \mathcal{F}) \to \overline{\mathbb{R}}$, $\int_\Omega \varrho d\mu$ is the usual Lebesgue integral, except when the Lebesgue integral $\int_\Omega \max\{\varrho, 0\} d\mu$ is $+\infty$, in which case $\int_\Omega \varrho d\mu = +\infty$.

Many problems in analysis and its applications require the evaluation of the infimum over $\mathcal{X}$ of the function $f\colon x \mapsto \int_\Omega \varphi(\cdot, x(\cdot)) d\mu$. A simpler task is to evaluate the function $\phi\colon \omega \mapsto \inf \varphi(\omega, \mathsf{X})$ and then compute $\int_\Omega \phi d\mu$. In general, this provides only a lower bound as $\inf f(\mathcal{X}) \geqslant \int_\Omega \phi d\mu$. Conditions under which the two quantities are equal have been established in [15], [25], and [31] under various hypotheses on X, $(\Omega, \mathcal{F}, \mu)$, $\mathcal{X}$, and φ. The resulting infimization-integration interchange rule is a central tool in areas such as plasticity theory [5], convex analysis [13], multivariate analysis [15], calculus of variations [17], economics [18], stochastic processes [22], optimal transport [23], stochastic optimization [24], finance [25], variational analysis [32], and stochastic programming [37]. Note that, in Assumption 1.1[A]–[C], we do not require that $(\mathsf{X}, \mathcal{T}_{\mathsf{X}})$ be a topological vector space to accommodate certain applications. For instance, in [25], X is the space of càdlàg functions on $[0, 1]$ and $\mathcal{T}_{\mathsf{X}}$ is the Skorokhod topology. In this context, $(\mathsf{X}, \mathcal{T}_{\mathsf{X}})$ is a Polish space [2, Chapter 3] which is not a topological vector space [26] but which satisfies Assumption 1.1[A]–[C].

Our first contribution is Theorem 1.2 below, which provides, under the umbrella of Assumption 1.1, a broad setting for the interchange of infimization and integration.

Theorem 1.2 (interchange principle) *Suppose that Assumption 1.1 and the following hold:*

(i) $\inf_{\mathsf{x} \in \mathsf{X}} \varphi(\cdot, \mathsf{x})$ *is $\mathcal{F}$-measurable.*

(ii) *There exists a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{L}(\Omega; \mathsf{X})$ such that the following are satisfied:*

  (a) $\inf_{\mathsf{x} \in \mathsf{X}} \varphi(\cdot, \mathsf{x}) = \inf_{n \in \mathbb{N}} \varphi(\cdot, x_n(\cdot) + \overline{x}(\cdot))$ μ-a.e.

  (b) *There exists an increasing sequence $(\Omega_k)_{k \in \mathbb{N}}$ of finite μ-measure sets in $\mathcal{F}$ such that $\bigcup_{k \in \mathbb{N}} \Omega_k = \Omega$ and*

$$(\forall n \in \mathbb{N})(\forall k \in \mathbb{N}) \quad \big\{ 1_A x_n \mid \mathcal{F} \ni A \subset \Omega_k \text{ and } \overline{x_n(A)} \text{ is compact} \big\} \subset \mathcal{X}. \tag{1.1}$$

Then

$$\inf_{x \in \mathcal{X}} \int_\Omega \varphi\big(\omega, x(\omega)\big) \mu(d\omega) = \int_\Omega \inf_{\mathsf{x} \in \mathsf{X}} \varphi(\omega, \mathsf{x})\, \mu(d\omega). \tag{1.2}$$

Theorem 1.2 is proved in Section 3. The second contribution is the introduction of two new tools — compliant spaces and an extended notion of normal integrands. This is done in Section 4, where these notions are illustrated through various examples. In Section 5, compliance and normality are utilized to build a pathway between the abstract interchange principle of Theorem 1.2 and separate conditions on $\mathcal{X}$ and φ that capture various application settings. The main result of that section is Theorem 5.1, which encompasses in particular the interchange rules of [15, 25, 31], as well as those implicitly present in [28, 29, 38]. These different frameworks have so far not been brought together and we improve them in several directions, for instance by not requiring the completeness of $(\Omega, \mathcal{F}, \mu)$ and by relaxing the assumptions on X. This leads to new concrete scenarios under which (1.2) holds. Our third contribution, presented in Section 6, concerns convex-analytical operations on integral functions. By combining Theorem 1.2, compliance, and normality, we broaden conditions for evaluating Legendre conjugates, subdifferentials, recessions, Moreau envelopes, and proximity operators of integral functions by bringing the corresponding operations under the integral sign. These results improve state-of-the-art convex calculus rules from [1, 22, 24, 29, 31, 38].

2 Notation and background

2.1 Measure theory

We set $\overline{\mathbb{R}} = [-\infty, +\infty]$. Let $(\Omega, \mathcal{F})$ be a measurable space and let A be a subset of Ω. The characteristic function of A is denoted by 1_A and the complement of A is denoted by $\complement A$. Now let $(\mathsf{X}, \mathcal{T}_{\mathsf{X}})$ be a Hausdorff topological space with Borel σ-algebra $\mathcal{B}_{\mathsf{X}}$. We denote by $\mathcal{L}(\Omega; \mathsf{X})$ the vector space of measurable mappings from $(\Omega, \mathcal{F})$ to $(\mathsf{X}, \mathcal{B}_{\mathsf{X}})$. Given a measure μ on $(\Omega, \mathcal{F})$, $\mathcal{L}^1(\Omega; \mathbb{R})$ is the subset of $\mathcal{L}(\Omega; \mathbb{R})$ of integrable functions, and $\mathcal{L}^1(\Omega; \overline{\mathbb{R}})$ is defined likewise. Given a separable Banach space $(\mathsf{X}, \|\cdot\|_{\mathsf{X}})$, we set $\mathcal{L}^\infty(\Omega; \mathsf{X}) = \big\{x \in \mathcal{L}(\Omega; \mathsf{X}) \mid \sup \|x(\Omega)\|_{\mathsf{X}} < +\infty\big\}$.

2.2 Topological spaces

Given topological spaces $(\mathsf{Y}, \mathcal{T}_{\mathsf{Y}})$ and $(\mathsf{Z}, \mathcal{T}_{\mathsf{Z}})$, $\mathcal{T}_{\mathsf{Y}} \boxtimes \mathcal{T}_{\mathsf{Z}}$ denotes the standard product topology.

Let $(\mathsf{X}, \mathcal{T}_{\mathsf{X}})$ be a Hausdorff topological space. The Borel σ-algebra of $(\mathsf{X}, \mathcal{T}_{\mathsf{X}})$ is denoted by $\mathcal{B}_{\mathsf{X}}$. Furthermore, $(\mathsf{X}, \mathcal{T}_{\mathsf{X}})$ is:

- regular [7, Section I.8.4] if, for every closed subset C of $(\mathsf{X}, \mathcal{T}_{\mathsf{X}})$ and every $\mathsf{x} \in \complement\mathsf{C}$, there exist $\mathsf{V} \in \mathcal{T}_{\mathsf{X}}$ and $\mathsf{W} \in \mathcal{T}_{\mathsf{X}}$ such that $\mathsf{C} \subset \mathsf{V}$, $\mathsf{x} \in \mathsf{W}$, and $\mathsf{V} \cap \mathsf{W} = \varnothing$;
- a Polish space [8, Section IX.6.1] if it is separable and there exists a distance d on X that induces the same topology as $\mathcal{T}_{\mathsf{X}}$ and such that (X, d) is a complete metric space;
- a Souslin space [8, Section IX.6.2] if there exist a Polish space $(\mathsf{Y}, \mathcal{T}_{\mathsf{Y}})$ and a continuous surjective mapping from $(\mathsf{Y}, \mathcal{T}_{\mathsf{Y}})$ to $(\mathsf{X}, \mathcal{T}_{\mathsf{X}})$;
- a Lusin space [8, Section IX.6.4] if there exists a topology $\widetilde{\mathcal{T}_{\mathsf{X}}}$ on X such that $\mathcal{T}_{\mathsf{X}} \subset \widetilde{\mathcal{T}_{\mathsf{X}}}$ and $(\mathsf{X}, \widetilde{\mathcal{T}_{\mathsf{X}}})$ is a Polish space;
- a Fréchet space [9, Section II.4.1] if it is a locally convex real topological vector space and there exists a distance d on X that induces the same topology as $\mathcal{T}_{\mathsf{X}}$ and such that (X, d) is a complete metric space.

Now let $\mathsf{f} \colon \mathsf{X} \to \overline{\mathbb{R}}$. The epigraph of f is

$$\operatorname{epi} \mathsf{f} = \big\{(\mathsf{x}, \xi) \in \mathsf{X} \times \mathbb{R} \mid \mathsf{f}(\mathsf{x}) \leqslant \xi\big\}, \tag{2.1}$$

f is proper if $-\infty \notin f(X) \neq \{+\infty\}$, and f is $\mathcal{T}_X$-lower semicontinuous if epi f is $\mathcal{T}_X \boxtimes \mathcal{T}_{\mathbb{R}}$-closed.

2.3 Duality

The dual of a real topological vector space $(X, \mathcal{T}_X)$, that is, the vector space of continuous linear functionals on $(X, \mathcal{T}_X)$, is denoted by $(X, \mathcal{T}_X)^*$.

Let X and Y be real vector spaces which are in separating duality via a bilinear form $\langle \cdot, \cdot \rangle_{X,Y} \colon X \times Y \to \mathbb{R}$, that is [9, Section II.6.1],

$$\begin{cases} (\forall x \in X) \quad \langle x, \cdot \rangle_{X,Y} = 0 \quad \Rightarrow \quad x = 0 \\ (\forall y \in Y) \quad \langle \cdot, y \rangle_{X,Y} = 0 \quad \Rightarrow \quad y = 0. \end{cases} \tag{2.2}$$

In addition, equip X with a locally convex topology $\mathcal{T}_X$ which is compatible with the pairing $\langle \cdot, \cdot \rangle_{X,Y}$ in the sense that $(X, \mathcal{T}_X)^* = \{\langle \cdot, y \rangle_{X,Y}\}_{y \in Y}$ and, likewise, equip Y with a locally convex topology $\mathcal{T}_Y$ which is compatible with the pairing $\langle \cdot, \cdot \rangle_{X,Y}$ in the sense that $(Y, \mathcal{T}_Y)^* = \{\langle x, \cdot \rangle_{X,Y}\}_{x \in X}$ [9, Section IV.1.1]. Following [20], the Legendre conjugate of $f \colon X \to \overline{\mathbb{R}}$ is

$$f^* \colon Y \to \overline{\mathbb{R}} \colon y \mapsto \sup_{x \in X} \big(\langle x, y \rangle_{X,Y} - f(x)\big) \tag{2.3}$$

and the Legendre conjugate of $g \colon Y \to \overline{\mathbb{R}}$ is

$$g^* \colon X \to \overline{\mathbb{R}} \colon x \mapsto \sup_{y \in Y} \big(\langle x, y \rangle_{X,Y} - g(y)\big). \tag{2.4}$$

Let $f \colon X \to \overline{\mathbb{R}}$. If f is proper, its subdifferential is the set-valued operator

$$\begin{aligned} \partial f \colon X &\to 2^Y \\ x &\mapsto \big\{y \in Y \mid (\forall z \in X)\ \langle z - x, y \rangle_{X,Y} + f(x) \leqslant f(z)\big\} = \big\{y \in Y \mid f(x) + f^*(y) = \langle x, y \rangle_{X,Y}\big\}. \end{aligned} \tag{2.5}$$

In addition, f is convex if epi f is a convex subset of $X \times \mathbb{R}$, and $\Gamma_0(X)$ denotes the class of proper lower semicontinuous convex functions from X to $]-\infty, +\infty]$. Suppose that $f \in \Gamma_0(X)$ and let $z \in \operatorname{dom} f$. The recession function of f is the function in $\Gamma_0(X)$ defined by

$$\operatorname{rec} f \colon X \to \left]-\infty, +\infty\right] \colon x \mapsto \lim_{0 < \alpha \uparrow +\infty} \frac{f(z + \alpha x) - f(z)}{\alpha}. \tag{2.6}$$

Now suppose that, in addition, $X = Y$ is Hilbertian and $\langle \cdot, \cdot \rangle_{X,Y}$ is the scalar product of X, and let $\gamma \in \left]0, +\infty\right[$. The Moreau envelope of f of index γ is the function in $\Gamma_0(X)$ defined by

$$^{\gamma}f \colon X \to \mathbb{R} \colon x \mapsto \min_{y \in X} \left(f(y) + \frac{1}{2\gamma}\|x - y\|_X^2\right) \tag{2.7}$$

and the proximal point of $x \in X$ relative to γf is the unique point $\operatorname{prox}_{\gamma f} x \in X$ such that

$$^{\gamma}f(x) = f(\operatorname{prox}_{\gamma f} x) + \frac{1}{2\gamma}\|x - \operatorname{prox}_{\gamma f} x\|_X^2. \tag{2.8}$$

The proximity operator $\operatorname{prox}_{\gamma f} \colon X \to X$ thus defined can be expressed as

$$\operatorname{prox}_{\gamma f} = (\mathrm{Id} + \gamma \partial f)^{-1}. \tag{2.9}$$

3 Proof of the interchange principle

Proving Theorem 1.2 necessitates a few technical facts.

Lemma 3.1 *Let $(\Omega,\mathcal{F})$ be a measurable space, let n be a strictly positive integer, and let $(\varrho_i)_{0\leqslant i\leqslant n}$ be a family in $\mathcal{L}(\Omega;\mathbb{R})$. Then there exists a family $(B_i)_{0\leqslant i\leqslant n}$ in $\mathcal{F}$ such that*

$$(B_i)_{0\leqslant i\leqslant n} \text{ are pairwise disjoint},\quad \bigcup_{i=0}^{n} B_i=\Omega,\quad\text{and}\quad \min_{0\leqslant i\leqslant n}\varrho_i=\sum_{i=0}^{n}1_{B_i}\varrho_i. \tag{3.1}$$

Proof. We proceed by induction on n. If $n=1$, we obtain (3.1) by choosing $B_0=[\varrho_0\leqslant\varrho_1]$ and $B_1=\complement B_0$. Now assume that the claim is true for n, let $\varrho_{n+1}\in\mathcal{L}(\Omega;\mathbb{R})$, and set

$$\varrho=\min_{0\leqslant i\leqslant n}\varrho_i,\quad D=[\varrho\leqslant\varrho_{n+1}],\quad C_{n+1}=\complement D,\quad\text{and}\quad \big(\forall i\in\{0,\ldots,n\}\big)\;\; C_i=B_i\cap D. \tag{3.2}$$

Then $(C_i)_{0\leqslant i\leqslant n+1}$ is a family of pairwise disjoint sets in $\mathcal{F}$. Additionally,

$$\bigcup_{i=0}^{n+1}C_i=C_{n+1}\cup\bigcup_{i=0}^{n}C_i=(\complement D)\cup\bigcup_{i=0}^{n}(B_i\cap D)=(\complement D)\cup D=\Omega \tag{3.3}$$

and

$$\min_{0\leqslant i\leqslant n+1}\varrho_i=\min\{\varrho,\varrho_{n+1}\}=1_D\varrho+1_{\complement D}\varrho_{n+1}=1_D\sum_{i=0}^{n}1_{B_i}\varrho_i+1_{C_{n+1}}\varrho_{n+1}=\sum_{i=0}^{n+1}1_{C_i}\varrho_i, \tag{3.4}$$

which concludes the induction argument. ☐

Lemma 3.2 *Let $(\Omega,\mathcal{F},\mu)$ be a σ-finite measure space such that $\mu(\Omega)\neq 0$ and let $\mathcal{R}$ be a nonempty subset of $\mathcal{L}(\Omega;\overline{\mathbb{R}})$. Then there exists an element in $\mathcal{L}(\Omega;\overline{\mathbb{R}})$, denoted by $\operatorname{ess\,inf}\mathcal{R}$ and unique up to a set of μ-measure zero, such that*

$$\big(\forall\vartheta\in\mathcal{L}(\Omega;\overline{\mathbb{R}})\big)\quad\big[\,(\forall\varrho\in\mathcal{R})\;\;\vartheta\leqslant\varrho\;\;\mu\text{-a.e.}\,\big]\quad\Leftrightarrow\quad\vartheta\leqslant\operatorname{ess\,inf}\mathcal{R}\;\;\mu\text{-a.e.} \tag{3.5}$$

Moreover, there exists a sequence $(\varrho_n)_{n\in\mathbb{N}}$ in $\mathcal{R}$ such that $\operatorname{ess\,inf}\mathcal{R}=\inf_{n\in\mathbb{N}}\varrho_n$.

