A Dykstra-like algorithm for two monotone operators

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Abstract

Dykstra's algorithm employs the projectors onto two closed convex sets in a Hilbert space to construct iteratively the projector onto their intersection. In this paper, we use a duality argument to devise an extension of this algorithm for constructing the resolvent of the sum of two maximal monotone operators from the individual resolvents. This result is sharpened to obtain the construction of the proximity operator of the sum of two proper lower semicontinuous convex functions.

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1 Introduction

Throughout this paper $\mathcal{H}$ is a real Hilbert space with scalar product $\langle \cdot \mid \cdot \rangle$ and norm $\|\cdot\|$. The projector onto a nonempty closed convex set $U \subset \mathcal{H}$ is denoted by P_U, and $\to$ denotes strong convergence.

A standard problem in applied mathematics is to find the projection of a point $z \in \mathcal{H}$ onto the intersection of two nonempty closed convex subsets U and V of $\mathcal{H}$. In the case when U and V are vector subspaces, an algorithmic solution to this problem was found in 1933 by von Neumann in the form of the classical alternating projection method (see [12] for historical background and applications in various disciplines).

Theorem 1.1 (von Neumann's algorithm) [23] *Let $z \in \mathcal{H}$, let U and V be closed vector subspaces of $\mathcal{H}$, and set*

$$x_0 = z \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad y_n = P_V x_n \quad \text{and} \quad x_{n+1} = P_U y_n. \tag{1.1}$$

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Then $x_n \to P_{U\cap V} z$ and $y_n \to P_{U\cap V} z$.

Unfortunately, this result fails in two respects when U and V are general intersecting closed convex sets: first, while weak convergence of the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in (1.1) holds [8], strong convergence can fail [16]; second, as simple examples show [9], the limit point need not be the projection of z onto $U \cap V$. In the case when U and V are closed convex cones in a Euclidean space, a modification of the iteration (1.1), proposed by Dykstra in [14] in the form of (1.2) below, provides convergence to $P_{U\cap V} z$. This result was then extended to closed convex sets as follows (for further analysis on this theorem, see [5, 13, 15, 22]).

Theorem 1.2 (Dykstra's algorithm) [7] *Let $z \in \mathcal{H}$, let U and V be closed convex subsets of $\mathcal{H}$ such that $U \cap V \neq \varnothing$, and set*

$$\begin{cases} x_0 = z \\ p_0 = 0 \\ q_0 = 0 \end{cases} \quad \textit{and} \quad (\forall n \in \mathbb{N}) \quad \begin{cases} y_n = P_V(x_n + p_n) \\ p_{n+1} = x_n + p_n - y_n \end{cases} \quad \textit{and} \quad \begin{cases} x_{n+1} = P_U(y_n + q_n) \\ q_{n+1} = y_n + q_n - x_{n+1}. \end{cases} \tag{1.2}$$

Then $x_n \to P_{U\cap V} z$ and $y_n \to P_{U\cap V} z$.

The objective of this paper is to propose a generalization of Theorem 1.2 for finding the resolvent of the sum of two maximal monotone operators. This generalization is proposed in Section 2. In the case of subdifferentials, our results are sharpened in Section 3, where we provide an algorithm for finding the proximity operator of the sum of two proper lower semicontinuous convex functions. This analysis captures in particular Theorem 1.2.

2 The resolvent of the sum of two monotone operators

We first recall some basic notation and definitions.

