L_1 *techniques in control problems*

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l_1-optimization

Key idea:

$$\text{minimize 1-norm of a vector} \implies \text{zero components.}$$

Simplest result: **Lemma.** *There exist a solution x^* of optimization problem*

$$\|x\|_1 \longrightarrow \min \quad \text{s.t.} \quad Ax = b,$$

with $A \in \mathbb{R}^{m\times n}$, $m < n$, with $\leq m$ nonzero components.

l_1 everywhere!

- Regression (Lasso)
- Optimization (Exact penalties, basis pursuit)
- Estimation (Least absolute values)
- Signal and image processing (Compressed Sensing)
- Classification and recognition (SVM)
- and beyond... Titles like "L_1-revolution".

l_1 in control

3 directions of research:

- Optimal control with l_1 performance index or constraints
- l_1-filtering
- Reducing the number of controls, states, outputs.

But there are many other applications of l_1 techniques in control.

Optimal control

We focus on discrete-time case. Simplest example is linear system with l_1 performance index.

$$x_k \in \mathbf{R}^n, \quad u_k \in \mathbf{R}^l, \quad x_0 = 0, \quad d \in \mathbf{R}^m$$

$$x_{k+1} = Ax_k + Bu_k, \quad Cx_N = d$$

$$\min ||u||_1 = \sum_{k=0}^{N-1} \sum_{i=1}^{l} |u_k^i|.$$

Here N is fixed, matrices A, B, C and vector d are known.

Assumptions — pair (A, B) is controllable, rank C equals m, $N \geq m$.

Properties of the solution

Theorem The solution u^* of the above problem with no more than m nonzero entries exists.

Proof. $x_N = Bu_{N-1} + ABu_{N-2} + ... + A^{N-1}Bu_0$, thus the basic problem is equivalent to l_1-optimization problem

$$\min ||u||_1, \quad CHu = d, \quad H = [B|AB|...|A^{N-1}B]$$

Matrix CH has rank m under assumptions, hence we are in the framework of the main lemma.

Explicit solution: the simplest case $l = 1, m = 1$ (scalar control, target set is a hyperplane). Then we find $i = argmax_k |CA^{N-k-1}B|, u_i^* = \frac{d}{CA^{N-i-1}B}, u_k^* = 0, k \neq i.$

Solution for $n = m = 2, l = 1$

Scalar control, $2D$ state, terminal point fixed:

$$\min ||u||_1 = \sum_{k=0}^{N-1} |u_k|$$

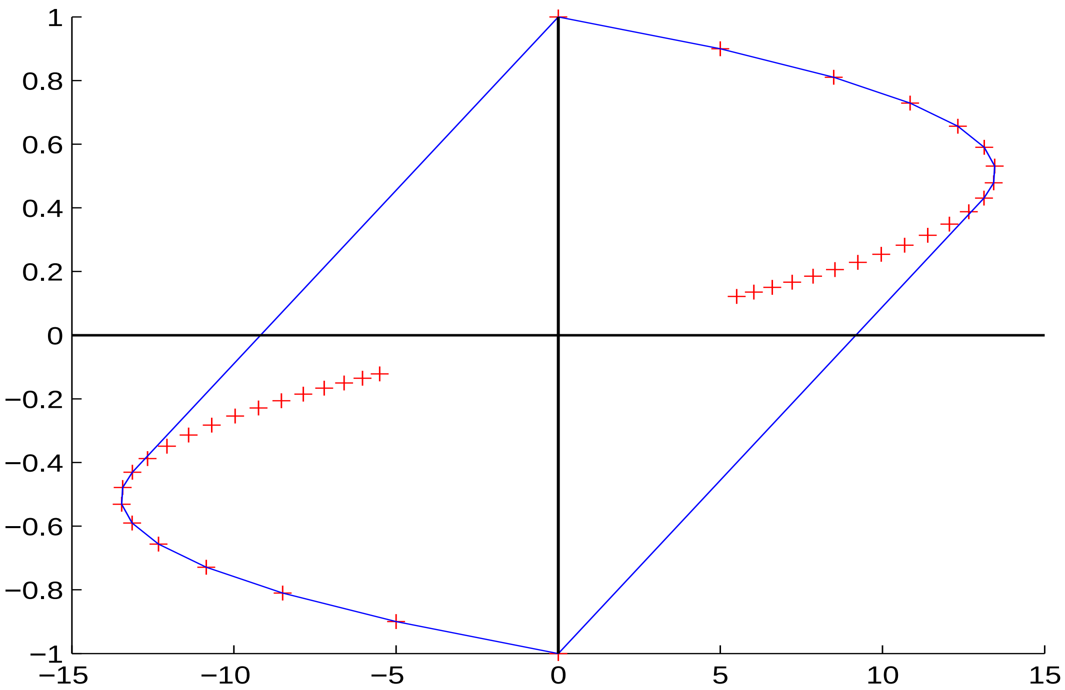
$$x_{k+1} = Ax_k + Bu_k, \quad x_0 = 0, \quad x_N = d.$$

Construct vectors $s_{i+1} = A^i B, \quad i = 0, ..., N-1, \quad s_{N+i+1} = -A^i B, \quad i = 0, ..., N-1,$ on the plane and their convex hull $S_N = conv(s_i, i = 1, ..., 2N)$. This is the attainable set S_N (for unit ball constraints). Intersection of the ray λd with this set defines the optimal value $||u^*||_1 = \min\{1/\lambda : \lambda d \in S_N\}$; the optimal control u^* can be easily constructed as well. It contains no more than 2 nonzero entries.