Proof. Using Assumption 1.1[D], construct $0<\chi\in\mathcal{L}^1(\Omega;\mathbb{R})$ such that $\int_\Omega\chi d\mu=1$ and define $\mathsf{P}\colon\mathcal{F}\to[0,1]\colon A\mapsto\int_A\chi d\mu$. Then $(\forall A\in\mathcal{F})\;\mu(A)=0\Leftrightarrow\mathsf{P}(A)=0$. Hence, the assertions follow from [21, Proposition II-4-1 and its proof] applied in the probability space $(\Omega,\mathcal{F},\mathsf{P})$. ☐

Lemma 3.3 *Let $(\Omega,\mathcal{F},\mu)$ be a measure space, let $(\mathsf{X},\mathcal{T}_{\mathsf{X}})$ be a Souslin space, let $z\colon(\Omega,\mathcal{F})\to(\mathsf{X},\mathcal{B}_{\mathsf{X}})$ be measurable, and let $E\in\mathcal{F}$ be such that $\mu(E)<+\infty$. Then there exists a sequence $(E_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that*

$$\big[\,(\forall n\in\mathbb{N})\;\;E_n\subset E\text{ and }\overline{z(E_n)}\text{ is compact}\,\big]\quad\text{and}\quad\mu(E)=\mu\bigg(\bigcup_{n\in\mathbb{N}}E_n\bigg). \tag{3.6}$$

Proof. A simple adaptation of the proof of [38, Lemma 5], where $(\mathsf{X},\mathcal{T}_{\mathsf{X}})$ is a locally convex Souslin topological vector space. ☐

Lemma 3.4 *Suppose that Assumption 1.1[A]–[D] hold. Let $\psi\colon (\Omega\times\mathsf{X},\mathcal{F}\otimes\mathcal{B}_{\mathsf{X}})\to\overline{\mathbb{R}}$ be measurable, let $\mathcal{Z}$ be a nonempty at most countable subset of $\mathcal{L}(\Omega;\mathsf{X})$, and let $(\Omega_k)_{k\in\mathbb{N}}$ be an increasing sequence of finite μ-measure sets in $\mathcal{F}$ such that $\bigcup_{k\in\mathbb{N}}\Omega_k=\Omega$. Define*

$$\mathcal{D}=\bigcup_{z\in\mathcal{Z}}\bigcup_{k\in\mathbb{N}}\big\{1_A z \mid \mathcal{F}\ni A\subset\Omega_k \text{ and } \overline{z(A)} \text{ is compact}\big\} \tag{3.7}$$

and

$$\mathcal{R}=\big\{\varrho\in\mathcal{L}^1(\Omega;\mathbb{R}) \mid (\exists\, x\in\mathcal{D})\ \psi\big(\cdot,x(\cdot)\big)\leqslant\varrho(\cdot)\ \mu\text{-a.e.}\big\}. \tag{3.8}$$

Suppose that

$$\psi(\cdot,0)\leqslant 0. \tag{3.9}$$

Then $\mathcal{R}\neq\varnothing$ and $\operatorname{ess\,inf}\mathcal{R}\leqslant\inf_{z\in\mathcal{Z}}\psi(\cdot,z(\cdot))$ μ-a.e.

Proof. Take $z\in\mathcal{Z}$ and note that $(\forall A\in\mathcal{F})\ 1_A z\in\mathcal{L}(\Omega;\mathsf{X})$. Since $\overline{z(\varnothing)}=\varnothing$ is compact, it results from (3.7) that $0=1_\varnothing z\in\mathcal{D}$. Hence, by (3.9), $0\in\mathcal{R}$. Next, thanks to Assumption 1.1[D], there exists $\chi\in\mathcal{L}^1(\Omega;\mathbb{R})$ such that $\chi>0$. Let us set

$$(\forall n\in\mathbb{N})\quad A_n=\Omega_n\cap\big[\psi\big(\cdot,z(\cdot)\big)\leqslant n\chi(\cdot)\big]. \tag{3.10}$$

Lemma 3.3 asserts that there exists a family $(A_{n,k})_{(n,k)\in\mathbb{N}^2}$ in $\mathcal{F}$ such that

$$(\forall n\in\mathbb{N})\quad \begin{cases}(\forall k\in\mathbb{N})\ \ A_{n,k}\subset A_n \text{ and } \overline{z(A_{n,k})} \text{ is compact}\\ \mu(A_n)=\mu\Big(\displaystyle\bigcup_{k\in\mathbb{N}}A_{n,k}\Big).\end{cases} \tag{3.11}$$

In turn, by (3.7) and (3.10),

$$(\forall n\in\mathbb{N})(\forall k\in\mathbb{N})\quad 1_{A_{n,k}}z\in\mathcal{D}. \tag{3.12}$$

Define

$$(\forall n\in\mathbb{N})(\forall k\in\mathbb{N})(\forall m\in\mathbb{N})\quad \varrho_{n,k,m}(\cdot)=\max\big\{\psi\big(\cdot,1_{A_{n,k}}(\cdot)z(\cdot)\big),-m\chi(\cdot)\big\}. \tag{3.13}$$

Fix temporarily $(n,k,m)\in\mathbb{N}^3$. We infer from (3.11), (3.10), and (3.9) that

$$\begin{aligned}(\forall\omega\in\Omega)\quad \psi\big(\omega,1_{A_{n,k}}(\omega)z(\omega)\big)&=\begin{cases}\psi\big(\omega,z(\omega)\big), & \text{if } \omega\in A_{n,k};\\ \psi(\omega,0), & \text{otherwise}\end{cases}\\ &\leqslant\begin{cases}n\chi(\omega), & \text{if } \omega\in A_{n,k};\\ 0, & \text{otherwise}\end{cases}\\ &\leqslant n\chi(\omega).\end{aligned} \tag{3.14}$$

Therefore, $-m\chi\leqslant\varrho_{n,k,m}\leqslant n\chi$, which entails that $\varrho_{n,k,m}\in\mathcal{L}^1(\Omega;\mathbb{R})$. In turn, we derive from (3.13), (3.12), and (3.8) that $\varrho_{n,k,m}\in\mathcal{R}$. Thus, Lemma 3.2 guarantees that there exists $B_{n,k,m}\in\mathcal{F}$ such that $\mu(B_{n,k,m})=0$ and

$$\big(\forall\omega\in\complement B_{n,k,m}\big)\quad (\operatorname{ess\,inf}\mathcal{R})(\omega)\leqslant\varrho_{n,k,m}(\omega). \tag{3.15}$$

Now set

$$A = \bigcap_{(n,k)\in\mathbb{N}^2} \complement A_{n,k}, \quad B = \bigcup_{(n,k,m)\in\mathbb{N}^3} B_{n,k,m}, \quad \text{and} \quad C = \big[\psi(\cdot\,,z(\cdot)) < +\infty\big] \cap (A \cup B). \tag{3.16}$$

Then $\mu(B) = 0$. Furthermore, since (3.10) yields $[\psi(\cdot\,,z(\cdot)) < +\infty] = \bigcup_{n\in\mathbb{N}} A_n$, it follows from (3.16) and (3.11) that

$$\mu\Big(\big[\psi(\cdot\,,z(\cdot)) < +\infty\big] \cap A\Big) \leqslant \sum_{n\in\mathbb{N}} \mu(A_n \cap A) \leqslant \sum_{n\in\mathbb{N}} \mu\bigg(A_n \cap \bigcap_{k\in\mathbb{N}} \complement A_{n,k}\bigg) = 0. \tag{3.17}$$

Hence, using (3.16), we obtain

$$\mu(C) = 0 \quad \text{and} \quad \complement C = \big[\psi(\cdot\,,z(\cdot)) = +\infty\big] \cup \big(\complement A \cap \complement B\big). \tag{3.18}$$

Now suppose that $\omega \in \complement A \cap \complement B$. Then it follows from (3.16) that there exists $(n,k) \in \mathbb{N}^2$ such that $\omega \in A_{n,k} \cap \complement B$. Therefore, we derive from (3.16), (3.15), and (3.13) that

$$(\forall m \in \mathbb{N}) \quad (\operatorname{ess\,inf} \mathcal{R})(\omega) \leqslant \varrho_{n,k,m}(\omega) = \max\big\{\psi\big(\omega, 1_{A_{n,k}}(\omega)z(\omega)\big), -m\chi(\omega)\big\}. \tag{3.19}$$

Hence, letting $m \uparrow +\infty$ yields $(\operatorname{ess\,inf} \mathcal{R})(\omega) \leqslant \psi(\omega, 1_{A_{n,k}}(\omega)z(\omega)) = \psi(\omega, z(\omega))$. We have thus shown that $\operatorname{ess\,inf} \mathcal{R} \leqslant \psi(\cdot\,,z(\cdot))$ μ-a.e. Since $\mathcal{Z}$ is at most countable, the proof is complete. ☐

Proof of Theorem 1.2. Define

$$\Phi\colon \mathcal{L}(\Omega;\mathsf{X}) \to \mathcal{L}(\Omega;\overline{\mathbb{R}})\colon x \mapsto \varphi\big(\cdot\,,x(\cdot)\big) \tag{3.20}$$

and note that, thanks to Assumption 1.1[G],

$$\int_\Omega \inf \varphi(\cdot\,,\mathsf{X})\, d\mu \leqslant \inf_{x\in\mathcal{X}} \int_\Omega \Phi(x) d\mu \leqslant \int_\Omega \Phi(\overline{x}) d\mu < +\infty. \tag{3.21}$$

Hence, the interchange rule (1.2) holds when $\inf_{x\in\mathcal{X}} \int_\Omega \Phi(x) d\mu = -\infty$ and we assume henceforth that

$$\inf_{x\in\mathcal{X}} \int_\Omega \Phi(x) d\mu \in \mathbb{R}. \tag{3.22}$$

Now define

$$\vartheta = \max\big\{\Phi(\overline{x}), 0\big\} \tag{3.23}$$

and

$$\psi\colon \Omega \times \mathsf{X} \to \overline{\mathbb{R}}\colon (\omega,\mathsf{x}) \mapsto \begin{cases} \varphi\big(\omega, \mathsf{x} + \overline{x}(\omega)\big) - \vartheta(\omega), & \text{if } \vartheta(\omega) < +\infty;\\ -\infty, & \text{if } \vartheta(\omega) = +\infty. \end{cases} \tag{3.24}$$

Then we derive from Assumption 1.1[G] that

$$\vartheta \in \mathcal{L}^1(\Omega;\overline{\mathbb{R}}) \tag{3.25}$$

and, therefore, that

$$\mu\big([\vartheta = +\infty]\big) = 0. \tag{3.26}$$

On the other hand, Assumption 1.1[B] ensures that the mapping $(\Omega \times \mathsf{X}, \mathcal{F} \otimes \mathcal{B}_{\mathsf{X}}) \to (\mathsf{X}, \mathcal{B}_{\mathsf{X}}) \colon (\omega, \mathsf{x}) \mapsto \mathsf{x} + \overline{x}(\omega)$ is measurable. Thus, it follows from Assumption 1.1[F], (3.25), and (3.24) that

$$\psi \text{ is } \mathcal{F} \otimes \mathcal{B}_{\mathsf{X}}\text{-measurable.} \tag{3.27}$$

At the same time, since

$$\inf_{\mathsf{x} \in \mathsf{X}} \psi(\cdot, \mathsf{x}) = \inf_{\mathsf{x} \in \mathsf{X}} \varphi\big(\cdot, \mathsf{x} + \overline{x}(\cdot)\big) - \vartheta(\cdot) = \inf_{\mathsf{x} \in \mathsf{X}} \varphi(\cdot, \mathsf{x}) - \vartheta(\cdot) \tag{3.28}$$

and since Assumption 1.1[F] yields $\inf \varphi(\cdot, \mathsf{X}) < +\infty$, it results from (i) that

$$\inf \psi(\cdot, \mathsf{X}) \in \mathcal{L}(\Omega; \overline{\mathbb{R}}). \tag{3.29}$$

Let us set

$$\Psi \colon \mathcal{L}(\Omega; \mathsf{X}) \to \mathcal{L}(\Omega; \overline{\mathbb{R}}) \colon x \mapsto \psi\big(\cdot, x(\cdot)\big). \tag{3.30}$$

By (3.24) and (3.26),

$$\big(\forall \omega \in \complement[\vartheta = +\infty]\big)(\forall x \in \mathcal{X}) \quad \big(\Psi(x)\big)(\omega) = \big(\Phi(x + \overline{x})\big)(\omega) - \vartheta(\omega). \tag{3.31}$$

Hence, upon invoking (3.25), we deduce from Assumption 1.1[E]&[G] that

$$\begin{aligned} \inf_{x \in \mathcal{X}} \int_\Omega \Psi(x) d\mu &= \inf_{x \in \mathcal{X}} \int_\Omega \big(\Phi(x + \overline{x}) - \vartheta\big) d\mu \\ &= \inf_{x \in \mathcal{X}} \int_\Omega \Phi(x + \overline{x}) d\mu - \int_\Omega \vartheta d\mu \\ &= \inf_{x \in \mathcal{X}} \int_\Omega \Phi(x) d\mu - \int_\Omega \vartheta d\mu \end{aligned} \tag{3.32}$$

and, likewise, from (3.28) that

$$\int_\Omega \inf \psi(\cdot, \mathsf{X})\, d\mu = \int_\Omega \inf \varphi(\cdot, \mathsf{X})\, d\mu - \int_\Omega \vartheta d\mu. \tag{3.33}$$

Now set

$$\mathcal{D} = \bigcup_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \big\{ 1_A x_n \mid \mathcal{F} \ni A \subset \Omega_k \text{ and } \overline{x_n(A)} \text{ is compact} \big\} \tag{3.34}$$

and

$$\mathcal{R} = \big\{ \varrho \in \mathcal{L}^1(\Omega; \mathbb{R}) \mid (\exists\, x \in \mathcal{D})\ \Psi(x) \leqslant \varrho \ \mu\text{-a.e.} \big\}, \tag{3.35}$$

and note that (ii)(b) states that

$$\mathcal{D} \subset \mathcal{X}. \tag{3.36}$$

Using (3.24) and (3.23), we infer from Lemma 3.4 applied to $\mathcal{Z} = \{x_n\}_{n \in \mathbb{N}}$ that $\operatorname{ess\,inf} \mathcal{R} \leqslant \inf_{n \in \mathbb{N}} \Psi(x_n)$ μ-a.e. In turn, we derive from (3.31), (ii)(a), and (3.28) that

$$\operatorname{ess\,inf} \mathcal{R} \leqslant \inf_{n \in \mathbb{N}} \Psi(x_n) = \inf_{n \in \mathbb{N}} \Phi(x_n + \overline{x}) - \vartheta = \inf \varphi(\cdot, \mathsf{X}) - \vartheta = \inf \psi(\cdot, \mathsf{X}) \ \mu\text{-a.e.} \tag{3.37}$$

On the other hand, (3.35) implies that $(\forall \varrho \in \mathcal{R})\ \inf \psi(\cdot,\mathsf{X}) \leqslant \varrho(\cdot)$ μ-a.e. Hence, (3.29) and Lemma 3.2 guarantee that $\inf \psi(\cdot,\mathsf{X}) \leqslant \operatorname{ess\,inf} \mathcal{R}$ μ-a.e. Altogether, $\operatorname{ess\,inf} \mathcal{R} = \inf \psi(\cdot,\mathsf{X})$ μ-a.e. Thus, we deduce from Lemma 3.2 that there exists a sequence $(\varrho_n)_{n\in\mathbb{N}}$ in $\mathcal{R}$ such that

$$\inf_{n\in\mathbb{N}} \varrho_n(\cdot) = \inf \psi(\cdot,\mathsf{X}) \ \mu\text{-a.e.} \tag{3.38}$$

For every $n \in \mathbb{N}$, it follows from (3.35) and (3.34) that there exist $\ell_n \in \mathbb{N}$, $k_n \in \mathbb{N}$, and $\mathcal{F} \ni A_n \subset \Omega_{k_n}$ such that

$$\overline{x_{\ell_n}(A_n)} \text{ is compact} \quad \text{and} \quad \Psi\big(1_{A_n} x_{\ell_n}\big) \leqslant \varrho_n \ \mu\text{-a.e.} \tag{3.39}$$

Let us set

$$(\forall n \in \mathbb{N}) \quad \chi_n = \min_{0\leqslant i\leqslant n} \varrho_i. \tag{3.40}$$

Fix temporarily $n \in \mathbb{N}$. Lemma 3.1 asserts that there exists a family $(B_{n,i})_{0\leqslant i\leqslant n}$ in $\mathcal{F}$ such that

$$(B_{n,i})_{0\leqslant i\leqslant n} \text{ are pairwise disjoint}, \quad \bigcup_{i=0}^{n} B_{n,i} = \Omega, \quad \text{and} \quad \chi_n = \sum_{i=0}^{n} 1_{B_{n,i}} \varrho_i. \tag{3.41}$$

Now set

$$y_n = \sum_{i=0}^{n} 1_{A_i \cap B_{n,i}} x_{\ell_i}. \tag{3.42}$$

For every $i \in \{0,\ldots,n\}$, since $A_i \cap B_{n,i} \subset A_i \subset \Omega_{k_i}$, (3.39) implies that $\overline{x_{\ell_i}(A_i \cap B_{n,i})}$ is compact and, therefore, (3.34) and (3.36) yield $1_{A_i\cap B_{n,i}} x_{\ell_i} \in \mathcal{D} \subset \mathcal{X}$. Consequently, (3.42) and Assumption 1.1[E] ensure that $y_n \in \mathcal{X}$. At the same time, we derive from (3.42), (3.41), and (3.39) that

$$\Psi(y_n) = \sum_{i=0}^{n} 1_{B_{n,i}} \Psi\big(1_{A_i} x_{\ell_i}\big) \leqslant \sum_{i=0}^{n} 1_{B_{n,i}} \varrho_i = \chi_n \ \mu\text{-a.e.} \tag{3.43}$$

Therefore, since $y_n \in \mathcal{X}$,

$$\inf_{x\in\mathcal{X}} \int_\Omega \Psi(x) d\mu \leqslant \int_\Omega \Psi(y_n) d\mu \leqslant \int_\Omega \chi_n d\mu. \tag{3.44}$$

On the other hand, it results from (3.32), (3.22), and (3.25) that $\inf_{x\in\mathcal{X}} \int_\Omega \Psi(x) d\mu \in \mathbb{R}$. Thus, since $\chi_n \downarrow \inf_{i\in\mathbb{N}} \varrho_i(\cdot) = \inf \psi(\cdot,\mathsf{X})$ μ-a.e. by virtue of (3.40) and (3.38), (3.44) and the monotone convergence theorem [4, Theorem 2.8.2 and Corollary 2.8.6] entail that

$$\inf_{x\in\mathcal{X}} \int_\Omega \Psi(x) d\mu \leqslant \lim \int_\Omega \chi_n d\mu = \int_\Omega \lim \chi_n \, d\mu = \int_\Omega \inf \psi(\cdot,\mathsf{X})\, d\mu. \tag{3.45}$$

Consequently, since $\int_\Omega \inf \psi(\cdot,\mathsf{X})\, d\mu \leqslant \inf_{x\in\mathcal{X}} \int_\Omega \Psi(x) d\mu$, we conclude that

$$\inf_{x\in\mathcal{X}} \int_\Omega \Psi(x) d\mu = \int_\Omega \inf \psi(\cdot,\mathsf{X})\, d\mu. \tag{3.46}$$

In view of (3.32), (3.33), and (3.25), the proof is complete. ☐

Remark 3.5 Replacing φ by $-\varphi$ in items [F] and [G] of Assumption 1.1 and in Theorem 1.2 provides conditions under which

$$\sup_{x\in\mathcal{X}} \int_\Omega \varphi\big(\omega, x(\omega)\big)\mu(d\omega) = \int_\Omega \sup_{\mathsf{x}\in\mathsf{X}} \varphi(\omega, \mathsf{x})\, \mu(d\omega), \tag{3.47}$$

with the convention that, given a measurable function $\varrho\colon (\Omega, \mathcal{F}) \to \overline{\mathbb{R}}$, $\int_\Omega \varrho d\mu$ is the usual Lebesgue integral, except when the Lebesgue integral $\int_\Omega \min\{\varrho, 0\}d\mu$ is $-\infty$, in which case $\int_\Omega \varrho d\mu = -\infty$.