Notation 2.1 Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued operator. Then $\operatorname{dom} A = \big\{x \in \mathcal{H} \mid Ax \neq \varnothing\big\}$ is the domain of A, $\operatorname{ran} A = \big\{u \in \mathcal{H} \mid (\exists\, x \in \mathcal{H})\ u \in Ax\big\}$ is its range, $\operatorname{zer} A = \big\{x \in \mathcal{H} \mid 0 \in Ax\big\}$ is its set of zeros, and $\operatorname{gra} A = \big\{(x,u) \in \mathcal{H}\times\mathcal{H} \mid u \in Ax\big\}$ is its graph. The inverse of A is the operator $A^{-1}\colon \mathcal{H} \to 2^{\mathcal{H}}$ with graph $\big\{(u,x) \in \mathcal{H}\times\mathcal{H} \mid u \in Ax\big\}$ and the resolvent of A is $J_A = (\operatorname{Id} + A)^{-1}$. We set

$$A^{\sim}\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto -A^{-1}(-x). \tag{2.1}$$

Now, suppose that A is monotone, i.e., for every (x,u) and (y,v) in $\operatorname{gra} A$,

$$\langle x - y \mid u - v\rangle \geq 0. \tag{2.2}$$

Then $J_A\colon \operatorname{ran}(\operatorname{Id} + A) \to \mathcal{H}$ is single-valued. Moreover, A is declared maximal monotone when the following property is satisfied for every $(x,u) \in \mathcal{H}\times\mathcal{H}$: if (2.2) holds for every $(y,v) \in \operatorname{gra} A$, then

$(x,u) \in \operatorname{gra} A$. Minty's theorem states that A is maximal monotone if and only if $\operatorname{ran}(\operatorname{Id}+A) = \mathcal{H}$. In this case, we have

$$J_{A^{-1}} = \operatorname{Id} - J_A \quad \text{and} \quad (J_{A^{-1}})^{\sim} = \operatorname{Id} + A^{\sim}. \tag{2.3}$$

Finally, the strong relative interior of a convex subset C of $\mathcal{H}$ is

$$\operatorname{sri} C = \left\{ x \in C \;\middle|\; \bigcup_{\lambda > 0} \lambda(C - x) = \overline{\operatorname{span}(C - x)} \right\}. \tag{2.4}$$

For background on monotone operators, see [4] and [21].

Let C and D be maximal monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$. As discussed in [3], pairing the inclusion $0 \in Cx + Dx$ with the dual inclusion $0 \in C^{-1}u + D^{\sim}u$ brings useful insights into the analysis of various problems in nonlinear analysis. This approach relies on the simple equivalence

$$\operatorname{zer}(C + D) \neq \varnothing \quad \Leftrightarrow \quad \operatorname{zer}(C^{-1} + D^{\sim}) \neq \varnothing. \tag{2.5}$$

In [6], this duality framework proved particularly useful in the investigation of the asymptotic behavior of the composition of two resolvents. Some of these results will be instrumental in the present paper.

Proposition 2.2 *Let C and D be maximal monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$. Then* $\operatorname{zer}(C + \operatorname{Id} - J_D) \neq \varnothing$ *if and only if $J_{C^{-1}+D^{\sim}} 0$ exists, in which case* $\operatorname{zer}(C^{-1} + \operatorname{Id} + D^{\sim}) = \{J_{C^{-1}+D^{\sim}} 0\}$.

Proof. By definition, $J_{C^{-1}+D^{\sim}} 0$ is the unique solution to the inclusion $0 \in (\operatorname{Id} + C^{-1} + D^{\sim})u$, i.e., the unique point in $\operatorname{zer}(C^{-1} + \operatorname{Id} + D^{\sim})$. On the other hand, we derive from (2.5) applied to the maximal monotone operators C and $J_{D^{-1}}$, and from (2.3) that

$$\operatorname{zer}(C + \operatorname{Id} - J_D) \neq \varnothing \quad \Leftrightarrow \quad \operatorname{zer}(C^{-1} + (J_{D^{-1}})^{\sim}) \neq \varnothing \quad \Leftrightarrow \quad \operatorname{zer}(C^{-1} + \operatorname{Id} + D^{\sim}) \neq \varnothing. \tag{2.6}$$

The proof is now complete. $\square$

Theorem 2.3 *Let C and D be maximal monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$, and set*

$$u_0 \in \mathcal{H} \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad v_n = J_D u_n \quad \text{and} \quad u_{n+1} = J_C v_n. \tag{2.7}$$

Then the following hold.