It is convenient to use codes *convhull*, *convhulln* in Matlab.

Example

$$A = \begin{pmatrix} 0.8 & 5 \\ 0 & 0.9 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x_{20} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



x are for $\pm A^k B$, blue line is a convex hull of these points (attainable set).

Matrix A is stable, the solution u^* for $N \geq 9$ does not depend on N :

$$||u^*||_1 = 0.1089, u^*_{N-1} = -0.0328, u^*_{N-9} = 0.0761.$$

Another example

D.Tabak, B.C.Kuo, Optimal control by mathematical programming, Prentice-Hall, 1971. Section 5.5 "Fuel-optimal rendezvous problem".

Space flight, 4 states, 2 controls, fixed terminal point: $n = 4, l = 2, m = 4$. Optimal solution found by LP, it has 4 nonzero impulses.

We have solved many similar examples.

Related problems

1. Weighted norm $||u||_1 = \sum_k \alpha_k |u_k|, \quad \alpha_k > 0$ — all results hold true.

2. $u_k \geq 0, \min \sum_k \alpha_k u_k$ — similar approach.

3. No terminal constraints, $||u||_1 \leq 1, \min F(x_N), F$ is a concave function. The solution is attained at a vertex of attainable set and contains one impulse.

4. Minimum-time problem: control is bounded $||u||_1 \leq r$, terminal point x_N is fixed, find minimal N. The problem can have no solution (e.g. if A is stable, and r small enough). If optimal solution exists, it has no more n nonzero components.

Comparison with l_∞ constrained control: the optimal solution always exists and bang-bang principle holds.

More complicated problems

Mixed performance index:

$$\min \left(\sum_k [(Px_k, x_k) + (Qu_k, u_k)] + \mu ||u||_1 \right)$$

$$x_{k+1} = Ax_k + Bu_k.$$

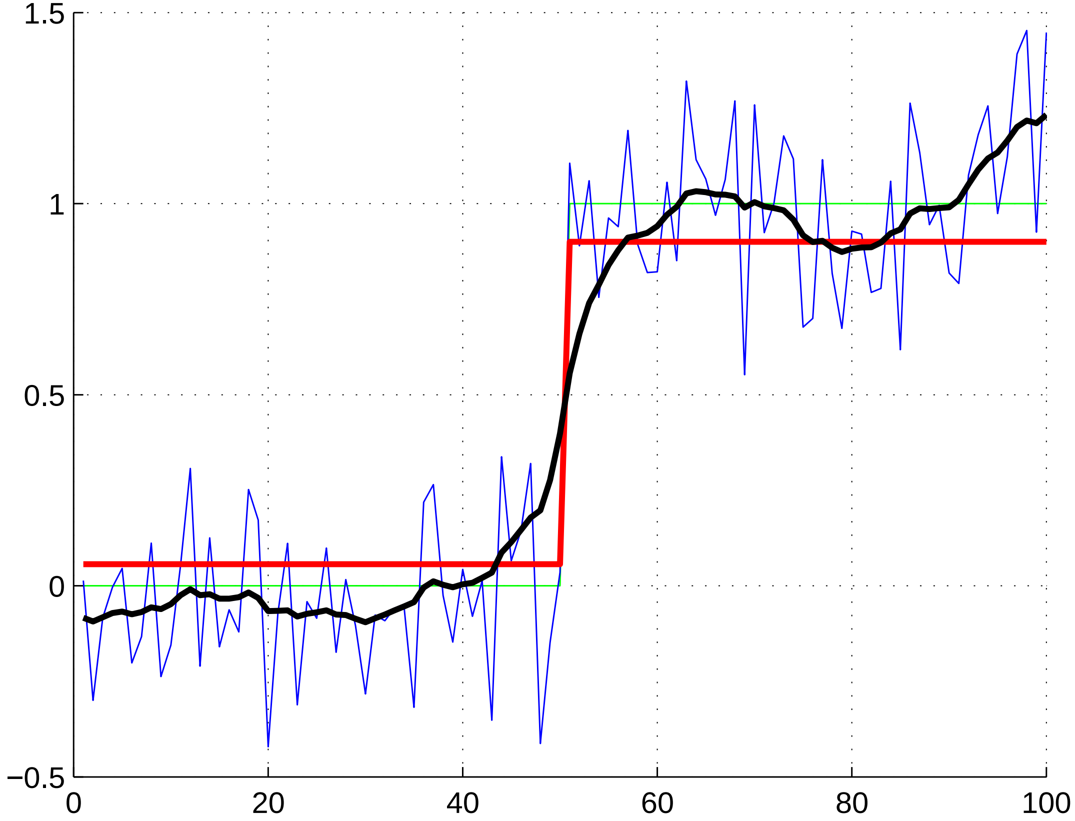
The number of nonzero variables depends on $\mu > 0$. Similar example will be discussed later (l_1-filtering).