Remark 3.6 In Theorem 1.2, suppose that $\inf_{x\in\mathcal{X}} \int_\Omega \varphi(\cdot, x(\cdot))d\mu > -\infty$ and let $z \in \mathcal{X}$. Then

$$\int_\Omega \varphi\big(\omega, z(\omega)\big)\mu(d\omega) = \min_{x\in\mathcal{X}} \int_\Omega \varphi\big(\omega, x(\omega)\big)\mu(d\omega) \quad \Leftrightarrow \quad \varphi\big(\cdot, z(\cdot)\big) = \min \varphi(\cdot, \mathsf{X})\ \mu\text{-a.e.} \tag{3.48}$$

4 Compliant spaces and normal integrands

The objective of this section is to develop tools to convert the interchange principle of Theorem 1.2 into interchange rules formulated in terms of explicit conditions on the ambient space $\mathcal{X}$ and the integrand φ. Our framework hinges on a notion of compliant spaces and a notion of normal integrands in an extended sense.

4.1 Compliant spaces

We introduce the following notion of a compliant space, which generalizes and unifies the notions of decomposability employed in the interchange rules of [24, 25, 29, 31, 32, 37, 38].

Definition 4.1 (compliance) Suppose that Assumption 1.1[A]–[E] holds. Then $\mathcal{X}$ is *compliant* if, for every $A \in \mathcal{F}$ such that $\mu(A) < +\infty$ and every $z \in \mathcal{L}(\Omega; \mathsf{X})$ such that $\overline{z(A)}$ is compact, $1_A z \in \mathcal{X}$.

Proposition 4.2 *Suppose that Assumption 1.1[A]–[E] holds, together with one of the following:*

(i) $(\mathsf{X}, \mathcal{T}_\mathsf{X})$ *is a Souslin topological vector space and, for every $A \in \mathcal{F}$ such that $\mu(A) < +\infty$ and every $z \in \mathcal{L}(\Omega; \mathsf{X})$ such that $z(A)$ is $\mathcal{T}_\mathsf{X}$-bounded (in the sense that, for every neighborhood $\mathsf{V} \in \mathcal{T}_\mathsf{X}$ of 0, there exists $\alpha \in \,]0, +\infty[$ such that $z(A) \subset \bigcap_{\beta>\alpha} \beta\mathsf{V}$ [33]), $1_A z \in \mathcal{X}$.*

(ii) X *is a separable Banach space with strong topology $\mathcal{T}_\mathsf{X}$ and, for every $A \in \mathcal{F}$ such that $\mu(A) < +\infty$ and every $z \in \mathcal{L}^\infty(\Omega; \mathsf{X})$, $1_A z \in \mathcal{X}$.*

(iii) X *is a separable Banach space with strong topology $\mathcal{T}_\mathsf{X}$, $\mu(\Omega) < +\infty$, and $\mathcal{L}^\infty(\Omega; \mathsf{X}) \subset \mathcal{X}$.*

(iv) X *is a separable Banach space with strong topology $\mathcal{T}_\mathsf{X}$ and $\mathcal{X}$ is* Rockafellar-decomposable *[29] in the sense that, for every $A \in \mathcal{F}$ such that $\mu(A) < +\infty$, every $z \in \mathcal{L}^\infty(\Omega; \mathsf{X})$, and every $x \in \mathcal{X}$, $1_A z + 1_{\complement A} x \in \mathcal{X}$.*

(v) $(\mathsf{X}, \mathcal{T}_\mathsf{X})$ *is a Souslin locally convex topological vector space and $\mathcal{X}$ is* Valadier-decomposable *[38] in the sense that, for every $A \in \mathcal{F}$ such that $\mu(A) < +\infty$, every $z \in \mathcal{L}(\Omega; \mathsf{X})$ such that $\overline{z(A)}$ is compact, and every $x \in \mathcal{X}$, $1_A z + 1_{\complement A} x \in \mathcal{X}$.*

(vi) X *is the standard Euclidean space $\mathbb{R}^N$ and, for every $A \in \mathcal{F}$ such that $\mu(A) < +\infty$ and every $z \in \mathcal{L}^\infty(\Omega; \mathsf{X})$, $1_A z \in \mathcal{X}$.*

Then $\mathcal{X}$ is compliant.

Proof. (i): Let $A \in \mathcal{F}$ be such that $\mu(A) < +\infty$ and let $z \in \mathcal{L}(\Omega; \mathsf{X})$ be such that $\overline{z(A)}$ is compact. It results from [33, Theorem 1.15(b)] that $z(A)$ is $\mathcal{T}_{\mathsf{X}}$-bounded. Thus $1_A z \in \mathcal{X}$.

(iii)⇒(ii)⇒(i): Clear.

(iv)⇒(ii): Clear.

(v): Clear.

(vi)⇒(ii): Clear. ☐

4.2 Normal integrands

We introduce a notion of a normal integrand which unifies and extends those of [28, 29, 31, 38].

Definition 4.3 (normality) Let $(\mathsf{X}, \mathcal{T}_{\mathsf{X}})$ be a Souslin space, let $(\Omega, \mathcal{F})$ be a measurable space, let $\varphi\colon (\Omega \times \mathsf{X}, \mathcal{F} \otimes \mathcal{B}_{\mathsf{X}}) \to \overline{\mathbb{R}}$ be measurable, and equip $\mathsf{X} \times \mathbb{R}$ with the topology $\mathcal{T}_{\mathsf{X}} \boxtimes \mathcal{T}_{\mathbb{R}}$. Then φ is a *normal integrand* if there exist sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{L}(\Omega; \mathsf{X})$ and $(\varrho_n)_{n\in\mathbb{N}}$ in $\mathcal{L}(\Omega; \mathbb{R})$ such that

$$(\forall \omega \in \Omega) \quad \big\{\big(x_n(\omega), \varrho_n(\omega)\big)\big\}_{n\in\mathbb{N}} \subset \operatorname{epi} \varphi_\omega \quad \text{and} \quad \overline{\operatorname{epi} \varphi_\omega} = \overline{\big\{\big(x_n(\omega), \varrho_n(\omega)\big)\big\}_{n\in\mathbb{N}}}. \tag{4.1}$$

The following theorem furnishes examples of normal integrands.

Theorem 4.4 *Let $(\mathsf{X}, \mathcal{T}_{\mathsf{X}})$ be a Souslin space, let $(\Omega, \mathcal{F})$ be a measurable space, and let $\varphi\colon \Omega \times \mathsf{X} \to \overline{\mathbb{R}}$ be such that $(\forall \omega \in \Omega)$ $\operatorname{epi} \varphi_\omega \neq \varnothing$. Suppose that one of the following holds:*

(i) *φ is $\mathcal{F} \otimes \mathcal{B}_{\mathsf{X}}$-measurable and one of the following is satisfied:*

  (a) *There exists a measure μ such that $(\Omega, \mathcal{F}, \mu)$ is complete and σ-finite.*

  (b) *Ω is a Borel subset of $\mathbb{R}^M$ and $\mathcal{F}$ is the associated Lebesgue σ-algebra.*

  (c) *For every $\omega \in \Omega$, there exists $\mathbf{V}_\omega \in \mathcal{T}_{\mathsf{X}} \boxtimes \mathcal{T}_{\mathbb{R}}$ such that $\mathbf{V}_\omega \subset \operatorname{epi} \varphi_\omega$ and $\overline{\mathbf{V}_\omega} = \overline{\operatorname{epi} \varphi_\omega}$.*

  (d) *The functions $(\varphi_\omega)_{\omega\in\Omega}$ are upper semicontinuous.*

(ii) *The functions $(\varphi(\cdot, \mathsf{x}))_{\mathsf{x}\in\mathsf{X}}$ are $\mathcal{F}$-measurable and one of the following is satisfied:*

  (a) *$(\mathsf{X}, \mathcal{T}_{\mathsf{X}})$ is metrizable and, for every $\omega \in \Omega$, there exists $\mathbf{V}_\omega \in \mathcal{T}_{\mathsf{X}} \boxtimes \mathcal{T}_{\mathbb{R}}$ such that $\mathbf{V}_\omega \subset \operatorname{epi} \varphi_\omega = \overline{\mathbf{V}_\omega}$.*

  (b) *$(\mathsf{X}, \mathcal{T}_{\mathsf{X}})$ is a Fréchet space and, for every $\omega \in \Omega$, $\varphi_\omega \in \Gamma_0(\mathsf{X})$ and $\operatorname{int} \operatorname{dom} \varphi_\omega \neq \varnothing$.*

  (c) *$(\mathsf{X}, \mathcal{T}_{\mathsf{X}})$ is the standard Euclidean line $\mathbb{R}$ and, for every $\omega \in \Omega$, $\varphi_\omega \in \Gamma_0(\mathbb{R})$ and $\operatorname{dom} \varphi_\omega$ is not a singleton.*

(iii) *$(\mathsf{X}, \mathcal{T}_{\mathsf{X}})$ is a regular Souslin space, the functions $(\varphi_\omega)_{\omega\in\Omega}$ are continuous, and the functions $(\varphi(\cdot, \mathsf{x}))_{\mathsf{x}\in\mathsf{X}}$ are $\mathcal{F}$-measurable.*

(iv) *For some separable Fréchet space $(\mathsf{Y}, \mathcal{T}_{\mathsf{Y}})$, $\mathsf{X} = (\mathsf{Y}, \mathcal{T}_{\mathsf{Y}})^*$, $\mathcal{T}_{\mathsf{X}}$ is the weak topology, the functions $(\varphi_\omega)_{\omega\in\Omega}$ are $\mathcal{T}_{\mathsf{X}}$-lower semicontinuous, and one of the following is satisfied:*

  (a) *For every closed subset $\mathbf{C}$ of $(\mathsf{X} \times \mathbb{R}, \mathcal{T}_{\mathsf{X}} \boxtimes \mathcal{T}_{\mathbb{R}})$, $\big\{\omega \in \Omega \mid \mathbf{C} \cap \operatorname{epi} \varphi_\omega \neq \varnothing\big\} \in \mathcal{F}$.*

  (b) *$(\Omega, \mathcal{T}_\Omega)$ is a Hausdorff topological space, $\mathcal{F} = \mathcal{B}_\Omega$, and φ is $\mathcal{T}_\Omega \boxtimes \mathcal{T}_{\mathsf{X}}$-lower semicontinuous.*

  (c) *$(\Omega, \mathcal{T}_\Omega)$ is a Lusin space, $\mathcal{F} = \mathcal{B}_\Omega$, and φ is $\mathcal{F} \otimes \mathcal{B}_{\mathsf{X}}$-measurable.*

(v) *X is a separable reflexive Banach space, $\mathcal{T}_{\mathsf{X}}$ is the weak topology, $(\Omega, \mathcal{T}_\Omega)$ is a Hausdorff topological space, $\mathcal{F} = \mathcal{B}_\Omega$, the functions $(\varphi_\omega)_{\omega\in\Omega}$ are $\mathcal{T}_{\mathsf{X}}$-lower semicontinuous, and one of the following is satisfied:*

  (a) *φ is $\mathcal{T}_\Omega \boxtimes \mathcal{T}_{\mathsf{X}}$-lower semicontinuous.*

(b) $(\Omega, \mathcal{T}_\Omega)$ *is a Lusin space and* φ *is* $\mathcal{F} \otimes \mathcal{B}_{\mathsf{X}}$*-measurable.*

(vi) $(\mathsf{X}, \mathcal{T}_{\mathsf{X}})$ *is the standard Euclidean space* $\mathbb{R}^N$*,* Ω *is a Borel subset of* $\mathbb{R}^M$*,* $\mathcal{F} = \mathcal{B}_\Omega$*,* φ *is* $\mathcal{F} \otimes \mathcal{B}_{\mathsf{X}}$*-measurable, and the functions* $(\varphi_\omega)_{\omega \in \Omega}$ *are lower semicontinuous.*

(vii) $(\mathsf{X}, \mathcal{T}_{\mathsf{X}})$ *is a Polish space, the functions* $(\varphi_\omega)_{\omega \in \Omega}$ *are lower semicontinuous, and one of the following is satisfied:*

(a) *For every* $\mathbf{V} \in \mathcal{T}_{\mathsf{X}} \boxtimes \mathcal{T}_{\mathbb{R}}$*,* $\big\{\omega \in \Omega \mid \mathbf{V} \cap \operatorname{epi} \varphi_\omega \neq \varnothing\big\} \in \mathcal{F}$*.*

(b) $(\mathsf{X}, \mathcal{T}_{\mathsf{X}})$ *is the standard Euclidean space* $\mathbb{R}^N$ *and, for every closed subset* $\mathbf{C}$ *of* $\mathsf{X} \times \mathbb{R}$*,* $\big\{\omega \in \Omega \mid \mathbf{C} \cap \operatorname{epi} \varphi_\omega \neq \varnothing\big\} \in \mathcal{F}$*.*

(viii) *There exists a measurable function* $\mathsf{f} \colon (\mathsf{X}, \mathcal{B}_{\mathsf{X}}) \to \overline{\mathbb{R}}$ *such that* $(\forall \omega \in \Omega)\ \varphi_\omega = \mathsf{f}$*.*

Then φ *is normal.*

Proof. Set $\boldsymbol{G} = \big\{(\omega, \mathsf{x}, \xi) \in \Omega \times \mathsf{X} \times \mathbb{R} \mid \varphi(\omega, \mathsf{x}) \leqslant \xi\big\}$. Then

$$\boldsymbol{G} = \big\{(\omega, \mathsf{x}, \xi) \in \Omega \times \mathsf{X} \times \mathbb{R} \mid (\mathsf{x}, \xi) \in \operatorname{epi} \varphi_\omega\big\}. \tag{4.2}$$

Further, [4, Lemma 6.4.2(i)] yields

$$\varphi \text{ is } \mathcal{F} \otimes \mathcal{B}_{\mathsf{X}}\text{-measurable} \quad \Leftrightarrow \quad \boldsymbol{G} \in \mathcal{F} \otimes \mathcal{B}_{\mathsf{X}} \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{F} \otimes \mathcal{B}_{\mathsf{X} \times \mathbb{R}}. \tag{4.3}$$

We also note that $(\mathsf{X} \times \mathbb{R}, \mathcal{T}_{\mathsf{X}} \boxtimes \mathcal{T}_{\mathbb{R}})$ is a Souslin space [8, Proposition IX.6.7].