(i) *Suppose that* $\operatorname{zer}(C + \operatorname{Id} - J_D) \neq \varnothing$. *Then $v_n - u_n \to J_{C^{-1}+D^{\sim}} 0$ and $v_n - u_{n+1} \to J_{C^{-1}+D^{\sim}} 0$.*

(ii) *Suppose that* $\operatorname{zer}(C + \operatorname{Id} - J_D) = \varnothing$. *Then $\|u_n\| \to +\infty$ and $\|v_n\| \to +\infty$.*

Proof. (i): [6, Theorem 3.3]. (ii): [6, Theorem 3.5]. $\square$

The main result of this section is the following theorem on the asymptotic behavior of the extension of (1.2) from projectors onto closed convex sets to resolvents of maximal monotone operators.

Theorem 2.4 *Let $z \in \mathcal{H}$, let A and B be maximal monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$, and set*

$$\begin{cases} x_0 = z \\ p_0 = 0 \\ q_0 = 0 \end{cases} \quad \textit{and} \quad (\forall n \in \mathbb{N}) \quad \begin{cases} y_n = J_B(x_n + p_n) \\ p_{n+1} = x_n + p_n - y_n \end{cases} \quad \textit{and} \quad \begin{cases} x_{n+1} = J_A(y_n + q_n) \\ q_{n+1} = y_n + q_n - x_{n+1}. \end{cases} \tag{2.8}$$

Then the following hold.

(i) *Suppose that $z \in \operatorname{ran}(\operatorname{Id} + A + B)$. Then $x_n \to J_{A+B}\, z$ and $y_n \to J_{A+B}\, z$.*

(ii) *Suppose that $z \notin \operatorname{ran}(\operatorname{Id} + A + B)$. Then $\|p_n\| \to +\infty$ and $\|q_n\| \to +\infty$.*

Proof. It follows from (2.8) that

$$(\forall n \in \mathbb{N}) \quad p_{n+1} + q_n + y_n = x_n + p_n - y_n + q_n + y_n = x_n + p_n + q_n. \tag{2.9}$$

On the other hand, $(\forall n \in \mathbb{N})$ $p_n + q_n = z - x_n$. Indeed, in view of (2.8), this identity is certainly true for $n = 0$ and, if $p_n + q_n = z - x_n$ for some $n \in \mathbb{N}$, then $p_{n+1} + q_{n+1} = x_n + p_n - y_n + y_n + q_n - x_{n+1} = z - x_{n+1}$. Altogether, we have

$$(\forall n \in \mathbb{N}) \quad z = p_{n+1} + q_n + y_n = p_n + q_n + x_n \tag{2.10}$$

and we can therefore rewrite (2.8) as

$$\begin{cases} x_0 = z \\ p_0 = 0 \\ q_0 = 0 \end{cases} \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad \begin{cases} y_n = J_B(z - q_n) \\ p_{n+1} = z - q_n - y_n \end{cases} \quad \text{and} \quad \begin{cases} x_{n+1} = J_A(z - p_{n+1}) \\ q_{n+1} = z - p_{n+1} - x_{n+1}. \end{cases} \tag{2.11}$$

Now set

$$u_0 = -z \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad u_n = q_n - z \quad \text{and} \quad v_n = -p_{n+1}. \tag{2.12}$$

We infer from (2.12) and (2.10) that

$$(\forall n \in \mathbb{N}) \quad v_n - u_n = -p_{n+1} - q_n + z = y_n \quad \text{and} \quad v_n - u_{n+1} = -p_{n+1} - q_{n+1} + z = x_{n+1}. \tag{2.13}$$