Another cases — mixed l_1 and l_∞ control constraints or state constraints.

l_1-filterimg

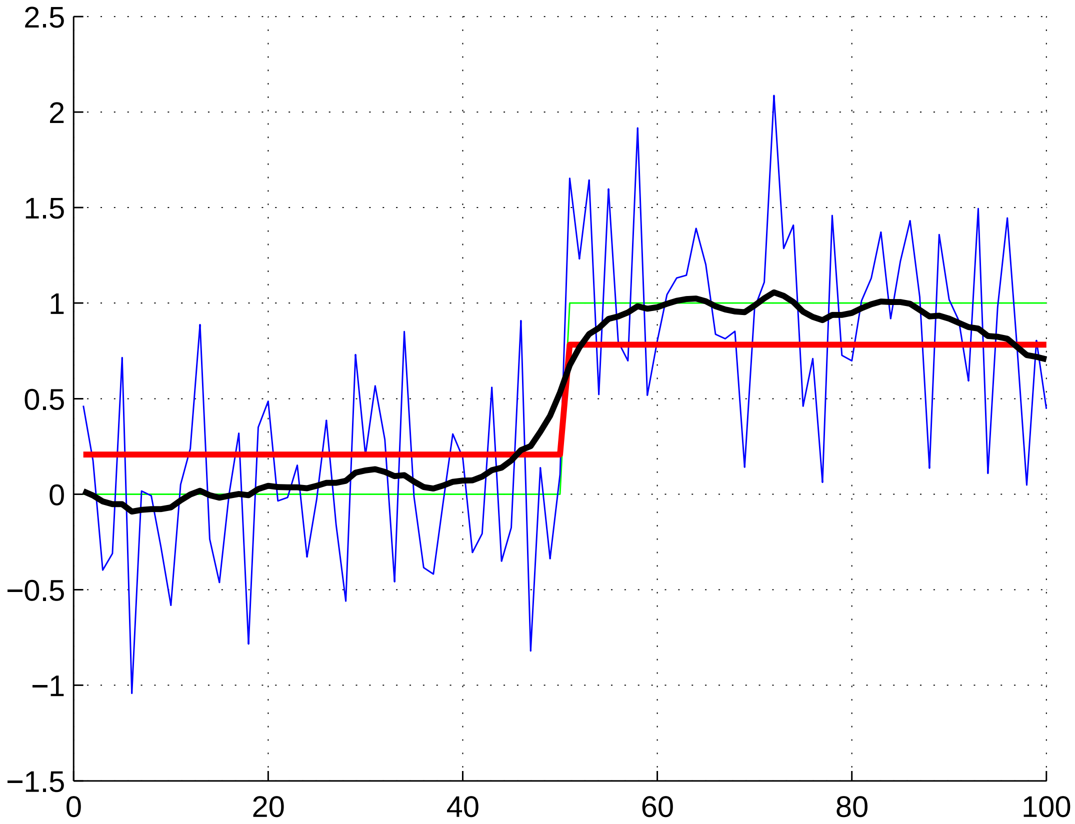
- Examples
- Time series
- General smoothing and filtering problem
- Discussion

Fault detection



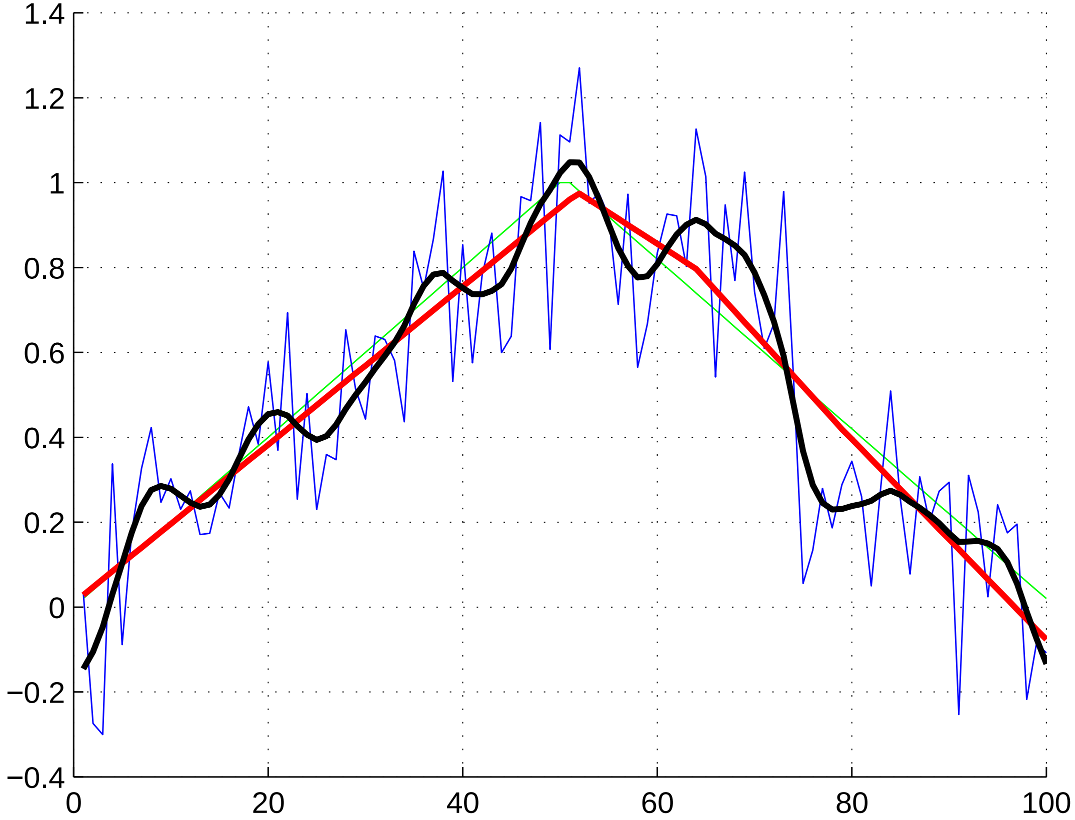
Green line — true signal (unit step), blue — signal + noise ($\sigma = 0.2$), black — smoothing by quadratic minimization, red — by l_1 technique.