(i)(a): Applying [11, Theorem III.22] to the mapping $\Upsilon \colon \Omega \to 2^{\mathsf{X} \times \mathbb{R}} \colon \omega \mapsto \operatorname{epi} \varphi_\omega$, we deduce from (4.2) and (4.3) that there exist a sequence $(x_n)_{n \in \mathbb{N}}$ of mappings from Ω to X and a sequence $(\varrho_n)_{n \in \mathbb{N}}$ of functions from Ω to $\mathbb{R}$ such that

$$(\forall n \in \mathbb{N}) \quad (\Omega, \mathcal{F}) \to (\mathsf{X} \times \mathbb{R}, \mathcal{B}_{\mathsf{X} \times \mathbb{R}}) \colon \omega \mapsto \big(x_n(\omega), \varrho_n(\omega)\big) \text{ is measurable} \tag{4.4}$$

and

$$(\forall \omega \in \Omega) \quad \big\{\big(x_n(\omega), \varrho_n(\omega)\big)\big\}_{n \in \mathbb{N}} \subset \Upsilon(\omega) \quad \text{and} \quad \overline{\Upsilon(\omega)} = \overline{\big\{\big(x_n(\omega), \varrho_n(\omega)\big)\big\}_{n \in \mathbb{N}}}. \tag{4.5}$$

Moreover, since $\mathcal{B}_{\mathsf{X} \times \mathbb{R}} = \mathcal{B}_{\mathsf{X}} \otimes \mathcal{B}_{\mathbb{R}}$ [4, Lemma 6.4.2(i)], it follows from (4.4) that, for every $n \in \mathbb{N}$, $x_n \colon (\Omega, \mathcal{F}) \to (\mathsf{X}, \mathcal{B}_{\mathsf{X}})$ and $\varrho_n \colon (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ are measurable. Altogether, φ is normal.

(i)(b)$\Rightarrow$(i)(a): Take μ to be the Lebesgue measure on Ω.

(i)(c): Let $\{(\mathsf{x}_n, \xi_n)\}_{n \in \mathbb{N}}$ be a dense set in $(\mathsf{X} \times \mathbb{R}, \mathcal{T}_{\mathsf{X}} \boxtimes \mathcal{T}_{\mathbb{R}})$ and define

$$(\forall n \in \mathbb{N}) \quad \Omega_n = \big[\varphi(\cdot, \mathsf{x}_n) \leqslant \xi_n\big]. \tag{4.6}$$

On the one hand, the $\mathcal{F} \otimes \mathcal{B}_{\mathsf{X}}$-measurability of φ ensures that $(\forall n \in \mathbb{N})\ \Omega_n \in \mathcal{F}$. On the other hand, for every $\omega \in \Omega$, since $\mathbf{V}_\omega$ is open, there exists $n \in \mathbb{N}$ such that $(\mathsf{x}_n, \xi_n) \in \mathbf{V}_\omega \subset \operatorname{epi} \varphi_\omega$, which yields $\omega \in \Omega_n$ and thus $\Omega = \bigcup_{k \in \mathbb{N}} \Omega_k$. This yields a sequence $(\Theta_n)_{n \in \mathbb{N}}$ of pairwise disjoint sets in $\mathcal{F}$ such that

$$\Theta_0 = \Omega_0, \quad \bigcup_{n \in \mathbb{N}} \Theta_n = \Omega, \quad \text{and} \quad (\forall n \in \mathbb{N})\ \Theta_n \subset \Omega_n. \tag{4.7}$$

For every $\omega \in \Omega$, there exists a unique $n_\omega \in \mathbb{N}$ such that $\omega \in \Theta_{n_\omega}$. Now define

$$z \colon \Omega \to \mathsf{X} \colon \omega \mapsto \mathsf{x}_{n_\omega} \quad \text{and} \quad \vartheta \colon \Omega \to \mathbb{R} \colon \omega \mapsto \xi_{n_\omega}. \tag{4.8}$$

Then

$$(\forall \mathsf{V} \in \mathcal{T}_{\mathsf{X}}) \quad z^{-1}(\mathsf{V}) = \bigcup_{\substack{n\in\mathbb{N}\\ \mathsf{x}_n\in\mathsf{V}}} \Theta_n \in \mathcal{F}, \tag{4.9}$$

which implies that $z \in \mathcal{L}(\Omega; \mathsf{X})$. Likewise, $\vartheta \in \mathcal{L}(\Omega; \mathbb{R})$. Next, define

$$(\forall n \in \mathbb{N}) \quad x_n\colon \Omega \to \mathsf{X}\colon \omega \mapsto \begin{cases} \mathsf{x}_n, & \text{if } \omega \in \Omega_n;\\ z(\omega), & \text{if } \omega \in \complement\Omega_n \end{cases} \tag{4.10}$$

and

$$(\forall n \in \mathbb{N}) \quad \varrho_n\colon \Omega \to \mathbb{R}\colon \omega \mapsto \begin{cases} \xi_n, & \text{if } \omega \in \Omega_n;\\ \vartheta(\omega), & \text{if } \omega \in \complement\Omega_n. \end{cases} \tag{4.11}$$

Then $(x_n)_{n\in\mathbb{N}}$ and $(\varrho_n)_{n\in\mathbb{N}}$ are sequences in $\mathcal{L}(\Omega;\mathsf{X})$ and $\mathcal{L}(\Omega;\mathbb{R})$, respectively. Moreover, we deduce from (4.10), (4.11), (4.6), and (4.7) that

$$(\forall \omega \in \Omega)(\forall n \in \mathbb{N}) \quad \big(x_n(\omega), \varrho_n(\omega)\big) \in \operatorname{epi}\varphi_\omega. \tag{4.12}$$

On the other hand, for every $\omega \in \Omega$, since $\{(\mathsf{x}_n, \xi_n)\}_{n\in\mathbb{N}}$ is dense in $(\mathsf{X}\times\mathbb{R}, \mathcal{T}_{\mathsf{X}} \boxtimes \mathcal{T}_{\mathbb{R}})$ and since $\mathbf{V}_\omega$ is open, we infer from (4.10), (4.11), and (4.6) that

$$\overline{\big\{\big(x_n(\omega), \varrho_n(\omega)\big)\big\}_{n\in\mathbb{N}}} = \overline{\big\{(\mathsf{x}_n, \xi_n)\big\}_{n\in\mathbb{N}} \cap \operatorname{epi}\varphi_\omega} \supset \overline{\big\{(\mathsf{x}_n, \xi_n)\big\}_{n\in\mathbb{N}} \cap \mathbf{V}_\omega} = \overline{\mathbf{V}_\omega} = \overline{\operatorname{epi}\varphi_\omega}. \tag{4.13}$$

Consequently, φ is normal.

(i)(d)⇒(i)(c): Set $(\forall \omega \in \Omega)\ \mathbf{V}_\omega = \big\{(\mathsf{x}, \xi) \in \mathsf{X}\times\mathbb{R} \mid \varphi(\omega, \mathsf{x}) < \xi\big\}$. Now fix $\omega \in \Omega$ and $(\mathsf{x}, \xi) \in \operatorname{epi}\varphi_\omega$. Since the sequence $(\mathsf{x}, \xi + 2^{-n})_{n\in\mathbb{N}}$ lies in $\mathbf{V}_\omega$ and $(\mathsf{x}, \xi + 2^{-n}) \to (\mathsf{x}, \xi)$, we obtain $(\mathsf{x}, \xi) \in \overline{\mathbf{V}_\omega}$. Hence $\overline{\mathbf{V}_\omega} = \overline{\operatorname{epi}\varphi_\omega}$. At the same time, the upper semicontinuity of φ_ω guarantees that $\mathbf{V}_\omega$ is open.

(ii)(a)⇒(i)(c): It suffices to show that φ is $\mathcal{F}\otimes\mathcal{B}_{\mathsf{X}}$-measurable. Let $\{(\mathsf{x}_n, \xi_n)\}_{n\in\mathbb{N}}$ be dense in $(\mathsf{X}\times\mathbb{R}, \mathcal{T}_{\mathsf{X}} \boxtimes \mathcal{T}_{\mathbb{R}})$, let $\mathbf{V} \in \mathcal{T}_{\mathsf{X}} \boxtimes \mathcal{T}_{\mathbb{R}}$, and set $\mathbb{K} = \big\{n \in \mathbb{N} \mid (\mathsf{x}_n, \xi_n) \in \mathbf{V}\big\}$. Then

$$\overline{\{(\mathsf{x}_n, \xi_n)\}_{n\in\mathbb{K}}} = \overline{\{(\mathsf{x}_n, \xi_n)\}_{n\in\mathbb{N}} \cap \mathbf{V}} = \overline{\mathbf{V}}. \tag{4.14}$$

Suppose that there exists $\omega \in \Omega$ such that

$$\mathbf{V} \cap \operatorname{epi}\varphi_\omega \neq \varnothing \quad \text{and} \quad (\forall n \in \mathbb{K})\ \ (\mathsf{x}_n, \xi_n) \notin \operatorname{epi}\varphi_\omega. \tag{4.15}$$

Since $\mathbf{V}$ is open and $\overline{\mathbf{V}_\omega} = \operatorname{epi}\varphi_\omega$, there exists $(\mathsf{y}, \eta) \in \mathbf{V} \cap \mathbf{V}_\omega$. Therefore, we infer from (4.14) that there exists a subnet $(\mathsf{x}_{k(b)}, \xi_{k(b)})_{b\in B}$ of $(\mathsf{x}_n, \xi_n)_{n\in\mathbb{K}}$ such that $(\mathsf{x}_{k(b)}, \xi_{k(b)}) \to (\mathsf{y}, \eta)$. This and (4.15) force $(\mathsf{y}, \eta) \in \overline{\complement\operatorname{epi}\varphi_\omega} = \overline{\complement\overline{\mathbf{V}_\omega}} = \complement\operatorname{int}\overline{\mathbf{V}_\omega}$, which is in contradiction with the inclusion $(\mathsf{y}, \eta) \in \mathbf{V}_\omega$. Hence, the $\mathcal{F}$-measurability of the functions $(\varphi(\cdot, \mathsf{x}))_{\mathsf{x}\in\mathsf{X}}$ yields

$$\big\{\omega \in \Omega \mid \mathbf{V} \cap \operatorname{epi}\varphi_\omega \neq \varnothing\big\} = \bigcup_{n\in\mathbb{K}} \big\{\omega \in \Omega \mid (\mathsf{x}_n, \xi_n) \in \operatorname{epi}\varphi_\omega\big\} = \bigcup_{n\in\mathbb{K}} \big[\varphi(\cdot, \mathsf{x}_n) \leqslant \xi_n\big] \in \mathcal{F}. \tag{4.16}$$

Therefore, since $(\mathsf{X}\times\mathbb{R}, \mathcal{T}_{\mathsf{X}} \boxtimes \mathcal{T}_{\mathbb{R}})$ is a separable metrizable space and the sets $(\operatorname{epi}\varphi_\omega)_{\omega\in\Omega}$ are closed, [16, Theorem 3.5(i)] and (4.2) imply that $\boldsymbol{G} \in \mathcal{F}\otimes\mathcal{B}_{\mathsf{X}\times\mathbb{R}}$. Consequently, (4.3) asserts that φ is $\mathcal{F}\otimes\mathcal{B}_{\mathsf{X}}$-measurable.

(ii)(b)⇒(ii)(a): Set $(\forall\omega\in\Omega)\ \mathbf{V}_\omega=\operatorname{int}\operatorname{epi}\varphi_\omega$. For every $\omega\in\Omega$, the assumption ensures that $\operatorname{epi}\varphi_\omega$ is closed and convex, and that $\mathbf{V}_\omega\neq\varnothing$ [40, Theorem 2.2.20 and Corollary 2.2.10]. Thus [40, Theorem 1.1.2(iv)] yields $(\forall\omega\in\Omega)\ \operatorname{epi}\varphi_\omega=\overline{\mathbf{V}_\omega}$.

(ii)(c)⇒(ii)(b): Clear.

(iii): It results from [34] that there exists a topology $\widetilde{\mathcal{T}_{\mathsf{X}}}$ on X such that

$$\mathcal{T}_{\mathsf{X}}\subset\widetilde{\mathcal{T}_{\mathsf{X}}} \tag{4.17}$$

and

$$\big(\mathsf{X},\widetilde{\mathcal{T}_{\mathsf{X}}}\big)\ \text{is a metrizable Souslin space.} \tag{4.18}$$

Set $(\forall\omega\in\Omega)\ \mathbf{V}_\omega=\big\{(\mathsf{x},\xi)\in\mathsf{X}\times\mathbb{R}\mid\varphi(\omega,\mathsf{x})<\xi\big\}$. Then, since (4.17) implies that

$$(\forall\omega\in\Omega)\quad\varphi_\omega\ \text{is}\ \widetilde{\mathcal{T}_{\mathsf{X}}}\text{-continuous,} \tag{4.19}$$

it follows that

$$(\forall\omega\in\Omega)\quad\mathbf{V}_\omega\in\widetilde{\mathcal{T}_{\mathsf{X}}}\boxtimes\mathcal{T}_{\mathbb{R}}\quad\text{and}\quad\overline{\mathbf{V}_\omega}^{\widetilde{\mathcal{T}_{\mathsf{X}}}\boxtimes\mathcal{T}_{\mathbb{R}}}=\overline{\operatorname{epi}\varphi_\omega}^{\widetilde{\mathcal{T}_{\mathsf{X}}}\boxtimes\mathcal{T}_{\mathbb{R}}}=\operatorname{epi}\varphi_\omega. \tag{4.20}$$

On the other hand, we derive from (4.18), (4.17), and [36, Corollary 2, p. 101] that the Borel σ-algebra of $(\mathsf{X},\widetilde{\mathcal{T}_{\mathsf{X}}})$ is $\mathcal{B}_{\mathsf{X}}$. Altogether, applying (ii)(a) to the metrizable Souslin space $(\mathsf{X},\widetilde{\mathcal{T}_{\mathsf{X}}})$, we deduce that φ is $\mathcal{F}\otimes\mathcal{B}_{\mathsf{X}}$-measurable and that there exist sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{L}(\Omega;\mathsf{X})$ and $(\varrho_n)_{n\in\mathbb{N}}$ in $\mathcal{L}(\Omega;\mathbb{R})$ such that

$$(\forall\omega\in\Omega)\quad\big\{\big(x_n(\omega),\varrho_n(\omega)\big)\big\}_{n\in\mathbb{N}}\subset\operatorname{epi}\varphi_\omega\quad\text{and}\quad\overline{\operatorname{epi}\varphi_\omega}^{\widetilde{\mathcal{T}_{\mathsf{X}}}\boxtimes\mathcal{T}_{\mathbb{R}}}=\overline{\big\{\big(x_n(\omega),\varrho_n(\omega)\big)\big\}_{n\in\mathbb{N}}}^{\widetilde{\mathcal{T}_{\mathsf{X}}}\boxtimes\mathcal{T}_{\mathbb{R}}}. \tag{4.21}$$

Hence, by (4.17) and (4.20),

$$\overline{\big\{\big(x_n(\omega),\varrho_n(\omega)\big)\big\}_{n\in\mathbb{N}}}\supset\overline{\big\{\big(x_n(\omega),\varrho_n(\omega)\big)\big\}_{n\in\mathbb{N}}}^{\widetilde{\mathcal{T}_{\mathsf{X}}}\boxtimes\mathcal{T}_{\mathbb{R}}}=\overline{\operatorname{epi}\varphi_\omega}^{\widetilde{\mathcal{T}_{\mathsf{X}}}\boxtimes\mathcal{T}_{\mathbb{R}}}=\operatorname{epi}\varphi_\omega. \tag{4.22}$$

Consequently, φ is normal.