Furthermore, it follows from (2.12) and (2.11) that the sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are coupled via the equations

$$\begin{aligned} (\forall n \in \mathbb{N}) \quad v_n &= -p_{n+1} = q_n - z + y_n = u_n + J_B(-u_n) \\ \text{and} \quad u_{n+1} &= q_{n+1} - z = -p_{n+1} - x_{n+1} = v_n - J_A(v_n + z). \end{aligned} \tag{2.14}$$

Now define two maximal monotone operators by

$$C \colon \mathcal{H} \to 2^{\mathcal{H}} \colon v \mapsto A^{-1}(v + z) \quad \text{and} \quad D = B^{\sim}. \tag{2.15}$$

It is easily seen that

$$C^{-1} = -z + A \quad \text{and} \quad D^{\sim} = B. \tag{2.16}$$

Moreover, (2.3) yields

$$(\forall u \in \mathcal{H})(\forall v \in \mathcal{H}) \quad u = v - J_A(v+z) \;\Leftrightarrow\; u + z = v + z - J_A(v+z) = J_{A^{-1}}(v+z)$$
$$\Leftrightarrow\; v - u \in A^{-1}(u+z) = Cu$$
$$\Leftrightarrow\; u = J_C v \tag{2.17}$$

and

$$(\forall u \in \mathcal{H}) \quad u + J_B(-u) = -(-u - J_B(-u)) = -J_{B^{-1}}(-u) = J_D u. \tag{2.18}$$

Thus, we can rewrite (2.14) as

$$(\forall n \in \mathbb{N}) \quad v_n = J_D u_n \quad \text{and} \quad u_{n+1} = J_C v_n, \tag{2.19}$$

which is precisely the iteration described in (2.7) with initial point $u_0 = -z$. On the other hand, it follows from (2.16) that, for every $x \in \mathcal{H}$,

$$x = J_{A+B}\, z \;\Leftrightarrow\; 0 \in x + (-z + Ax + Bx) \;\Leftrightarrow\; x = J_{(-z+A)+B}\, 0 = J_{C^{-1}+D^{\sim}}\, 0. \tag{2.20}$$

Hence,

$$z \in \operatorname{ran}(\operatorname{Id} + A + B) \;\Leftrightarrow\; z \in \operatorname{dom}(\operatorname{Id} + A + B)^{-1}$$
$$\Leftrightarrow\; J_{A+B}\, z \text{ exists}$$
$$\Leftrightarrow\; J_{C^{-1}+D^{\sim}}\, 0 \text{ exists}. \tag{2.21}$$

Therefore, we deduce from Proposition 2.2 and Theorem 2.3 the following.

(i): If $z \in \operatorname{ran}(\operatorname{Id} + A + B)$, then (2.13) yields $y_n = v_n - u_n \to J_{C^{-1}+D^{\sim}}\, 0 = J_{A+B}\, z$ and $x_{n+1} = v_n - u_{n+1} \to J_{C^{-1}+D^{\sim}}\, 0 = J_{A+B}\, z$.

(ii): If $z \notin \operatorname{ran}(\operatorname{Id} + A + B)$, then (2.12) yields $\|p_{n+1}\| = \|v_n\| \to +\infty$ and $\|q_n\| = \|u_n + z\| \geq \|u_n\| - \|z\| \to +\infty$. □

Remark 2.5 Suppose that in Theorem 2.4 we make the additional assumption that $A+B$ is maximal monotone, as is true when $0 \in \operatorname{sri}(\operatorname{dom} A - \operatorname{dom} B)$; see [2, Corollaire 1] or [21, Theorem 23.2]. Then item (ii) never occurs in Theorem 2.4 and therefore $(\forall z \in \mathcal{H})\ x_n \to J_{A+B}\, z$ and $y_n \to J_{A+B}\, z$.

3 The proximity operator of the sum of two convex functions

In this section, we turn our attention to the intermediate situation between (1.2) and (2.11) in which proximity operators are used.