Large noise



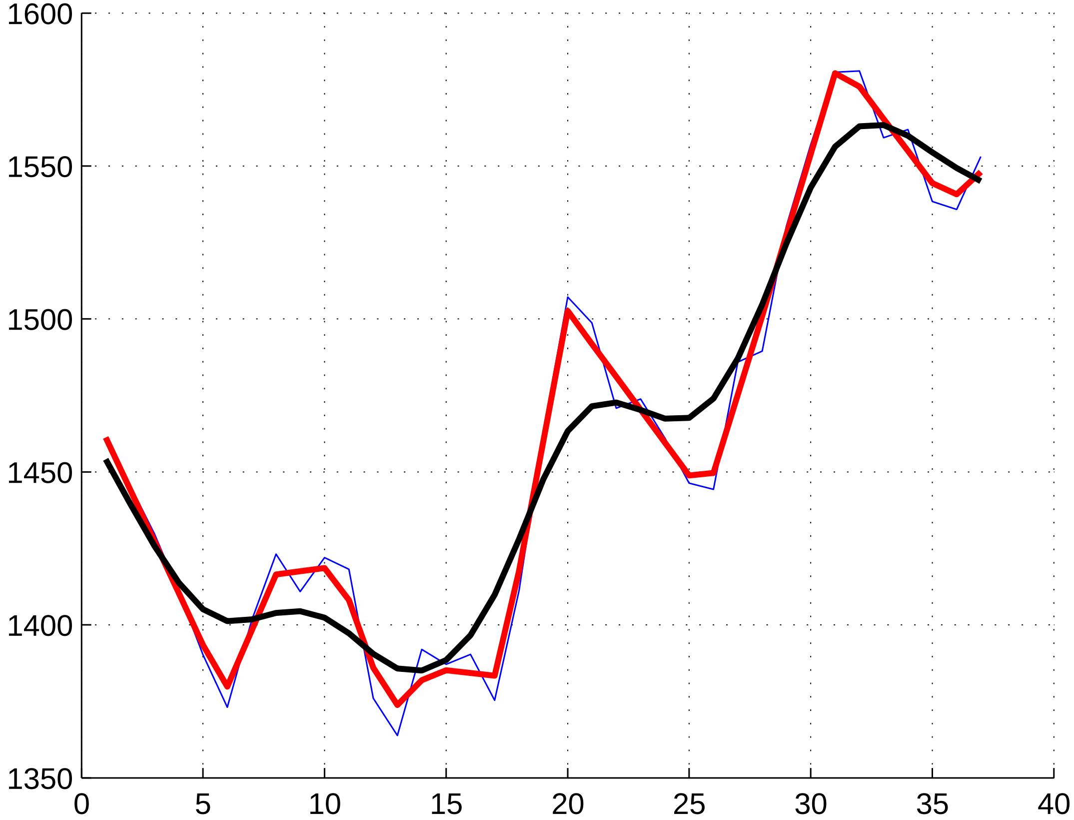
The same for $\sigma = 0.5$.

Piece-wise linear approximation



True signal is triangular.

RTS index



Real data — RTS index for January-February, 2010.

Time series — quadratic filtering

Basic problem: given time series $y_k, k = 1, ..., N$, present it as $y_k = x_k + v_k$, where x_k is trend, v_k is noise.

Typical is Hodrick-Prescott (HP) smoother: x_k is the solution of quadratic optimization problem

$$\min \left(\sum_{k=1}^{N} (x_k - y_k)^2 + \mu \sum_{k=2}^{N-1} (x_{k+1} - 2x_k + x_{k-1})^2 \right)$$

or

$$\min \left(\sum_{k=1}^{N} (x_k - y_k)^2 + \mu \sum_{k=1}^{N-1} (x_{k+1} - x_k)^2 \right),$$

here $\mu > 0$ is a parameter.