(iv): It follows from [9, Section II.4.3] that $(\mathsf{Y}\times\mathbb{R},\mathcal{T}_{\mathsf{Y}}\boxtimes\mathcal{T}_{\mathbb{R}})$ is a separable Fréchet space. Moreover, by [9, Proposition II.6.8], $\mathsf{X}\times\mathbb{R}=(\mathsf{Y}\times\mathbb{R},\mathcal{T}_{\mathsf{Y}}\boxtimes\mathcal{T}_{\mathbb{R}})^*$ and the weak topology of $\mathsf{X}\times\mathbb{R}$ is $\mathcal{T}_{\mathsf{X}}\boxtimes\mathcal{T}_{\mathbb{R}}$. In turn, arguing as in [35, Section IV-1.7], we deduce that there exists a covering $(\mathbf{C}_n)_{n\in\mathbb{N}}$ of $\mathsf{X}\times\mathbb{R}$, with respective $\mathcal{T}_{\mathsf{X}}\boxtimes\mathcal{T}_{\mathbb{R}}$-induced topologies $(\mathcal{T}_{\mathbf{C}_n})_{n\in\mathbb{N}}$, such that, for every $n\in\mathbb{N}$, $(\mathbf{C}_n,\mathcal{T}_{\mathbf{C}_n})$ is a compact separable metrizable space, hence a Polish space. We also introduce

$$(\forall n\in\mathbb{N})\quad Q_n\colon\Omega\times\mathbf{C}_n\to\Omega\colon(\omega,\mathsf{x},\xi)\mapsto\omega. \tag{4.23}$$

Note that, for every subset $\mathbf{C}$ of $\mathsf{X}\times\mathbb{R}$,

$$\{\omega\in\Omega\mid\mathbf{C}\cap\operatorname{epi}\varphi_\omega\neq\varnothing\}=\bigcup_{n\in\mathbb{N}}\{\omega\in\Omega\mid\mathbf{C}\cap\mathbf{C}_n\cap\operatorname{epi}\varphi_\omega\neq\varnothing\}=\bigcup_{n\in\mathbb{N}}Q_n\Big(G\cap\big(\Omega\times(\mathbf{C}\cap\mathbf{C}_n)\big)\Big). \tag{4.24}$$

(iv)(a): For every $n\in\mathbb{N}$, set

$$\Omega_n=\big\{\omega\in\Omega\mid\mathbf{C}_n\cap\operatorname{epi}\varphi_\omega\neq\varnothing\big\}, \tag{4.25}$$

denote by $\mathcal{F}_n$ the trace σ-algebra of $\mathcal{F}$ on Ω_n, and observe that

$$\Omega_n\in\mathcal{F}\quad\text{and}\quad\mathcal{F}_n\subset\mathcal{F}. \tag{4.26}$$

Now define

$$\mathbb{K} = \big\{n \in \mathbb{N} \mid \Omega_n \neq \varnothing\big\} \quad \text{and} \quad (\forall n \in \mathbb{K}) \;\; K_n \colon \Omega_n \to 2^{\mathsf{C}_n} \colon \omega \mapsto \mathsf{C}_n \cap \operatorname{epi} \varphi_\omega. \tag{4.27}$$

Then

$$\mathbb{K} \neq \varnothing \quad \text{and} \quad \bigcup_{n\in\mathbb{K}} \Omega_n = \Omega. \tag{4.28}$$

Furthermore, the $\mathcal{T}_{\mathsf{X}} \boxtimes \mathcal{T}_{\mathbb{R}}$-closedness of $(\operatorname{epi} \varphi_\omega)_{\omega\in\Omega}$ guarantees that

$$(\forall n \in \mathbb{K})(\forall \omega \in \Omega) \quad K_n(\omega) \text{ is } \mathcal{T}_{\mathsf{C}_n}\text{-closed.} \tag{4.29}$$

On the other hand, for every $n \in \mathbb{K}$ and every closed subset D of $(\mathsf{C}_n, \mathcal{T}_{\mathsf{C}_n})$, there exists a closed subset E of $(\mathsf{X} \times \mathbb{R}, \mathcal{T}_{\mathsf{X}} \boxtimes \mathcal{T}_{\mathbb{R}})$ such that $\mathsf{D} = \mathsf{C}_n \cap \mathsf{E}$ [7, Section I.3.1] and therefore, since C_n is $\mathcal{T}_{\mathsf{X}} \boxtimes \mathcal{T}_{\mathbb{R}}$-closed, we deduce from (4.26) that

$$\big\{\omega \in \Omega_n \mid \mathsf{D} \cap K_n(\omega) \neq \varnothing\big\} = \Omega_n \cap \big\{\omega \in \Omega \mid \mathsf{C}_n \cap \mathsf{E} \cap \operatorname{epi} \varphi_\omega \neq \varnothing\big\} \in \mathcal{F}_n. \tag{4.30}$$

Hence, for every $n \in \mathbb{K}$, since $(\mathsf{C}_n, \mathcal{T}_{\mathsf{C}_n})$ is a Polish space, we deduce from [16, Theorem 3.5(i), Theorem 5.1, and Theorem 5.6] that there exist measurable mappings $\boldsymbol{y}_n$ and $(\boldsymbol{z}_{n,k})_{k\in\mathbb{N}}$ from $(\Omega_n, \mathcal{F}_n)$ to $(\mathsf{C}_n, \mathcal{B}_{\mathsf{C}_n})$ such that

$$(\forall \omega \in \Omega_n) \quad \boldsymbol{y}_n(\omega) \in K_n(\omega) \quad \text{and} \quad K_n(\omega) = \overline{\big\{\boldsymbol{z}_{n,k}(\omega)\big\}_{k\in\mathbb{N}}}^{\mathcal{T}_{\mathsf{C}_n}} = \mathsf{C}_n \cap \overline{\big\{\boldsymbol{z}_{n,k}(\omega)\big\}_{k\in\mathbb{N}}}. \tag{4.31}$$

In addition, since [16, Theorem 3.5(i)] asserts that

$$\begin{aligned}(\forall n \in \mathbb{K}) \quad &\big\{(\omega, \mathsf{x}, \xi) \in \Omega_n \times \mathsf{C}_n \mid (\mathsf{x}, \xi) \in \mathsf{C}_n \cap \operatorname{epi} \varphi_\omega\big\} \\ &\quad = \big\{(\omega, \mathsf{x}, \xi) \in \Omega_n \times \mathsf{C}_n \mid (\mathsf{x}, \xi) \in K_n(\omega)\big\} \\ &\quad \in \mathcal{F}_n \otimes \mathcal{B}_{\mathsf{C}_n} \\ &\quad \subset \mathcal{F} \otimes \mathcal{B}_{\mathsf{X}\times\mathbb{R}},\end{aligned} \tag{4.32}$$

we get from (4.2) that

$$\boldsymbol{G} = \bigcup_{n\in\mathbb{K}} \big\{(\omega, \mathsf{x}, \xi) \in \Omega_n \times \mathsf{C}_n \mid (\mathsf{x}, \xi) \in \mathsf{C}_n \cap \operatorname{epi} \varphi_\omega\big\} \in \mathcal{F} \otimes \mathcal{B}_{\mathsf{X}\times\mathbb{R}}. \tag{4.33}$$

Thus, in the light of (4.3), φ is $\mathcal{F} \otimes \mathcal{B}_{\mathsf{X}}$-measurable. Next, using (4.28), we construct a family $(\Theta_n)_{n\in\mathbb{K}}$ of pairwise disjoint sets in $\mathcal{F}$ such that

$$\Theta_{\min \mathbb{K}} = \Omega_{\min \mathbb{K}}, \quad \bigcup_{n\in\mathbb{K}} \Theta_n = \Omega, \quad \text{and} \quad (\forall n \in \mathbb{K}) \;\; \Theta_n \subset \Omega_n. \tag{4.34}$$

In turn, for every $\omega \in \Omega$, there exists a unique $\ell_\omega \in \mathbb{K}$ such that $\omega \in \Theta_{\ell_\omega}$. Therefore, appealing to (4.34), the mapping

$$\boldsymbol{y} \colon \Omega \to \mathsf{X} \times \mathbb{R} \colon \omega \mapsto \boldsymbol{y}_{\ell_\omega}(\omega) \tag{4.35}$$

is well defined and, in view of (4.31),

$$(\forall \omega \in \Omega) \quad \boldsymbol{y}(\omega) = \boldsymbol{y}_{\ell_\omega}(\omega) \in K_{\ell_\omega}(\omega) \subset \operatorname{epi} \varphi_\omega. \tag{4.36}$$

Let $\mathsf{V} \in \mathcal{T}_{\mathsf{X}} \boxtimes \mathcal{T}_{\mathbb{R}}$. Then, for every $n \in \mathbb{K}$, $\mathsf{V} \cap \mathsf{C}_n$ is $\mathcal{T}_{\mathsf{C}_n}$-open and thus the measurability of $\boldsymbol{y}_n \colon (\Omega_n, \mathcal{F}_n) \to (\mathsf{C}_n, \mathcal{B}_{\mathsf{C}_n})$ and (4.26) ensure that $\boldsymbol{y}_n^{-1}(\mathsf{V} \cap \mathsf{C}_n) \in \mathcal{F}_n \subset \mathcal{F}$. Hence, we infer from (4.34), (4.35), and (4.31) that

$$\begin{aligned} \boldsymbol{y}^{-1}(\mathsf{V}) &= \bigcup_{n\in\mathbb{K}} \{\omega \in \Theta_n \mid \boldsymbol{y}(\omega) \in \mathsf{V}\} \\ &= \bigcup_{n\in\mathbb{K}} \{\omega \in \Theta_n \mid \boldsymbol{y}_n(\omega) \in \mathsf{C}_n \cap \mathsf{V}\} \\ &= \bigcup_{n\in\mathbb{K}} \big(\Theta_n \cap \boldsymbol{y}_n^{-1}(\mathsf{C}_n \cap \mathsf{V})\big) \\ &\in \mathcal{F}. \end{aligned} \tag{4.37}$$

This verifies that $\boldsymbol{y} \colon (\Omega, \mathcal{F}) \to (\mathsf{X} \times \mathbb{R}, \mathcal{B}_{\mathsf{X}\times\mathbb{R}})$ is measurable. We now define

$$(\forall n \in \mathbb{K})(\forall k \in \mathbb{N}) \quad \boldsymbol{x}_{n,k} \colon \Omega \to \mathsf{X} \times \mathbb{R} \colon \omega \mapsto \begin{cases} \boldsymbol{z}_{n,k}(\omega), & \text{if } \omega \in \Omega_n; \\ \boldsymbol{y}(\omega), & \text{if } \omega \in \complement\Omega_n. \end{cases} \tag{4.38}$$

It results from (4.26) that $(\boldsymbol{x}_{n,k})_{n\in\mathbb{K},k\in\mathbb{N}}$ are measurable mappings from $(\Omega, \mathcal{F})$ to $(\mathsf{X} \times \mathbb{R}, \mathcal{B}_{\mathsf{X}\times\mathbb{R}})$. Furthermore, (4.31) and (4.36) give

$$(\forall n \in \mathbb{K})(\forall k \in \mathbb{N})(\forall \omega \in \Omega) \quad \boldsymbol{x}_{n,k}(\omega) \in \operatorname{epi} \varphi_\omega. \tag{4.39}$$

Fix $\omega \in \Omega$ and let $\mathsf{x} \in \operatorname{epi} \varphi_\omega$. Since $\bigcup_{n\in\mathbb{K}} (\mathsf{C}_n \cap \operatorname{epi} \varphi_\omega) = \operatorname{epi} \varphi_\omega$, there exists $N \in \mathbb{K}$ such that $\omega \in \Omega_N$ and $\mathsf{x} \in \mathsf{C}_N \cap \operatorname{epi} \varphi_\omega = K_N(\omega)$. Thus, it results from (4.31) and (4.38) that

$$\mathsf{x} \in \overline{\{\boldsymbol{z}_{N,k}(\omega)\}_{k\in\mathbb{N}}} = \overline{\{\boldsymbol{x}_{N,k}(\omega)\}_{k\in\mathbb{N}}} \subset \overline{\{\boldsymbol{x}_{n,k}(\omega)\}_{n\in\mathbb{K},k\in\mathbb{N}}}. \tag{4.40}$$

Therefore, since $\operatorname{epi} \varphi_\omega$ is closed, it follows from (4.39) and [7, Section I.3.1] that

$$\operatorname{epi} \varphi_\omega = \overline{\{\boldsymbol{x}_{n,k}(\omega)\}_{n\in\mathbb{K},k\in\mathbb{N}}}. \tag{4.41}$$

At the same time, for every $n \in \mathbb{K}$ and every $k \in \mathbb{N}$, since $\mathcal{B}_{\mathsf{X}\times\mathbb{R}} = \mathcal{B}_{\mathsf{X}} \otimes \mathcal{B}_{\mathbb{R}}$ [4, Lemma 6.4.2(i)] and since $\boldsymbol{x}_{n,k} \colon (\Omega, \mathcal{F}) \to (\mathsf{X} \times \mathbb{R}, \mathcal{B}_{\mathsf{X}\times\mathbb{R}})$ is measurable, there exist $x_{n,k} \in \mathcal{L}(\Omega; \mathsf{X})$ and $\varrho_{n,k} \in \mathcal{L}(\Omega; \mathbb{R})$ such that $(\forall \omega \in \Omega)$ $\boldsymbol{x}_{n,k}(\omega) = (x_{n,k}(\omega), \varrho_{n,k}(\omega))$. Altogether, φ is normal.

(iv)(b)⇒(iv)(a): Let C be a nonempty closed subset of $(\mathsf{X} \times \mathbb{R}, \mathcal{T}_{\mathsf{X}} \boxtimes \mathcal{T}_{\mathbb{R}})$. Note that the lower semicontinuity of φ ensures that $\boldsymbol{G}$ is closed. For every $n \in \mathbb{N}$, since $\boldsymbol{G} \cap (\Omega \times (\mathsf{C} \cap \mathsf{C}_n))$ is closed in $(\Omega \times \mathsf{C}_n, \mathcal{T}_\Omega \boxtimes \mathcal{T}_{\mathsf{C}_n})$, it follows from (4.23) and [7, Corollaire I.10.5 and Théorème I.10.1] that $Q_n(\boldsymbol{G} \cap (\Omega \times (\mathsf{C} \cap \mathsf{C}_n)))$ is closed in $(\Omega, \mathcal{T}_\Omega)$ and, therefore, that it belongs to $\mathcal{B}_\Omega = \mathcal{F}$. Thus, by (4.24), $\{\omega \in \Omega \mid \mathsf{C} \cap \operatorname{epi} \varphi_\omega \neq \varnothing\} \in \mathcal{F}$.

(iv)(c)⇒(iv)(a): There exists a topology $\widetilde{\mathcal{T}_\Omega}$ on Ω such that

$$\mathcal{T}_\Omega \subset \widetilde{\mathcal{T}_\Omega} \text{ and } \big(\Omega, \widetilde{\mathcal{T}_\Omega}\big) \text{ is a Polish space.} \tag{4.42}$$

In addition, by [36, Corollary 2, p. 101], the Borel σ-algebra of $(\Omega, \widetilde{\mathcal{T}_\Omega})$ is $\mathcal{B}_\Omega = \mathcal{F}$. Let C be a closed subset of $(\mathsf{X} \times \mathbb{R}, \mathcal{T}_{\mathsf{X}} \boxtimes \mathcal{T}_{\mathbb{R}})$ and fix temporarily $n \in \mathbb{N}$. Since the $\mathcal{F} \otimes \mathcal{B}_{\mathsf{X}}$-measurability of φ and (4.3) ensure that $\boldsymbol{G} \in \mathcal{F} \otimes \mathcal{B}_{\mathsf{X}\times\mathbb{R}}$, we have $\boldsymbol{G} \cap (\Omega \times (\mathsf{C} \cap \mathsf{C}_n)) = \boldsymbol{G} \cap (\Omega \times \mathsf{C}) \cap (\Omega \times \mathsf{C}_n) \in \mathcal{B}_{\Omega\times\mathsf{C}_n}$. At the same time, for every $\omega \in \Omega$,

$$\big\{(\mathsf{x}, \xi) \in \mathsf{X} \times \mathbb{R} \mid (\omega, \mathsf{x}, \xi) \in \boldsymbol{G} \cap \big(\Omega \times (\mathsf{C} \cap \mathsf{C}_n)\big)\big\}$$

$$= \big\{(\mathsf{x},\xi) \in \mathsf{X}\times\mathbb{R} \mid (\mathsf{x},\xi) \in \mathbf{C}\cap\mathbf{C}_n \text{ and } (\mathsf{x},\xi) \in \operatorname{epi}\varphi_\omega\big\},$$
$$= \mathbf{C}\cap\mathbf{C}_n\cap\operatorname{epi}\varphi_\omega \tag{4.43}$$

is a closed subset of the compact space $(\mathbf{C}_n, \mathcal{T}_{\mathbf{C}_n})$. In turn, since $(\Omega, \widetilde{\mathcal{T}_\Omega})$ and $(\mathbf{C}_n, \mathcal{T}_{\mathbf{C}_n})$ are Polish spaces, [10, Theorem 1] guarantees that $Q_n(\boldsymbol{G}\cap(\Omega\times(\mathbf{C}\cap\mathbf{C}_n))) \in \mathcal{B}_\Omega = \mathcal{F}$. Consequently, we infer from (4.24) that $\big\{\omega\in\Omega \mid \mathbf{C}\cap\operatorname{epi}\varphi_\omega \neq \varnothing\big\} \in \mathcal{F}$.

(v): Let $(\mathsf{Y}, \mathcal{T}_\mathsf{Y})$ be the strong dual of X. Then $(\mathsf{Y}, \mathcal{T}_\mathsf{Y})$ is a separable reflexive Banach space. Consequently, (v)(a) follows from (iv)(b), and (v)(b) follows from (iv)(c).

(vi)⇒(v)(b): Let $\mathcal{T}_\Omega$ be the topology on Ω induced by the standard topology on $\mathbb{R}^M$. By [36, Corollary 1, p. 102], $(\Omega, \mathcal{T}_\Omega)$ is a Lusin space.

(vii)(a): The lower semicontinuity of $(\varphi_\omega)_{\omega\in\Omega}$ ensures that the sets $(\operatorname{epi}\varphi_\omega)_{\omega\in\Omega}$ are closed. Hence, since $(\mathsf{X}\times\mathbb{R}, \mathcal{T}_\mathsf{X}\boxtimes\mathcal{T}_\mathbb{R})$ is a Polish space, [16, Theorem 3.5(i)] and (4.2) yield $\boldsymbol{G} \in \mathcal{F}\otimes\mathcal{B}_{\mathsf{X}\times\mathbb{R}}$. Therefore, by (4.3), φ is $\mathcal{F}\otimes\mathcal{B}_\mathsf{X}$-measurable. Consequently, we deduce the assertion from [16, Theorem 5.6].

(vii)(b)⇒(vii)(a): This follows from [16, Theorem 3.2(ii)].

(viii): The $\mathcal{B}_\mathsf{X}$-measurability of f implies that φ is $\mathcal{F}\otimes\mathcal{B}_\mathsf{X}$-measurable. At the same time, since $(\mathsf{X}\times\mathbb{R}, \mathcal{T}_\mathsf{X}\boxtimes\mathcal{T}_\mathbb{R})$ is a Souslin space, we deduce from [36, Proposition II.0] that there exists a sequence $\{(\mathsf{x}_n,\xi_n)\}_{n\in\mathbb{N}}$ in epi f such that $\overline{\{(\mathsf{x}_n,\xi_n)\}_{n\in\mathbb{N}}} = \overline{\operatorname{epi}\mathsf{f}}$. Altogether, upon setting

$$(\forall n\in\mathbb{N})\quad x_n\colon \Omega\to\mathsf{X}\colon \omega\mapsto \mathsf{x}_n \quad\text{and}\quad \varrho_n\colon\Omega\to\mathbb{R}\colon\omega\mapsto\xi_n, \tag{4.44}$$

we conclude that φ is normal. ☐

Remark 4.5 Here are a few observations about Definition 4.3.