Notation 3.1 A function $f\colon \mathcal{H} \to\ ]-\infty, +\infty]$ is proper if $\operatorname{dom} f = \{x \in \mathcal{H} \mid f(x) < +\infty\} \neq \varnothing$. The class of proper lower semicontinuous convex functions from $\mathcal{H}$ to $]-\infty, +\infty]$ is denoted by

$\Gamma_0(\mathcal{H})$. Now let $f \in \Gamma_0(\mathcal{H})$. The conjugate of f is the function $f^* \in \Gamma_0(\mathcal{H})$ defined by $f^* \colon u \mapsto \sup_{x\in\mathcal{H}} \langle x \mid u\rangle - f(x)$, the subdifferential of f is the maximal monotone operator

$$\partial f \colon \mathcal{H} \to 2^{\mathcal{H}} \colon x \mapsto \big\{u \in \mathcal{H} \mid (\forall y \in \mathcal{H}) \quad \langle y - x \mid u\rangle + f(x) \leq f(y)\big\}, \tag{3.1}$$

the set of minimizers of f is $\operatorname{argmin} f = \operatorname{zer} \partial f$, the Moreau envelope of f is the continuous convex function

$$\operatorname{env} f \colon \mathcal{H} \to \mathbb{R} \colon x \mapsto \inf_{y\in\mathcal{H}} f(y) + \frac{1}{2}\|x - y\|^2, \tag{3.2}$$

and the reflection of f is the function $f^\vee \colon x \mapsto f(-x)$. For every $x \in \mathcal{H}$, the function $y \mapsto f(y) + \|x-y\|^2/2$ admits a unique minimizer, which is denoted by $\operatorname{prox}_f x$. Alternatively, $\operatorname{prox}_f = J_{\partial f}$. See [19, 24] for background on convex analysis, and [11, 18, 20] for background on proximity operators.

We require an additional result from [6], which sharpens Theorem 2.3.

Theorem 3.2 *Let φ and ψ be functions in $\Gamma_0(\mathcal{H})$ such that*

$$\inf\,(\varphi + \operatorname{env}\psi)(\mathcal{H}) > -\infty, \tag{3.3}$$

and set

$$u_0 \in \mathcal{H} \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad v_n = \operatorname{prox}_\psi u_n \quad \text{and} \quad u_{n+1} = \operatorname{prox}_\varphi v_n. \tag{3.4}$$

Then the following hold.

(i) *The function $\varphi^* + \psi^{*\vee} + \|\cdot\|^2/2$ admits a unique minimizer w. Moreover, $v_n - u_n \to w$ and $v_n - u_{n+1} \to w$.*

(ii) *Suppose that $\operatorname{argmin} \varphi + \operatorname{env}\psi = \varnothing$. Then $\|u_n\| \to +\infty$ and $\|v_n\| \to +\infty$.*

Proof. [6, Theorem 4.6]. ☐

Theorem 3.3 *Let $z \in \mathcal{H}$, let f and g be functions in $\Gamma_0(\mathcal{H})$ such that*

$$\operatorname{dom} f \cap \operatorname{dom} g \neq \varnothing, \tag{3.5}$$

and set

$$\begin{cases} x_0 = z \\ p_0 = 0 \\ q_0 = 0 \end{cases} \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad \begin{cases} y_n = \operatorname{prox}_g(x_n + p_n) \\ p_{n+1} = x_n + p_n - y_n \end{cases} \quad \text{and} \quad \begin{cases} x_{n+1} = \operatorname{prox}_f(y_n + q_n) \\ q_{n+1} = y_n + q_n - x_{n+1}. \end{cases} \tag{3.6}$$

Then the following hold.