Time series — l_1 filtering

Boyd (Boyd, Vanderberghe, Convex Optimization, 2004; Boyd a.o. SIAM Review, 2009, 51, No 2, 339-360) proposed the same estimates, but with l_2 norm replaced with l_1 in the second term:

$$\min\left(\sum_{k=1}^{N}(x_k - y_k)^2 + \mu\sum_{k=2}^{N-1}|x_{k+1} - 2x_k + x_{k-1}|\right)$$

$$\min\left(\sum_{k=1}^{N}(x_k - y_k)^2 + \mu\sum_{k=1}^{N-1}|x_{k+1} - x_k|\right).$$

We call them Boyd estimates of the first and zero order respectively. Their properties:

- The solution is piece-wise linear (piece-wise constant) function of k. The number of brakes and their location depends on data y_k and on μ.
- The solution can be found either by standard quadratic programming or by special methods tailored for such optimization problems. For instance Boyd developed l1-ls software. We exploited CVX, also developed by Boyd.

Kalman filter

System dynamics:

$$x_{k+1} = A_k x_k + B_k u_k + G_k w_k, \quad k = 1, ..., N$$

measurements

$$y_k = C_k x_k + D_k u_k + H_k w_k + v_k, \quad k = 1, ..., N$$

matrices $A_k, B_k, C_k, D_k, G_k, H_k$ are known, inputs u_k are available, noises w_k, v_k are Gaussian, mutually independent, zero mean with covariance matrices Q_k, R_k. The goal is to estimate states $\{x_k\}$ under measurements $\{y_k\}$. Then the best linear unbiased estimate coincides with least squares estimate. Its recurrent form defines Kalman filter.

In simplest case with $u_k = 0, G_k = G, H_k = 0, A_k = A, C_k = C, Q_k = \sigma_1^2, R_k = \sigma_2^2$ Kalman filtering is equivalent to MLS

$$\min \left(\sum_k (y_k - Cx_k)^2 + \mu \sum_k w_k^2 \right), \quad \mu = \sigma_2^2 / \sigma_1^2$$

subject to

$$x_{k+1} = Ax_k + Gw_k \quad k = 1, ..., N.$$

This is linear-quadratic regulator problem with free terminal point and its solution can be found explicitly; let's denote it $\hat{x}_k, \hat{w}_k$.

Properties of Kalman filter

Advantages: explicit formula for $\hat{x}_k$, recurrent form of the estimate (however the best $\hat{x}_k$ requires all N measurements, not the first k ones). Code *zkalman* in Matlab implements the filter.

Disadvantages: covariation matrices and initial conditions are needed. But the main objection are too restricted assumptions on noises. Are they really unbiased? Are they Gaussian? These assumptions are very unnatural for noises w_k in state equations. For instance, these terms can be caused by another player's actions; then they are not random.

l_1 - smoothing

For the same problem as above

$$x_{k+1} = Ax_k + Gw_k, \quad y_k = Cx_k + v_k, \quad k = 1, ..., N$$

we propose l_1 alternative to Kalman filter. Solve optimization problem

$$\min_{x,w} \left(\sum_k ||y_k - Cx_k||^2 + \mu \sum_k |w_k| \right),$$

subject to

$$x_{k+1} = Ax_k + Gw_k \quad k = 1, ..., N$$

and denote its solution $\hat{x}_k, \hat{w}_k$. Then $\hat{x}_k$ is the desired estimate for x_k.

Time-series filtering can be treated as a particular case of this general scheme for state equations

$$x_{k+1} = x_k + w_k, \quad y_k = x_k + v_k$$

or

$$x_{k+1} = 2x_k - x_{k-1} + w_k, \quad y_k = x_k + v_k$$

after exclusion of variables w_k.

Discussion

1. Motivation. Noises in measurements v_k in most cases are random and approximately Gaussian,thus they can be treated by LS method. However perturbations w_k in state equation are typically non random. We can assume them bounded in l_1 norm: $||w||_1 \leq r$. Translating this constraint into performance index by use of Lagrange multipliers, we arrive to above considered optimization problem.

2. Properties of the solution. Estimates $\hat{x}_k$ have the same structure as in l_1 control problem — many $\hat{w}_k$ are equal zero. That means that $\hat{x}_{k+1} = A\hat{x}_k$ for many k — no perturbations in state equation.

3. Computationally the problem is not hard, for instance CVX software is convenient.

4. In contrast with Kalman filtering the problem should be solved off-line. However its on-line versions can be designed.