(i) The setting of Theorem 4.4(vii)(b) corresponds to the definition of normality in [31].

(ii) The setting of Theorem 4.4(i)(a) corresponds to the definition of normality in [38], which itself contains that of [29].

(iii) The frameworks of (i) and (ii) above are distinct since the former does not require that $(\Omega,\mathcal{F},\mu)$ be complete. Definition 4.3 unifies them and, as seen in Theorem 4.4, goes beyond. For the importance of noncompleteness in applications, see for instance [27] and [32, p. 649].

5 Interchange rules with compliant spaces and normal integrands

The main result of this section is the following interchange theorem, which brings together the abstract principle of Theorem 1.2, the notion of compliance of Definition 4.1, and the notion of normality of Definition 4.3.

Theorem 5.1 *Suppose that Assumption 1.1 holds, that $\mathcal{X}$ is compliant, and that φ is normal. Then*

$$\inf_{x\in\mathcal{X}}\int_\Omega \varphi\big(\omega, x(\omega)\big)\mu(d\omega) = \int_\Omega \inf_{\mathsf{x}\in\mathsf{X}}\varphi(\omega,\mathsf{x})\,\mu(d\omega). \tag{5.1}$$

Proof. We apply Theorem 1.2. By virtue of the normality of φ, per Definition 4.3, we choose sequences $(z_n)_{n\in\mathbb{N}}$ in $\mathcal{L}(\Omega;\mathsf{X})$ and $(\vartheta_n)_{n\in\mathbb{N}}$ in $\mathcal{L}(\Omega;\mathbb{R})$ such that

$$(\forall\omega\in\Omega)\quad \big\{\big(z_n(\omega),\vartheta_n(\omega)\big)\big\}_{n\in\mathbb{N}} \subset \operatorname{epi}\varphi_\omega \quad\text{and}\quad \overline{\operatorname{epi}\varphi_\omega} = \overline{\big\{\big(z_n(\omega),\vartheta_n(\omega)\big)\big\}_{n\in\mathbb{N}}}. \tag{5.2}$$

On the other hand, Assumption 1.1[F] ensures that $(\forall\omega\in\Omega)\ \inf\varphi(\omega,\mathsf{X})<+\infty$. Now fix $\omega\in\Omega$ and let $\xi\in\,]\inf\varphi(\omega,\mathsf{X}),+\infty[$. Then there exits $\mathsf{x}\in\mathsf{X}$ such that $(\mathsf{x},\xi)\in\operatorname{epi}\varphi_\omega$. Thus, in view of (5.2), we obtain a subnet $(\vartheta_{k(b)}(\omega))_{b\in B}$ of $(\vartheta_n(\omega))_{n\in\mathbb{N}}$ such that $\vartheta_{k(b)}(\omega)\to\xi$. On the other hand,

$$(\forall b\in B)\quad \inf\varphi(\omega,\mathsf{X})\leqslant\inf_{n\in\mathbb{N}}\varphi\big(\omega,z_n(\omega)\big)\leqslant\varphi\big(\omega,z_{k(b)}(\omega)\big)\leqslant\vartheta_{k(b)}(\omega). \tag{5.3}$$

Hence $\inf\varphi(\omega,\mathsf{X})\leqslant\inf_{n\in\mathbb{N}}\varphi(\omega,z_n(\omega))\leqslant\xi$. In turn, letting $\xi\downarrow\inf\varphi(\omega,\mathsf{X})$ yields $\inf\varphi(\omega,\mathsf{X})=\inf_{n\in\mathbb{N}}\varphi(\omega,z_n(\omega))$. Therefore, property (ii)(a) in Theorem 1.2 is satisfied with $(\forall n\in\mathbb{N})\ x_n=z_n-\overline{x}$. At the same time, property (ii)(b) in Theorem 1.2 follows from Assumption 1.1[D] and the compliance of $\mathcal{X}$. Finally, since the functions $(\varphi(\cdot,z_n(\cdot)))_{n\in\mathbb{N}}$ are $\mathcal{F}$-measurable by Assumption 1.1[F], so is $\inf_{n\in\mathbb{N}}\varphi(\cdot,z_n(\cdot))=\inf\varphi(\cdot,\mathsf{X})$. ☐

In the remainder of this section, we construct new scenarios for the validity of the interchange rule as instantiations of Theorem 5.1.

Example 5.2 Let X be a separable real Banach space with strong topology $\mathcal{T}_\mathsf{X}$, let $(\Omega,\mathcal{F},\mu)$ be a σ-finite measure space such that $\mu(\Omega)\neq 0$, let $\mathcal{X}$ be a vector subspace of $\mathcal{L}(\Omega;\mathsf{X})$, and let $\varphi\colon(\Omega\times\mathsf{X},\mathcal{F}\otimes\mathcal{B}_\mathsf{X})\to\overline{\mathbb{R}}$ be measurable. Suppose that the following are satisfied:

(i) For every $A\in\mathcal{F}$ such that $\mu(A)<+\infty$ and every $z\in\mathcal{L}^\infty(\Omega;\mathsf{X})$, $1_A z\in\mathcal{X}$.

(ii) φ is normal.

(iii) There exists $\overline{x}\in\mathcal{X}$ such that $\int_\Omega\max\{\varphi(\cdot,\overline{x}(\cdot)),0\}d\mu<+\infty$.

Then the interchange rule (5.1) holds.

Proof. Note that Assumption 1.1 is satisfied. Hence, the assertion follows from Proposition 4.2(ii) and Theorem 5.1. ☐

Example 5.3 Suppose that Assumption 1.1 holds, that $(\Omega,\mathcal{F},\mu)$ is complete, and that $\mathcal{X}$ is compliant. Then the interchange rule (5.1) holds.

Proof. Combine Theorem 4.4(i)(a) and Theorem 5.1. ☐

When specialized to probability in separable Banach spaces, Theorem 5.1 yields conditions for the interchange of infimization and expectation. Here is an illustration.

Example 5.4 Let X be a separable real Banach space, let $(\Omega,\mathcal{F},\mathsf{P})$ be a probability space, let $\mathcal{X}$ be a vector subspace of $\mathcal{L}(\Omega;\mathsf{X})$ which contains $\mathcal{L}^\infty(\Omega;\mathsf{X})$, and let $\varphi\colon(\Omega\times\mathsf{X},\mathcal{F}\otimes\mathcal{B}_\mathsf{X})\to\overline{\mathbb{R}}$ be normal. In addition, set $\phi=\inf\varphi(\cdot,\mathsf{X})$ and $\Phi\colon\mathcal{L}(\Omega;\mathsf{X})\to\mathcal{L}(\Omega;\overline{\mathbb{R}})\colon x\mapsto\varphi(\cdot,x(\cdot))$, and suppose that there exists $\overline{x}\in\mathcal{X}$ such that $\mathsf{E}\max\{\Phi(\overline{x}),0\}<+\infty$. Then

$$\inf_{x\in\mathcal{X}}\mathsf{E}\Phi(x)=\mathsf{E}\phi. \tag{5.4}$$

Proof. This is a special case of Example 5.2. ☐

Example 5.5 Suppose that Assumption 1.1 holds, that $\mathcal{X}$ is compliant, and that the functions $(\varphi_\omega)_{\omega\in\Omega}$ are upper semicontinuous. Then the interchange rule (5.1) holds.

Proof. We deduce from Assumption 1.1[F] and Theorem 4.4(i)(d) that φ is normal. Thus, the conclusion follows from Theorem 5.1. ☐

An important realization of Example 5.5 is the case of Carathéodory integrands.

Example 5.6 (Carathéodory integrand) Let $(\mathsf{X}, \mathcal{T}_{\mathsf{X}})$ be a Souslin topological vector space, let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space such that $\mu(\Omega) \neq 0$, let $\mathcal{X}$ be a compliant vector subspace of $\mathcal{L}(\Omega; \mathsf{X})$, and let $\varphi\colon \Omega \times \mathsf{X} \to \overline{\mathbb{R}}$ be a Carathéodory integrand in the sense that, for every $(\omega, \mathsf{x}) \in \Omega \times \mathsf{X}$, $\varphi(\omega, \cdot)$ is continuous with $\operatorname{epi} \varphi_\omega \neq \varnothing$, and $\varphi(\cdot, \mathsf{x})$ is $\mathcal{F}$-measurable. Suppose that there exists $\overline{x} \in \mathcal{X}$ such that $\int_\Omega \max\{\varphi(\cdot, \overline{x}(\cdot)), 0\} d\mu < +\infty$. Then the interchange rule (5.1) holds.

Proof. Since $(\mathsf{X}, \mathcal{T}_{\mathsf{X}})$ is a Souslin topological vector space, [39, Section 35F, p. 244] implies that it is a regular Souslin space. Thus, we deduce from Theorem 4.4(iii) that φ is normal and, in particular, it is $\mathcal{F} \otimes \mathcal{B}_{\mathsf{X}}$-measurable. Hence, Assumption 1.1 is satisfied. Consequently, Example 5.5 yields the conclusion. ☐

Remark 5.7 Here are connections with existing work.

(i) Example 5.2 unifies and extends the classical results of [15, 29, 31]:

- It captures [31, Theorem 3A], where X is a Euclidean space and $\mathcal{X}$ is assumed to be Rockafellar-decomposable (see Proposition 4.2(iv) for definition).
- It covers the setting of [29], where $(\Omega, \mathcal{F}\, \mu)$ is assumed to be complete and where (i) and (ii) in Example 5.2 are specialized to:

  (i') $\mathcal{X}$ is Rockafellar-decomposable.

  (ii') The functions $(\varphi_\omega)_{\omega \in \Omega}$ are lower semicontinuous.

  The fact that property (ii) in Example 5.2 is satisfied when $(\Omega, \mathcal{F}, \mu)$ is complete is shown in Theorem 4.4(i)(a).
- It captures [15, Theorem 2.2], where $\mathcal{X} = \big\{x \in \mathcal{L}(\Omega; \mathsf{X}) \mid \int_\Omega \|x(\omega)\|_{\mathsf{X}}^p \, \mu(d\omega) < +\infty\big\}$ with $p \in [1, +\infty[$.

(ii) An important contribution of Theorem 5.1 and, in particular, of Example 5.2 is that completeness of the measure space $(\Omega, \mathcal{F}, \mu)$ is not required.

(iii) In the special case when X is a Banach space, an alternative framework that recovers the interchange rules of [15, 29, 31] was proposed in [14, Theorem 6.1], where the right-hand side of (1.2) is replaced by the integral of an abstract essential infimum. However, [14] does not provide new scenarios for (1.2) beyond the known cases in Banach spaces. An interpretation of the framework of [14] from the view point of monotone relations between partially ordered sets is proposed in [12].

(iv) Example 5.3 captures [25, Theorem 4], where $\mu(\Omega) < +\infty$ and $\mathcal{X}$ is Valadier-decomposable (see Proposition 4.2(v) for definition). It also covers the setting of [38], where X is a Souslin topological vector space and $\mathcal{X}$ is Valadier-decomposable.

(v) Example 5.4 contains the interchange rule of [24, 37], where X is the standard Euclidean space $\mathbb{R}^N$ and $\mathcal{X}$ is Rockafellar-decomposable.

(vi) Example 5.6 extends [31, Theorem 3A], where X is the standard Euclidean space $\mathbb{R}^N$ and $\mathcal{X}$ is Rockafellar-decomposable.

6 Interchanging convex-analytical operations and integration

We put the interchange principle of Theorem 1.2, compliance, and normality in action to evaluate convex-analytical objects associated with integral functions, namely conjugate functions, subdifferential operators, recession functions, Moreau envelopes, and proximity operators. This analysis results in new interchange rules for the convex calculus of integral functions. Throughout this section, we adopt the following notation.

Notation 6.1 Let $(\mathsf{X},\mathcal{T}_{\mathsf{X}})$ be a real topological vector space, let $(\Omega,\mathcal{F},\mu)$ be a σ-finite measure space such that $\mu(\Omega)\neq 0$, let $\mathcal{X}$ be a vector subspace of $\mathcal{L}(\Omega;\mathsf{X})$, and let $\varphi\colon(\Omega\times\mathsf{X},\mathcal{F}\otimes\mathcal{B}_{\mathsf{X}})\to\overline{\mathbb{R}}$ be an integrand. Then:

(i) $\widetilde{\mathcal{X}}$ is the vector space of equivalence classes of μ-a.e. equal mappings in $\mathcal{X}$.

(ii) The equivalence class in $\widetilde{\mathcal{X}}$ of $x\in\mathcal{X}$ is denoted by $\widetilde{x}$. Conversely, an arbitrary representative in $\mathcal{X}$ of $\widetilde{x}\in\widetilde{\mathcal{X}}$ is denoted by x.

(iii) $\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}\colon\widetilde{\mathcal{X}}\to\overline{\mathbb{R}}\colon\widetilde{x}\mapsto\int_\Omega\varphi(\omega,x(\omega))\mu(d\omega)$.

We shall require the following result. Its item (i) appears in [38, Lemma 4] in the special case when $(\Omega,\mathcal{F},\mu)$ is complete.

Lemma 6.2 *Let $(\Omega,\mathcal{F},\mu)$ be a σ-finite measure space such that $\mu(\Omega)\neq 0$, let $(\mathsf{X},\mathcal{T}_{\mathsf{X}})$ be a Souslin locally convex real topological vector space, and let $(\mathsf{Y},\mathcal{T}_{\mathsf{Y}})$ be a separable locally convex real topological vector space. Suppose that X and Y are placed in separating duality via a bilinear form $\langle\cdot,\cdot\rangle_{\mathsf{X},\mathsf{Y}}\colon\mathsf{X}\times\mathsf{Y}\to\mathbb{R}$ with which $\mathcal{T}_{\mathsf{X}}$ and $\mathcal{T}_{\mathsf{Y}}$ are compatible. Then the following hold:*

(i) *$\langle\cdot,\cdot\rangle_{\mathsf{X},\mathsf{Y}}\colon(\mathsf{X}\times\mathsf{Y},\mathcal{B}_{\mathsf{X}}\otimes\mathcal{B}_{\mathsf{Y}})\to\mathbb{R}$ is measurable.*

(ii) *Let $\mathcal{X}\subset\mathcal{L}(\Omega;\mathsf{X})$ and $\mathcal{Y}\subset\mathcal{L}(\Omega;\mathsf{Y})$ be vector subspaces such that the following are satisfied:*

  (a) $(\forall x\in\mathcal{X})(\forall y\in\mathcal{Y})\ \int_\Omega|\langle x(\omega),y(\omega)\rangle_{\mathsf{X},\mathsf{Y}}|\mu(d\omega)<+\infty$.

  (b) $\bigcup_{\mathsf{x}\in\mathsf{X}}\big\{1_A\mathsf{x}\mid A\in\mathcal{F}\text{ and }\mu(A)<+\infty\big\}\subset\mathcal{X}$.

  (c) $\bigcup_{\mathsf{y}\in\mathsf{Y}}\big\{1_A\mathsf{y}\mid A\in\mathcal{F}\text{ and }\mu(A)<+\infty\big\}\subset\mathcal{Y}$.

Then $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{Y}}$ are in separating duality via the bilinear form $\langle\cdot,\cdot\rangle$ defined by

$$(\forall\widetilde{x}\in\widetilde{\mathcal{X}})(\forall\widetilde{y}\in\widetilde{\mathcal{Y}})\quad\langle\widetilde{x},\widetilde{y}\rangle=\int_\Omega\big\langle x(\omega),y(\omega)\big\rangle_{\mathsf{X},\mathsf{Y}}\mu(d\omega).\tag{6.1}$$

Proof. (i): We deduce from [39, Section 35F, p. 244] that $(\mathsf{X},\mathcal{T}_{\mathsf{X}})$ is a regular Souslin space. On the other hand, since $\mathcal{T}_{\mathsf{Y}}$ and $\mathcal{T}_{\mathsf{X}}$ are compatible with $\langle\cdot,\cdot\rangle_{\mathsf{X},\mathsf{Y}}$, the functions $(\langle\mathsf{x},\cdot\rangle_{\mathsf{X},\mathsf{Y}})_{\mathsf{x}\in\mathsf{X}}$ are $\mathcal{B}_{\mathsf{Y}}$-measurable and the functions $(\langle\cdot,\mathsf{y}\rangle_{\mathsf{X},\mathsf{Y}})_{\mathsf{y}\in\mathsf{Y}}$ are continuous. Hence, Theorem 4.4(iii) implies that $\langle\cdot,\cdot\rangle_{\mathsf{X},\mathsf{Y}}\colon(\mathsf{X}\times\mathsf{Y},\mathcal{B}_{\mathsf{X}}\otimes\mathcal{B}_{\mathsf{Y}})\to\mathbb{R}$ is measurable.