(i) *$x_n \to \operatorname{prox}_{f+g} z$ and $y_n \to \operatorname{prox}_{f+g} z$.*

(ii) *Suppose that $\operatorname{argmin} f^*(\cdot + z) + \operatorname{env} g^{*\vee} = \varnothing$. Then $\|p_n\| \to +\infty$ and $\|q_n\| \to +\infty$.*

Proof. Set $A = \partial f$ and $B = \partial g$. Then $J_A = \operatorname{prox}_f$, $J_B = \operatorname{prox}_g$, and (3.6) is therefore a special case of (2.8). Let us set, as in (2.12),

$$u_0 = -z \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad u_n = q_n - z \quad \text{and} \quad v_n = -p_{n+1}. \tag{3.7}$$

Then we obtain, as in (2.13),

$$(\forall n \in \mathbb{N}) \quad v_n - u_n = y_n \quad \text{and} \quad v_n - u_{n+1} = x_{n+1} \tag{3.8}$$

and, as in (2.14),

$$(\forall n \in \mathbb{N}) \quad v_n = u_n + \operatorname{prox}_g(-u_n) \quad \text{and} \quad u_{n+1} = v_n - \operatorname{prox}_f(v_n + z). \tag{3.9}$$

Now define two functions in $\Gamma_0(\mathcal{H})$ by

$$\varphi\colon \mathcal{H} \to \left]-\infty, +\infty\right] : v \mapsto f^*(v+z) - \frac{1}{2}\|z\|^2 \quad \text{and} \quad \psi = g^{*\vee}. \tag{3.10}$$

Then

$$\varphi^* = f - \langle \cdot \mid z\rangle + \frac{1}{2}\|z\|^2, \ \psi^* = g^\vee, \text{ and } (\operatorname{env}\psi)^* = g^\vee + \frac{1}{2}\|\cdot\|^2. \tag{3.11}$$

Therefore

$$\begin{aligned}(\forall x \in \mathcal{H}) \quad \varphi^*(x) + \psi^{*\vee}(x) + \frac{1}{2}\|x\|^2 &= f(x) - \langle x \mid z\rangle + \frac{1}{2}\|z\|^2 + g(x) + \frac{1}{2}\|x\|^2 \\ &= (f+g)(x) + \frac{1}{2}\|z - x\|^2. \end{aligned} \tag{3.12}$$

Moreover, if we set $C = \partial\varphi$ and $D = \partial\psi$, it results from (3.10) that

$$(\forall v \in \mathcal{H}) \quad Cv = \partial f^*(v+z) = A^{-1}(v+z) \quad \text{and} \quad Dv = -\partial g^*(-v) = B^{\sim}v. \tag{3.13}$$

Hence, (2.17) and (2.18) yield

$$(\forall v \in \mathcal{H}) \quad \operatorname{prox}_\varphi v = v - \operatorname{prox}_f(v+z) \quad \text{and} \quad \operatorname{prox}_\psi v = v + \operatorname{prox}_g(-v), \tag{3.14}$$

which shows that (3.9) reduces to the iterative scheme (3.4) initialized with $u_0 = -z$. On the other hand, (3.10) yields

$$\varphi + \operatorname{env}\psi = f^*(\cdot + z) - \frac{1}{2}\|z\|^2 + \operatorname{env} g^{*\vee}. \tag{3.15}$$

It therefore follows from (3.11) and Fenchel duality that

$$\begin{aligned}(3.5) \quad &\Leftrightarrow \quad f + g \in \Gamma_0(\mathcal{H}) \\ &\Rightarrow \quad \operatorname{prox}_{f+g} z \text{ exists} \\ &\Leftrightarrow \quad \operatorname{argmin}\, f + g + \frac{1}{2}\|z - \cdot\|^2 \neq \varnothing \\ &\Leftrightarrow \quad \operatorname{argmin} f - \langle \cdot \mid z\rangle + \frac{1}{2}\|z\|^2 + g + \frac{1}{2}\|\cdot\|^2 \neq \varnothing \\ &\Leftrightarrow \quad \operatorname{argmin} \varphi^* + (\operatorname{env}\psi)^{*\vee} \neq \varnothing \\ &\Leftrightarrow \quad \inf\big(\varphi + \operatorname{env}\psi\big)(\mathcal{H}) > -\infty \\ &\Leftrightarrow \quad (3.3). \end{aligned} \tag{3.16}$$

We are now in a position to draw the following conclusions.