5. The choice of μ requires some a-priori information.

6. We have many test problems solved; however we have no experience in real-life filtering problems.

7. Extension and theoretical validation can be found in: *A.Yuditsky, A.Nemirovski a.o. "On the accuracy of l_1-filtering of signals with block-sparse structure", Conference "Neural Information Processing Systems", Granada, Spain, December 2011.*

Reducing the number of controls, states, outputs

Linear system:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$x \in \mathbb{R}^n$ — state $\quad y \in \mathbb{R}^l$ — output $\quad u \in \mathbb{R}^k$ — control $\quad (A, B)$ — controllable, $\quad (A, C)$ — observable

Goal — design controller K as linear state feedback

$$u = Kx$$

or static output feedback

$$u = Ky$$

- stabilizing closed-loop system
- optimizing one of the following criteria:
  - number of controls $\Longrightarrow$ number of actuators
  - number of states exploited for feedback $\Longrightarrow$ number of sensors
  - number of outputs $\Longrightarrow$ "minimal" information transmitted

$$\boxed{\text{Controller}} \xleftrightarrow{\text{information}} \boxed{\text{Plant}}$$

LMI approach

Function

$$V(x) = x^{\mathrm{T}} Q x, \quad Q \succ 0$$

is quadratic Lyapunov function for closed-loop system if $\iff A_c^{\mathrm{T}} Q + Q A_c \prec 0$ or

$$A_c P + P A_c^{\mathrm{T}} \prec 0, \quad P = Q^{-1}$$

State feedback: $u = Kx \implies A_c = A + BK$

$$AP + PA^{\mathrm{T}} + BKP + PB^{\mathrm{T}} K^{\mathrm{T}} \prec 0, \quad P \succ 0$$

Output feedback: $u = Ky \implies A_c = A + BKC$

$$AP + PA^{\mathrm{T}} + BKCP + PC^{\mathrm{T}} B^{\mathrm{T}} K^{\mathrm{T}} \prec 0, \quad P \succ 0$$

K and P are variables!

- However matrix inequality is nonlinear in K, P.
- Constraints are nonconvex.
- Static output feedback does not exist in general.

Design of gain

Let's try to reduce the number of nonzero rows of K

$$u = \underbrace{\begin{pmatrix} \cdots\cdots\cdots\cdots\cdots \\ 0 \quad 0 \quad \cdots \quad 0 \\ \cdots\cdots\cdots\cdots\cdots \\ 0 \quad 0 \quad \cdots \quad 0 \\ \cdots\cdots\cdots\cdots\cdots \end{pmatrix}}_{K} x$$

$\Longrightarrow$ use small number of controls

or — reduce the number of nonzero columns of K

$$u = \underbrace{\begin{pmatrix} \cdots & 0 & \cdots & 0 & \cdots \\ \cdots & 0 & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & \cdots & 0 & \cdots \end{pmatrix}}_{K} x$$

$\Longrightarrow$ use small number of states

Special matrix norms

Let $M \in \mathbb{R}^{m \times n}$

Introduce norms:

$$\|M\|_{r_1} = \sum_{i=1}^{m} \max_{1 \leqslant j \leqslant n} |m_{ij}|$$

$$\|M\|_{c_1} = \sum_{j=1}^{n} \max_{1 \leqslant i \leqslant m} |m_{ij}|$$

Theorem Solution of

$$\min ||M||$$

$$(A_i, M) = b_i, \quad i = 1, \ldots, l$$

contains $\leq l$ nonzero rows (columns).

Seeking controller:

- stabilizing closed-loop system
- with minimal r_1-нормой $\implies$ with reduced number of nonzero rows (r_1-optimization)

or

- with minimal c_1-norm $\implies$ with reduced number of nonzero columns. (c_1-optimization)

Preserving matrix structure

Left multiplication:

$$\begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \times \begin{pmatrix} \cdots & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \mathbf{0} & \cdots \end{pmatrix} = \begin{pmatrix} \cdots & \mathbf{0} & \cdots \\ \cdots & \mathbf{0} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \mathbf{0} & \cdots \end{pmatrix}$$

Right multiplication:

$$\begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \times \begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

r_1-optimization: reducing number of controls

Control system:

$$\dot{x} = Ax + Bu$$

Goal: stabilizing controller $u = Kx$, reducing number of controls

Lyapunov:

$$AP + PA^{\mathrm{T}} + BKP + PK^{\mathrm{T}}B^{\mathrm{T}} \prec 0, \quad P \succ 0$$

Introduce $Y = KP \implies$ LMI

$$AP + PA^{\mathrm{T}} + BY + Y^{\mathrm{T}}B^{\mathrm{T}} \prec 0, \quad P \succ 0$$

matrix Y with reduced number of nonzero rows

$\Downarrow$

gain $\quad K = YP^{-1} \quad$ with reduced number of nonzero rows

$\Downarrow$

reduced number of controls!

r_1-optimization: reducing number of controls

Proposition 1. *Let* $\widehat{Y}, \widehat{P}$ *— the solution of minimization problem*

$$\|Y\|_{r_1} \longrightarrow \min$$

subject to

$$AP + PA^{\mathrm{T}} + BY + Y^{\mathrm{T}}B^{\mathrm{T}} \prec 0, \qquad P \succ 0.$$

Typically $\widehat{Y}$ *has some zero rows, then the same number zero rows is in*

$$\widehat{K} = \widehat{Y}\widehat{P}^{-1}$$

and $u = \widehat{K}x$ *is the stabilizing controller.*

- We distinguish controls which are sufficient to design a stabilizing controller

- Constraint $AP + PA^{\mathrm{T}} + BY + Y^{\mathrm{T}}B^{\mathrm{T}} \preccurlyeq -2\alpha P \implies \alpha$ allows to fix stability degree of the system.