(ii): Note that (i) guarantees that, for every $x\in\mathcal{X}$ and every $y\in\mathcal{Y}$, $\langle x(\cdot),y(\cdot)\rangle_{\mathsf{X},\mathsf{Y}}$ is $\mathcal{F}$-measurable. Now let $\{\mathsf{y}_n\}_{n\in\mathbb{N}}$ be a dense subset of $(\mathsf{Y},\mathcal{T}_{\mathsf{Y}})$ and let $\widetilde{x}\in\widetilde{\mathcal{X}}$ be such that $(\forall\widetilde{y}\in\widetilde{\mathcal{Y}})\ \langle\widetilde{x},\widetilde{y}\rangle=0$. Then, for every $n\in\mathbb{N}$ and every $A\in\mathcal{F}$ such that $\mu(A)<+\infty$, since (ii)(c) ensures that $1_A\mathsf{y}_n\in\mathcal{Y}$, we deduce from (6.1) that $\int_A\langle x(\omega),\mathsf{y}_n\rangle_{\mathsf{X},\mathsf{Y}}\mu(d\omega)=\int_\Omega\langle x(\omega),1_A(\omega)\mathsf{y}_n\rangle_{\mathsf{X},\mathsf{Y}}\mu(d\omega)=0$. Therefore, since $(\Omega,\mathcal{F},\mu)$ is σ-finite, it follows that $(\forall n\in\mathbb{N})\ \langle x(\cdot),\mathsf{y}_n\rangle_{\mathsf{X},\mathsf{Y}}=0$ μ-a.e. Thus $\widetilde{x}=0$. Likewise, $(\forall\widetilde{y}\in\widetilde{\mathcal{Y}})\ \langle\cdot,\widetilde{y}\rangle=0\Rightarrow\widetilde{y}=0$, which completes the proof. ☐

The main result of this section is set in the following environment, which is well defined by virtue of Lemma 6.2.

Assumption 6.3

[A] $(\mathsf{X},\mathcal{T}_{\mathsf{X}})$ is a Souslin locally convex real topological vector space and $(\mathsf{Y},\mathcal{T}_{\mathsf{Y}})$ is a separable locally convex real topological vector space. In addition, X and Y are placed in separating duality via a bilinear form $\langle\cdot,\cdot\rangle_{\mathsf{X},\mathsf{Y}}\colon\mathsf{X}\times\mathsf{Y}\to\mathbb{R}$ with which $\mathcal{T}_{\mathsf{X}}$ and $\mathcal{T}_{\mathsf{Y}}$ are compatible.

[B] $(\Omega, \mathcal{F}, \mu)$ is a σ-finite measure space such that $\mu(\Omega) \neq 0$.

[C] $\mathcal{X} \subset \mathcal{L}(\Omega; \mathsf{X})$ and $\mathcal{Y} \subset \mathcal{L}(\Omega; \mathsf{Y})$ are vector subspaces such that $(\forall x \in \mathcal{X})(\forall y \in \mathcal{Y})$ $\int_\Omega |\langle x(\omega), y(\omega)\rangle_{\mathsf{X},\mathsf{Y}}|\mu(d\omega) < +\infty$. In addition,

$$\mathcal{X} \text{ is compliant and } \bigcup_{\mathsf{y}\in\mathsf{Y}} \{1_A \mathsf{y} \mid A \in \mathcal{F} \text{ and } \mu(A) < +\infty\} \subset \mathcal{Y}. \tag{6.2}$$

[D] $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{Y}}$ are placed in separating duality via the bilinear form $\langle \cdot, \cdot \rangle$ defined by

$$(\forall \widetilde{x} \in \widetilde{\mathcal{X}})(\forall \widetilde{y} \in \widetilde{\mathcal{Y}}) \quad \langle \widetilde{x}, \widetilde{y}\rangle = \int_\Omega \big\langle x(\omega), y(\omega)\big\rangle_{\mathsf{X},\mathsf{Y}} \mu(d\omega), \tag{6.3}$$

and they are equipped with locally convex Hausdorff topologies which are compatible with $\langle \cdot, \cdot \rangle$.

[E] $\varphi \colon (\Omega \times \mathsf{X}, \mathcal{F} \otimes \mathcal{B}_{\mathsf{X}}) \to \left]-\infty, +\infty\right]$ is normal and we write $\varphi^* \colon \Omega \times \mathsf{Y} \to \overline{\mathbb{R}} \colon (\omega, \mathsf{y}) \mapsto \varphi^*_\omega(\mathsf{y})$.

[F] $\operatorname{dom} \mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}} \neq \varnothing$.

Proposition 6.4 *Suppose that Assumption 6.3 holds. Then φ^* is $\mathcal{F} \otimes \mathcal{B}_{\mathsf{Y}}$-measurable.*

Proof. According to Assumption 6.3[E] and Definition 4.3, there exist sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{L}(\Omega; \mathsf{X})$ and $(\varrho_n)_{n\in\mathbb{N}}$ in $\mathcal{L}(\Omega; \mathbb{R})$ such that

$$(\forall \omega \in \Omega) \quad \big\{\big(x_n(\omega), \varrho_n(\omega)\big)\big\}_{n\in\mathbb{N}} \subset \operatorname{epi} \varphi_\omega \quad \text{and} \quad \overline{\operatorname{epi} \varphi_\omega} = \overline{\big\{\big(x_n(\omega), \varrho_n(\omega)\big)\big\}_{n\in\mathbb{N}}}. \tag{6.4}$$

Set

$$(\forall n \in \mathbb{N}) \quad \psi_n \colon \Omega \times \mathsf{Y} \to \mathbb{R} \colon (\omega, \mathsf{y}) \mapsto \langle x_n(\omega), \mathsf{y}\rangle_{\mathsf{X},\mathsf{Y}} - \varrho_n(\omega). \tag{6.5}$$

Then, for every $n \in \mathbb{N}$, Assumption 6.3[A]–[C] and Lemma 6.2(i) ensure that ψ_n is $\mathcal{F} \otimes \mathcal{B}_{\mathsf{Y}}$-measurable. On the other hand, since the functions $(\langle \cdot, \mathsf{y}\rangle_{\mathsf{X},\mathsf{Y}})_{\mathsf{y}\in\mathsf{Y}}$ are continuous, we derive from Assumption 6.3[E], (2.3), and (6.4) that

$$\begin{aligned}
\big(\forall (\omega, \mathsf{y}) \in \Omega \times \mathsf{Y}\big) \quad \varphi^*(\omega, \mathsf{y}) &= \sup_{(\mathsf{x},\xi)\in\operatorname{epi}\varphi_\omega} \big(\langle \mathsf{x}, \mathsf{y}\rangle_{\mathsf{X},\mathsf{Y}} - \xi\big) \\
&= \sup_{(\mathsf{x},\xi)\in\overline{\operatorname{epi}\varphi_\omega}} \big(\langle \mathsf{x}, \mathsf{y}\rangle_{\mathsf{X},\mathsf{Y}} - \xi\big) \\
&= \sup_{n\in\mathbb{N}} \big(\langle x_n(\omega), \mathsf{y}\rangle_{\mathsf{X},\mathsf{Y}} - \varrho_n(\omega)\big) \\
&= \sup_{n\in\mathbb{N}} \psi_n(\omega, \mathsf{y}).
\end{aligned} \tag{6.6}$$

Thus φ^* is $\mathcal{F} \otimes \mathcal{B}_{\mathsf{Y}}$-measurable. ☐

We first investigate the conjugate and the subdifferential of integral functions.

Theorem 6.5 *Suppose that Assumption 6.3 holds. Then the following are satisfied:*

(i) $\mathfrak{I}^*_{\varphi,\widetilde{\mathcal{X}}} = \mathfrak{I}_{\varphi^*,\widetilde{\mathcal{Y}}}$.

(ii) *Suppose that $\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}$ is proper, let $\widetilde{x} \in \widetilde{\mathcal{X}}$, and let $\widetilde{y} \in \widetilde{\mathcal{Y}}$. Then $\widetilde{y} \in \partial \mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}(\widetilde{x}) \Leftrightarrow y(\omega) \in \partial \varphi_\omega(x(\omega))$ for μ-almost every $\omega \in \Omega$.*

Proof. (i): In view of Assumption 6.3[E] and Proposition 6.4, $\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}$ and $\mathfrak{I}_{\varphi^*,\widetilde{\mathcal{Y}}}$ are well defined. Further, there exist sequences $(z_n)_{n\in\mathbb{N}}$ in $\mathcal{L}(\Omega;\mathsf{X})$ and $(\vartheta_n)_{n\in\mathbb{N}}$ in $\mathcal{L}(\Omega;\mathbb{R})$ such that

$$(\forall\omega\in\Omega)\quad \big\{\big(z_n(\omega),\vartheta_n(\omega)\big)\big\}_{n\in\mathbb{N}}\subset\operatorname{epi}\varphi_\omega \quad\text{and}\quad \overline{\operatorname{epi}\varphi_\omega}=\overline{\big\{\big(z_n(\omega),\vartheta_n(\omega)\big)\big\}_{n\in\mathbb{N}}}. \tag{6.7}$$

Let $\widetilde{y}\in\widetilde{\mathcal{Y}}$, define $\psi\colon\Omega\times\mathsf{X}\to\left]-\infty,+\infty\right]:(\omega,\mathsf{x})\mapsto\varphi_\omega(\mathsf{x})-\langle\mathsf{x},y(\omega)\rangle_{\mathsf{X},\mathsf{Y}}$, and note that $(\forall\omega\in\Omega)$ $\operatorname{epi}\psi_\omega\neq\varnothing$. Assumption 6.3[E] and Lemma 6.2(i) imply that

$$\psi\ \text{is}\ \mathcal{F}\otimes\mathcal{B}_{\mathsf{X}}\text{-measurable}. \tag{6.8}$$

Moreover, using the continuity of the linear functionals $(\langle\cdot,\mathsf{y}\rangle_{\mathsf{X},\mathsf{Y}})_{\mathsf{y}\in\mathsf{Y}}$, we derive from (6.7) that

$$\begin{aligned}(\forall\omega\in\Omega)\quad \inf\psi(\omega,\mathsf{X}) &= \inf_{(\mathsf{x},\xi)\in\operatorname{epi}\varphi_\omega}\big(\xi-\langle\mathsf{x},y(\omega)\rangle_{\mathsf{X},\mathsf{Y}}\big)\\ &= \inf_{(\mathsf{x},\xi)\in\overline{\operatorname{epi}\varphi_\omega}}\big(\xi-\langle\mathsf{x},y(\omega)\rangle_{\mathsf{X},\mathsf{Y}}\big)\\ &= \inf_{n\in\mathbb{N}}\big(\vartheta_n(\omega)-\langle z_n(\omega),y(\omega)\rangle\big)\\ &\geqslant \inf_{n\in\mathbb{N}}\big(\varphi_\omega\big(z_n(\omega)\big)-\langle z_n(\omega),y(\omega)\rangle\big)\\ &= \inf_{n\in\mathbb{N}}\psi\big(\omega,z_n(\omega)\big)\\ &\geqslant \inf\psi(\omega,\mathsf{X}).\end{aligned} \tag{6.9}$$

Hence, $(\forall\omega\in\Omega)$ $\inf\psi(\omega,\mathsf{X})=\inf_{n\in\mathbb{N}}\psi(\omega,z_n(\omega))$. Combining this with (6.8), we infer that $\inf\psi(\cdot,\mathsf{X})$ is $\mathcal{F}$-measurable and that ψ fulfills property (ii)(a) in Theorem 1.2 with $(\forall n\in\mathbb{N})$ $x_n=z_n-\overline{x}$. In turn, thanks to Assumption 6.3[B] and the compliance of $\mathcal{X}$, property (ii)(b) in Theorem 1.2 is fulfilled. Thus, by invoking (6.3) and Theorem 1.2, we obtain

$$\begin{aligned}\mathfrak{I}^*_{\varphi,\widetilde{\mathcal{X}}}(\widetilde{y}) &= \sup_{\widetilde{x}\in\widetilde{\mathcal{X}}}\big(\langle\widetilde{x},\widetilde{y}\rangle-\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}(\widetilde{x})\big)\\ &= \sup_{x\in\mathcal{X}}\left(\int_\Omega\big\langle x(\omega),y(\omega)\big\rangle_{\mathsf{X},\mathsf{Y}}\mu(d\omega)-\int_\Omega\varphi\big(\omega,x(\omega)\big)\mu(d\omega)\right)\\ &= -\inf_{x\in\mathcal{X}}\int_\Omega\psi\big(\omega,x(\omega)\big)\mu(d\omega)\\ &= -\int_\Omega\inf_{\mathsf{x}\in\mathsf{X}}\psi(\omega,\mathsf{x})\,\mu(d\omega)\\ &= \int_\Omega\varphi^*_\omega\big(y(\omega)\big)\mu(d\omega),\end{aligned} \tag{6.10}$$

as desired.

(ii): Since the functions $(\varphi_\omega)_{\omega\in\Omega}$ are proper by Assumption 6.3[E], we derive from (2.5), (i), (6.3), and the Fenchel–Young inequality that

$$\begin{aligned}\widetilde{y}\in\partial\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}(\widetilde{x}) &\Leftrightarrow \mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}(\widetilde{x})+\mathfrak{I}_{\varphi^*,\widetilde{\mathcal{Y}}}(\widetilde{y})=\langle\widetilde{x},\widetilde{y}\rangle\\ &\Leftrightarrow \int_\Omega\varphi_\omega\big(x(\omega)\big)\mu(d\omega)+\int_\Omega\varphi^*_\omega\big(y(\omega)\big)\mu(d\omega)=\int_\Omega\big\langle x(\omega),y(\omega)\big\rangle_{\mathsf{X},\mathsf{Y}}\mu(d\omega)\\ &\Leftrightarrow \varphi_\omega\big(x(\omega)\big)+\varphi^*_\omega\big(y(\omega)\big)=\big\langle x(\omega),y(\omega)\big\rangle_{\mathsf{X},\mathsf{Y}}\ \mu\text{-a.e.}\\ &\Leftrightarrow y(\omega)\in\partial\varphi_\omega\big(x(\omega)\big)\ \mu\text{-a.e.},\end{aligned} \tag{6.11}$$

which completes the proof. ☐

A first important consequence of Theorem 6.5(i) is the following.

Proposition 6.6 *Suppose that Assumption 6.3 holds, that* $(\mathsf{Y},\mathcal{T}_{\mathsf{Y}})$ *is a Souslin space, that* $\operatorname{dom}\mathfrak{I}_{\varphi^*,\widetilde{\mathcal{Y}}}\neq\varnothing$*, that* $\mathcal{Y}$ *is compliant, and that* $(\forall\omega\in\Omega)$ $\varphi_\omega\in\Gamma_0(\mathsf{X})$*. Then the following are satisfied:*

(i) $\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}\in\Gamma_0(\widetilde{\mathcal{X}})$.

(ii) *Set* $\operatorname{rec}\varphi\colon\Omega\times\mathsf{X}\to\left]-\infty,+\infty\right]:(\omega,\mathsf{x})\mapsto(\operatorname{rec}\varphi_\omega)(\mathsf{x})$*. Then* $\operatorname{rec}\varphi$ *is* $\mathcal{F}\otimes\mathcal{B}_{\mathsf{X}}$*-measurable and* $\operatorname{rec}\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}=\mathfrak{I}_{\operatorname{rec}\varphi,\widetilde{\mathcal{X}}}$.