(i): It view of (3.12), the minimizer of $\varphi^* + \psi^{*\vee} + \|\cdot\|^2/2$ is $w = \operatorname{prox}_{f+g} z$. Consequently, Theorem 3.2(i) and (3.8) yield $y_n = v_n - u_n \to \operatorname{prox}_{f+g} z$ and $x_{n+1} = v_n - u_{n+1} \to \operatorname{prox}_{f+g} z$.

(ii): It follows from (3.15) that $\operatorname{argmin} f^*(\cdot + z) + \operatorname{env} g^{*\vee} = \varnothing \Rightarrow \operatorname{argmin} \varphi + \operatorname{env} \psi = \varnothing$. In turn, it results from Theorem 3.2(ii) and (3.7) that $\|p_{n+1}\| = \|v_n\| \to +\infty$ and $\|q_n\| = \|u_n + z\| \geq \|u_n\| - \|z\| \to +\infty$. $\square$

Remark 3.4 Theorem 3.3 is sharper than Theorem 2.4 applied to $A = \partial f$ and $B = \partial g$. Indeed, since $\partial f + \partial g \subset \partial(f+g)$, we have, for every $p \in \mathcal{H}$,

$$p = J_{\partial f + \partial g} z \Leftrightarrow z - p \in \partial f(p) + \partial g(p) \Rightarrow z - p \in \partial(f+g)(p) \Leftrightarrow p = \operatorname{prox}_{f+g} z. \tag{3.17}$$

As a result, via Theorem 2.4(i), we obtain

$$x_n \to \operatorname{prox}_{f+g} z \quad \text{and} \quad y_n \to \operatorname{prox}_{f+g} z \tag{3.18}$$

provided that

$$z \in \operatorname{ran}(\operatorname{Id} + \partial f + \partial g). \tag{3.19}$$

A standard sufficient condition for this inclusion to hold for every $z \in \mathcal{H}$ is

$$0 \in \operatorname{sri}(\operatorname{dom} f - \operatorname{dom} g), \tag{3.20}$$

see [1] or [24, Theorem 2.8.7]. On the other hand, in Theorem 3.3(i), we obtain (3.18) for every $z \in \mathcal{H}$ with merely (3.5), i.e.,

$$0 \in (\operatorname{dom} f - \operatorname{dom} g). \tag{3.21}$$

Since (3.21) is less restrictive than (3.19), it is clear that Theorem 3.3 is sharper than Theorem 2.4 in the present subdifferential operator setting.

Let us note that, under condition (3.20), an alternative method for computing $p = \operatorname{prox}_{f+g} z$ for an arbitrary $z \in \mathcal{H}$ is the Douglas-Rachford algorithm for computing a zero of the sum of two maximal monotone operators [10, 17]. Indeed, it follows from (3.20) that p is characterized by the inclusion $0 \in \partial(f+g)(p) = \partial f(p) + \partial g(p) + p - z = Cp + Dp$, where $C = -z + \partial f$ and $D = \operatorname{Id} + \partial g$ are maximal monotone.

Remark 3.5 Let f and g be the indicator functions of closed convex subsets U and V of $\mathcal{H}$, respectively. Then Theorem 3.3(i) reduces to Theorem 1.2. If we further assume that U and V are closed vector subspaces, then we obtain Theorem 1.1 since in this case (1.2) yields $(\forall n \in \mathbb{N})$ $y_n = P_V(x_n + p_n) = P_V x_n + P_V p_n = P_V x_n$ and $x_{n+1} = P_U(y_n + q_n) = P_U y_n + P_U q_n = P_U y_n$.

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