- We arrive to SDP

- LMI techniques is exploited

Example 1: stabilizing helicopter Bell201-A

$$A = \begin{pmatrix} -0.0046 & 0.038 & 0{,}3259 & -0{,}0045 & -0{,}402 & -0{,}073 & -9{,}81 & 0 \\ -0.1978 & -0.5667 & 0{,}357 & -0{,}0378 & -0{,}2149 & 0{,}5683 & 0 & 0 \\ 0.0039 & -0.0029 & -0{,}2947 & 0{,}007 & 0{,}2266 & 0{,}0148 & 0 & 0 \\ 0.0133 & -0.0014 & -0{,}4076 & -0{,}0654 & -0{,}4093 & 0{,}2674 & 0 & 9{,}81 \\ 0.0127 & -0.01 & -0{,}8152 & -0{,}0397 & -0{,}821 & 0{,}1442 & 0 & 0 \\ -0.0285 & -0.0232 & 0.1064 & 0.0709 & -0.2786 & -0.7396 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.0676 & 0.1221 & -0.0001 & -0.0016 \\ -1.1151 & 0.1055 & 0.0039 & 0.0035 \\ 0.0062 & -0.0682 & 0.001 & -0.0035 \\ -0.017 & 0.0049 & 0.1067 & 0.1692 \\ -0.0129 & 0.0106 & 0.2227 & 0.143 \\ 0.139 & 0.0059 & 0.0326 & -0.407 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Leibfritz F., Lipinski W. Description of the benchmark examples in COMPleib 1.0. Technical report. University of Trier, 2003. URL: `www.complib.de`

Example 1 (cont)

$$\widehat{Y} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -0.0368 & -0.0368 & 0.0370 & 0.0351 & -0.0333 & 0.0369 & 0.0370 & -0.0370 \\ -0.0067 & 0.0075 & -0.0076 & 0.0074 & -0.0076 & -0.0076 & -0.0076 & -0.0076 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$\Downarrow$

$$\widehat{K} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -0.0383 & 0.0046 & 4.7623 & -0.0178 & -2.3057 & -0.2035 & 2.2435 & -3.5635 \\ 0.0091 & 0.0094 & -0.7096 & -0.0057 & -0.7954 & -0.0939 & -0.8332 & -0.7581 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$\Longrightarrow$ use controls u_2 и u_3

$$\max_i \operatorname{Re} \lambda_i(A + B\widehat{K}) = -0.0500$$

c_1-optimization: state feedback

Control system:

$$\dot{x} = Ax + u$$

(dimensions of state and control coincide)

Goal: control $u = Kx$, exploiting reduced number of states.

Lyapunov function provides:

$$A^{\mathrm{T}}Q + QA + QK + K^{\mathrm{T}}Q \prec 0, \quad Q \succ 0$$

Introduce $Y = QK \implies$ LMI

$$A^{\mathrm{T}}Q + QA + Y + Y^{\mathrm{T}} \prec 0, \quad Q \succ 0$$

Proposition 2. *Let $\widehat{Y}, \widehat{Q}$ be the solution of*

$$\|Y\|_{c_1} \longrightarrow \min \quad \text{s.t.} \quad A^{\mathrm{T}}Q + QA + Y + Y^{\mathrm{T}} \prec 0, \quad Q \succ 0.$$

Typically $\widehat{Y}$ has zero columns, then the same number of zero columns has gain

$$\widehat{K} = \widehat{Q}^{-1}\widehat{Y}$$

of state feedback stabilizing controller.

- That is we find the states which are sufficient for stabilization

Example 2

$\dot{x} = Ax + u$

$$A = \begin{pmatrix} 0 & 13 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 13 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$\widehat{Y} = \begin{pmatrix} -0.1930 & \mathbf{0} & \mathbf{0} & -0.0144 \\ -0.1930 & \mathbf{0} & \mathbf{0} & -0.0719 \\ -0.1930 & \mathbf{0} & \mathbf{0} & 0.0721 \\ -0.0878 & \mathbf{0} & \mathbf{0} & -0.0721 \end{pmatrix} \Longrightarrow \widehat{K} = \begin{pmatrix} -2.5325 & \mathbf{0} & \mathbf{0} & 0.0319 \\ -0.4642 & \mathbf{0} & \mathbf{0} & -0.0143 \\ -1.3742 & \mathbf{0} & \mathbf{0} & 0.2368 \\ -0.7718 & \mathbf{0} & \mathbf{0} & -0.8593 \end{pmatrix}$$

$$\max_i \operatorname{Re} \lambda_i(A + \widehat{K}) = -0.0503$$

Control uses x_1 and $x_4$only!