Proof. (i): Let $\widetilde{x}\in\widetilde{\mathcal{X}}$ and set

$$\psi\colon\Omega\times\mathsf{Y}\to\left]-\infty,+\infty\right]:(\omega,\mathsf{y})\mapsto\varphi_\omega^*(\mathsf{y})-\langle x(\omega),\mathsf{y}\rangle_{\mathsf{X},\mathsf{Y}}\quad\text{and}\quad\vartheta=\inf\psi(\cdot,\mathsf{Y}).\tag{6.12}$$

By Assumption 6.3[E],

$$\varphi\big(\cdot,x(\cdot)\big)\ \text{is }\mathcal{F}\text{-measurable,}\tag{6.13}$$

while it results from Proposition 6.4 and Lemma 6.2(i) that

$$\psi\ \text{is }\mathcal{F}\otimes\mathcal{B}_{\mathsf{Y}}\text{-measurable.}\tag{6.14}$$

Moreover, for every $\omega\in\Omega$, since $\varphi_\omega\in\Gamma_0(\mathsf{X})$, φ_ω^* is proper and hence $\operatorname{epi}\psi_\omega\neq\varnothing$. On the other hand, the Fenchel–Moreau biconjugation theorem yields

$$(\forall\omega\in\Omega)\quad\vartheta(\omega)=-\varphi_\omega^{**}\big(x(\omega)\big)=-\varphi_\omega\big(x(\omega)\big)\tag{6.15}$$

and it thus follows from (6.13) that ϑ is $\mathcal{F}$-measurable. Now define

$$(\forall n\in\mathbb{N})\quad M_n\colon\Omega\to 2^{\mathsf{Y}}:\omega\mapsto\begin{cases}\big\{\mathsf{y}\in\mathsf{Y}\mid\psi(\omega,\mathsf{y})\leqslant -n\big\}, & \text{if }\vartheta(\omega)=-\infty;\\ \big\{\mathsf{y}\in\mathsf{Y}\mid\psi(\omega,\mathsf{y})\leqslant\vartheta(\omega)+2^{-n}\big\}, & \text{if }\vartheta(\omega)\in\mathbb{R}.\end{cases}\tag{6.16}$$

Fix temporarily $n\in\mathbb{N}$. By (6.14), $\big\{(\omega,\mathsf{y})\mid\mathsf{y}\in M_n(\omega)\big\}\in\mathcal{F}\otimes\mathcal{B}_{\mathsf{Y}}$. Hence, since $(\mathsf{Y},\mathcal{T}_{\mathsf{Y}})$ is a Souslin space, [16, Theorem 5.7] guarantees that there exist $y_n\in\mathcal{L}(\Omega;\mathsf{Y})$ and $B_n\in\mathcal{F}$ such that $\mu(B_n)=0$ and $(\forall\omega\in\complement B_n)$ $y_n(\omega)\in M_n(\omega)$. Now set $B=\bigcup_{n\in\mathbb{N}}B_n$. Then $\mu(B)=0$ and, by virtue of (6.12) and (6.16),

$$\big(\forall\omega\in\complement B\big)(\forall n\in\mathbb{N})\quad\vartheta(\omega)\leqslant\inf_{k\in\mathbb{N}}\psi\big(\omega,y_k(\omega)\big)\leqslant\psi\big(\omega,y_n(\omega)\big)\leqslant\begin{cases}-n, & \text{if }\vartheta(\omega)=-\infty;\\ \vartheta(\omega)+2^{-n}, & \text{if }\vartheta(\omega)\in\mathbb{R}.\end{cases}\tag{6.17}$$

Thus, letting $n\uparrow+\infty$ yields $(\forall\omega\in\complement B)$ $\vartheta(\omega)=\inf_{n\in\mathbb{N}}\psi(\omega,y_n(\omega))$. Consequently, since $\mathcal{Y}$ is compliant, property (ii) in Theorem 1.2 is satisfied. In turn, we deduce from (6.15), Theorem 1.2, (6.3), and Theorem 6.5(i) that

$$\begin{aligned}\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}(\widetilde{x})&=\int_\Omega\varphi\big(\omega,x(\omega)\big)\mu(d\omega)\\&=-\int_\Omega\inf_{\mathsf{y}\in\mathsf{Y}}\psi(\omega,\mathsf{y})\,\mu(d\omega)\\&=-\inf_{y\in\mathcal{Y}}\int_\Omega\psi\big(\omega,y(\omega)\big)\mu(d\omega)\\&=\sup_{y\in\mathcal{Y}}\left(\int_\Omega\big\langle x(\omega),y(\omega)\big\rangle_{\mathsf{X},\mathsf{Y}}\mu(d\omega)-\int_\Omega\varphi_\omega^*\big(y(\omega)\big)\mu(d\omega)\right)\\&=\sup_{\widetilde{y}\in\widetilde{\mathcal{Y}}}\big(\langle\widetilde{x},\widetilde{y}\rangle-\mathfrak{I}^*_{\varphi,\widetilde{\mathcal{X}}}(\widetilde{y})\big)\end{aligned}$$

$$= \mathfrak{I}^{**}_{\varphi,\widetilde{\mathcal{X}}}(\widetilde{x}). \tag{6.18}$$

Thus $\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}} = \mathfrak{I}^{**}_{\varphi,\widetilde{\mathcal{X}}}$ and, since $\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}$ is proper, we conclude that $\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}} \in \Gamma_0(\widetilde{\mathcal{X}})$.

(ii): The normality of φ implies that it is $\mathcal{F} \otimes \mathcal{B}_{\mathsf{X}}$-measurable and that there exists $u \in \mathcal{L}(\Omega;\mathsf{X})$ such that $(\forall \omega \in \Omega)\ u(\omega) \in \operatorname{dom} \varphi_\omega$. Hence, for every $n \in \mathbb{N}$, the function $(\Omega \times \mathsf{X}, \mathcal{F} \otimes \mathcal{B}_{\mathsf{X}}) \to \left]-\infty,+\infty\right] : (\omega,\mathsf{x}) \mapsto \varphi_\omega(u(\omega)+n\mathsf{x}) - \varphi_\omega(u(\omega))$ is measurable. Since, by (2.6),

$$(\forall \omega \in \Omega)(\forall \mathsf{x} \in \mathsf{X}) \quad (\operatorname{rec}\varphi)(\omega,\mathsf{x}) = (\operatorname{rec}\varphi_\omega)(\mathsf{x}) = \lim_{\mathbb{N}\ni n\uparrow+\infty} \frac{\varphi_\omega\big(u(\omega)+n\mathsf{x}\big) - \varphi_\omega\big(u(\omega)\big)}{n}, \tag{6.19}$$

it follows that $\operatorname{rec}\varphi$ is $\mathcal{F}\otimes\mathcal{B}_{\mathsf{X}}$-measurable. Now let $\widetilde{x} \in \widetilde{\mathcal{X}}$ and $\widetilde{z} \in \operatorname{dom}\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}$. Then, for μ-almost every $\omega \in \Omega$, $z(\omega) \in \operatorname{dom}\varphi_\omega$ and it thus follows from the convexity of φ_ω that the function $\theta\colon \left]0,+\infty\right[\to \left]-\infty,+\infty\right] : \alpha \mapsto (\varphi_\omega(z(\omega)+\alpha x(\omega)) - \varphi_\omega(z(\omega)))/\alpha$ is increasing. Thus, appealing to (2.6) and the monotone convergence theorem, we deduce from (i) that

$$\begin{aligned}
\big(\operatorname{rec}\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}\big)(\widetilde{x}) &= \lim_{\alpha\uparrow+\infty} \frac{\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}(\widetilde{z}+\alpha\widetilde{x}) - \mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}(\widetilde{z})}{\alpha} \\
&= \lim_{\alpha\uparrow+\infty} \int_\Omega \frac{\varphi_\omega\big(z(\omega)+\alpha x(\omega)\big) - \varphi_\omega\big(z(\omega)\big)}{\alpha}\mu(d\omega) \\
&= \int_\Omega \lim_{\alpha\uparrow+\infty} \frac{\varphi_\omega\big(z(\omega)+\alpha x(\omega)\big) - \varphi_\omega\big(z(\omega)\big)}{\alpha}\mu(d\omega) \\
&= \int_\Omega (\operatorname{rec}\varphi_\omega)\big(x(\omega)\big)\mu(d\omega),
\end{aligned} \tag{6.20}$$

as claimed. □

Two key ingredients in Hilbertian convex analysis are the Moreau envelope of (2.7) and the proximity operator of (2.9) [1, 19]. To compute them for integral functions, we first observe that, in the case of Hilbert spaces identified with their duals, Assumption 6.3 can be simplified as follows.

Assumption 6.7

[A] X is a separable real Hilbert space with scalar product $\langle\cdot \mid \cdot\rangle_{\mathsf{X}}$, associated norm $\|\cdot\|_{\mathsf{X}}$, and strong topology $\mathcal{T}_{\mathsf{X}}$.

[B] $(\Omega,\mathcal{F},\mu)$ is a σ-finite measure space such that $\mu(\Omega) \neq 0$.

[C] $\mathcal{X} = \big\{x \in \mathcal{L}(\Omega;\mathsf{X}) \mid \int_\Omega \|x(\omega)\|_{\mathsf{X}}^2\,\mu(d\omega) < +\infty\big\}$ and $\widetilde{\mathcal{X}}$ is the usual real Hilbert space $L^2(\Omega;\mathsf{X})$ with scalar product

$$(\forall \widetilde{x} \in \widetilde{\mathcal{X}})(\forall \widetilde{y} \in \widetilde{\mathcal{X}}) \quad \langle \widetilde{x} \mid \widetilde{y}\rangle = \int_\Omega \langle x(\omega) \mid y(\omega)\rangle_{\mathsf{X}}\,\mu(d\omega). \tag{6.21}$$

[D] $\varphi\colon (\Omega\times\mathsf{X}, \mathcal{F}\otimes\mathcal{B}_{\mathsf{X}}) \to \left]-\infty,+\infty\right]$ is a normal integrand such that $(\forall\omega\in\Omega)\ \varphi_\omega \in \Gamma_0(\mathsf{X})$.

[E] $\operatorname{dom}\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}} \neq \varnothing$ and $\operatorname{dom}\mathfrak{I}_{\varphi^*,\widetilde{\mathcal{X}}} \neq \varnothing$.

Proposition 6.8 *Suppose that Assumption 6.7 holds and let $\gamma \in \left]0,+\infty\right[$. Then the following are satisfied:*

(i) *Let $\widetilde{x} \in \widetilde{\mathcal{X}}$ and $\widetilde{p} \in \widetilde{\mathcal{X}}$. Then $\widetilde{p} = \operatorname{prox}_{\gamma\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}}\widetilde{x} \Leftrightarrow p(\omega) = \operatorname{prox}_{\gamma\varphi_\omega}(x(\omega))$ for μ-almost every $\omega \in \Omega$.*

(ii) *Set ${}^{\gamma}\varphi\colon \Omega\times\mathsf{X} \to \left]-\infty,+\infty\right] : (\omega,\mathsf{x}) \mapsto {}^{\gamma}(\varphi_\omega)(\mathsf{x})$. Then ${}^{\gamma}\varphi$ is normal and ${}^{\gamma}\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}} = \mathfrak{I}_{{}^{\gamma}\varphi,\widetilde{\mathcal{X}}}$.*

Proof. Since Assumption 6.7 is an instance of Assumption 6.3, we first infer from Proposition 6.6(i) that $\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}} \in \Gamma_0(\widetilde{\mathcal{X}})$.

(i): We derive from (2.9) and Theorem 6.5(ii) that

$$\begin{aligned}\widetilde{p} = \operatorname{prox}_{\gamma\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}} \widetilde{x} &\Leftrightarrow \widetilde{x} - \widetilde{p} \in \gamma\partial\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}(\widetilde{p}) \\ &\Leftrightarrow x(\omega) - p(\omega) \in \gamma\partial\varphi_\omega\big(p(\omega)\big) \text{ for } \mu\text{-almost every } \omega \in \Omega \\ &\Leftrightarrow p(\omega) = \operatorname{prox}_{\gamma\varphi_\omega} x(\omega) \text{ for } \mu\text{-almost every } \omega \in \Omega. \end{aligned} \tag{6.22}$$

(ii): Since $\mathcal{B}_{\mathsf{X}\times\mathbb{R}} = \mathcal{B}_{\mathsf{X}} \otimes \mathcal{B}_{\mathbb{R}}$, it results from Assumption 6.7[D] and Definition 4.3 that there exists a sequence $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ in $\mathcal{L}(\Omega;\mathsf{X}\times\mathbb{R})$ such that

$$(\forall\omega\in\Omega)\quad \operatorname{epi}\varphi_\omega = \overline{\big\{\boldsymbol{x}_n(\omega)\big\}_{n\in\mathbb{N}}}. \tag{6.23}$$

Set $\mathbf{V} = \big\{(\mathsf{x},\xi)\in\mathsf{X}\times\mathbb{R} \mid \|\mathsf{x}\|_{\mathsf{X}}^2/(2\gamma) < \xi\big\}$. Then $\mathbf{V}$ is open and therefore, for every $\mathbf{C}\subset\mathsf{X}\times\mathbb{R}$, $\mathbf{C}+\mathbf{V} = \overline{\mathbf{C}}+\mathbf{V}$. Thus, we derive from (2.7) and (6.23) that

$$\begin{aligned}(\forall\omega\in\Omega)\quad \big\{(\mathsf{x},\xi)\in\mathsf{X}\times\mathbb{R} \mid {}^{\gamma}(\varphi_\omega)(\mathsf{x}) < \xi\big\} &= \big\{(\mathsf{x},\xi)\in\mathsf{X}\times\mathbb{R} \mid \varphi_\omega(\mathsf{x}) < \xi\big\} + \mathbf{V} \\ &= \overline{\big\{(\mathsf{x},\xi)\in\mathsf{X}\times\mathbb{R} \mid \varphi_\omega(\mathsf{x}) < \xi\big\}} + \mathbf{V} \\ &= \operatorname{epi}\varphi_\omega + \mathbf{V} \\ &= \overline{\big\{\boldsymbol{x}_n(\omega)\big\}_{n\in\mathbb{N}}} + \mathbf{V} \\ &= \big\{\boldsymbol{x}_n(\omega)\big\}_{n\in\mathbb{N}} + \mathbf{V} \\ &= \bigcup_{n\in\mathbb{N}} \big(\boldsymbol{x}_n(\omega) + \mathbf{V}\big). \end{aligned} \tag{6.24}$$

Hence, for every $\mathsf{x}\in\mathsf{X}$ and every $\xi\in\mathbb{R}$, since $(\mathsf{x},\xi)-\mathbf{V}\in\mathcal{B}_{\mathsf{X}\times\mathbb{R}}$ and $\{\boldsymbol{x}_n\}_{n\in\mathbb{N}}\subset\mathcal{L}(\Omega;\mathsf{X}\times\mathbb{R})$, we obtain

$$\big\{\omega\in\Omega \mid {}^{\gamma}(\varphi_\omega)(\mathsf{x}) < \xi\big\} = \left\{\omega\in\Omega \;\middle|\; (\mathsf{x},\xi)\in\bigcup_{n\in\mathbb{N}}\big(\boldsymbol{x}_n(\omega)+\mathbf{V}\big)\right\} = \bigcup_{n\in\mathbb{N}} \boldsymbol{x}_n^{-1}\big((\mathsf{x},\xi)-\mathbf{V}\big) \in \mathcal{F}, \tag{6.25}$$

which shows that $({}^{\gamma}\varphi)(\cdot,\mathsf{x})$ is $\mathcal{F}$-measurable. Hence, since $(\mathsf{X},\mathcal{T}_{\mathsf{X}})$ is a Fréchet space, Theorem 4.4(ii)(b) ensures that ${}^{\gamma}\varphi$ is normal. It remains to show that ${}^{\gamma}\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}} = \mathfrak{I}_{{}^{\gamma}\varphi,\widetilde{\mathcal{X}}}$. Let $\widetilde{x}\in\widetilde{\mathcal{X}}$ and set $\widetilde{p} = \operatorname{prox}_{\gamma\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}}\widetilde{x}$. Then, by (i), for μ-almost every $\omega\in\Omega$, $p(\omega) = \operatorname{prox}_{\gamma\varphi_\omega}(x(\omega))$ and, therefore, (2.8) yields ${}^{\gamma}(\varphi_\omega)(x(\omega)) = \varphi_\omega(p(\omega)) + \|x(\omega)-p(\omega)\|_{\mathsf{X}}^2/(2\gamma)$. Hence

$$\begin{aligned}{}^{\gamma}\mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}(\widetilde{x}) &= \mathfrak{I}_{\varphi,\widetilde{\mathcal{X}}}(\widetilde{p}) + \frac{1}{2\gamma}\|\widetilde{x}-\widetilde{p}\|_{\widetilde{\mathcal{X}}}^2 \\ &= \int_\Omega \varphi_\omega\big(p(\omega)\big)\mu(d\omega) + \frac{1}{2\gamma}\int_\Omega \|x(\omega)-p(\omega)\|_{\mathsf{X}}^2\mu(d\omega) \\ &= \int_\Omega {}^{\gamma}(\varphi_\omega)\big(x(\omega)\big)\mu(d\omega) \\ &= \mathfrak{I}_{{}^{\gamma}\varphi,\widetilde{\mathcal{X}}}(\widetilde{x}), \end{aligned} \tag{6.26}$$

which concludes the proof. ☐

Remark 6.9 Theorem 6.5, Proposition 6.6, and Proposition 6.8 extend the state of the art on several fronts, in particular by removing completeness of $(\Omega,\mathcal{F},\mu)$ when X is infinite-dimensional.

(i) The conclusion of Theorem 6.5(i) first appeared in [28, Theorem 2] in the special case when X is the standard Euclidean space $\mathbb{R}^N$ and $\mathcal{X}$ is Rockafellar-decomposable (see Proposition 4.2(iv) for definition).

(ii) In view of Proposition 4.2(iv) and Theorem 4.4(i)(a), Theorem 6.5 subsumes [29, Theorem 2 and Equation (25)] (see also [30, Theorem 21]), where X is a separable Banach space, $\mathcal{X}$ is Rockafellar-decomposable, and $(\Omega, \mathcal{F}, \mu)$ is complete.

(iii) The conclusion of Theorem 6.5(i) appears in [38] in the special case when $\mathcal{X}$ is Valadier-decomposable (see Proposition 4.2(v) for definition) and $(\Omega, \mathcal{F}, \mu)$ is complete.

(iv) Proposition 6.6(i) subsumes [29, Corollary p. 227], where X is a separable Banach space, $\mathcal{X}$ is Rockafellar-decomposable, and $(\Omega, \mathcal{F}, \mu)$ is complete.

(v) The conclusion of Proposition 6.6(ii) first appeared in [3, Proposition 1] in the context where X is a separable reflexive Banach space, $\mathcal{X}$ is Rockafellar-decomposable, and $(\Omega, \mathcal{F}, \mu)$ is a complete probability space. Another special case is [22, Theorem 2], where $\mathcal{X}$ is Valadier-decomposable and either $X = \mathbb{R}^N$ or $(\Omega, \mathcal{F}, \mu)$ is complete.

(vi) Proposition 6.8(i) appears in [1, Proposition 24.13] in the special case when $(\Omega, \mathcal{F}, \mu)$ is complete, for every $\omega \in \Omega$ $\varphi_\omega = f$, and either $\mu(\Omega) < +\infty$ or $f \geqslant f(0) \geqslant 0$.

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