$$\widehat{Y} = \begin{pmatrix} -0.5661 & 0 & 0 & \mathbf{0} \\ -0.5654 & 0 & 0 & \mathbf{0} \\ -0.5660 & 0 & 0 & \mathbf{0} \\ 0.0482 & 0 & 0 & \mathbf{0} \end{pmatrix} \Longrightarrow \widehat{K} = \begin{pmatrix} -3.4039 & 0 & 0 & \mathbf{0} \\ -0.6387 & 0 & 0 & \mathbf{0} \\ -1.6738 & 0 & 0 & \mathbf{0} \\ -1.7367 & 0 & 0 & \mathbf{0} \end{pmatrix}, \quad \max_i \operatorname{Re} \lambda_i(A_c) = -0.0503$$

c_1-optimization: output feedback

System considered:

$$\dot{x} = Ax + u$$
$$y = Cx$$

Goal: control $u = Ky$, with reduced number of outputs exploited.

As above:

$$A^{\mathrm{T}}Q + QA + QKC + C^{\mathrm{T}}K^{\mathrm{T}}Q \prec 0, \quad Q \succ 0$$

Introduce $Y = QK \implies$ LMI

$$A^{\mathrm{T}}Q + QA + YC + C^{\mathrm{T}}Y^{\mathrm{T}} \prec 0, \quad Q \succ 0$$

Proposition 3. *Let $\widehat{Y}, \widehat{Q}$ be the solution of*

$$\|Y\|_{c_1} \longrightarrow \min \quad \text{s.t.} \quad A^{\mathrm{T}}Q + QA + YC + C^{\mathrm{T}}Y^{\mathrm{T}} \prec 0, \quad Q \succ 0.$$

Matrix $\widehat{Y}$ has typically some zero columns, then gain

$$\widehat{K} = \widehat{Q}^{-1}\widehat{Y}$$

has the same number zero columns.

- Of course for $B = I$ stabilizing static output feedback exists.
- That is we distinguish outputs which allow to stabilize the system

Example 3

$$A = \begin{pmatrix} 0 & 13 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 13 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\widehat{Y} = \begin{pmatrix} -0.1167 & \mathbf{0} & -0.0729 \\ -0.1167 & \mathbf{0} & -0.0729 \\ -0.1167 & \mathbf{0} & 0.0729 \\ -0.0397 & \mathbf{0} & -0.0729 \end{pmatrix} \quad \Longrightarrow \quad \widehat{K} = \begin{pmatrix} -1.5180 & \mathbf{0} & -0.3871 \\ -0.2754 & \mathbf{0} & -0.0898 \\ -0.8551 & \mathbf{0} & -0.0050 \\ -0.3385 & \mathbf{0} & -0.9296 \end{pmatrix}$$

$$\max_i \operatorname{Re} \lambda_i(A + \widehat{K}C) = -0.0506$$

Control exploits outputs y_1 и y_3 only!

$$\widehat{Y} = \begin{pmatrix} -0.3013 & 0 & \mathbf{0} \\ -0.3009 & 0 & \mathbf{0} \\ -0.3012 & 0 & \mathbf{0} \\ 0.0881 & 0 & \mathbf{0} \end{pmatrix} \quad \Longrightarrow \quad \widehat{K} = \begin{pmatrix} -2.2317 & 0 & \mathbf{0} \\ -0.4130 & 0 & \mathbf{0} \\ -1.1678 & 0 & \mathbf{0} \\ -0.8547 & 0 & \mathbf{0} \end{pmatrix}, \quad \max_i \operatorname{Re} \lambda_i(A_c) = -0.0508$$

Design of linear output

System:

$$\dot{x} = Ax + Bu$$

Goal:

- find matrix C with reduced number of rows

$$y = Cx$$

such that it is possible to design

- stabilizing static output feedback $u = Ky$.

Motivation

Low-dimensional output $\Longrightarrow$ reduces information quantity transmitted from plant to controller

Design of linear output (cont)

Zero columns in Y

$\Downarrow$

$$u = Kx = \underbrace{\begin{pmatrix} \times & 0 & \times & 0 & \times \\ \times & 0 & \times & 0 & \times \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \times & 0 & \times & 0 & \times \end{pmatrix}}_{Y} \underbrace{\begin{pmatrix} \times & \times & \cdots & \times \\ \cdots & \cdots & \cdots & \cdots \\ \times & \times & \cdots & \times \\ \cdots & \cdots & \cdots & \cdots \\ \times & \times & \cdots & \times \end{pmatrix}}_{P^{-1}} x = \widetilde{K}\, \underbrace{\widetilde{C}x}_{y}$$

Proposition 4. *Suppose $\widehat{Y}, \widehat{P}$ is a solution of*

$$\|Y\|_{c_1} \longrightarrow \min \quad \text{s.t.} \quad AP + PA^{\mathrm{T}} + BY + Y^{\mathrm{T}}B^{\mathrm{T}} \prec 0, \quad P \succ 0.$$

Then gain $\widetilde{K}$ consists of nonzero columns of $\widehat{Y}$, and outputs $\widetilde{C}$ coincide with columns of $\widehat{P}^{-1}$.

- We design low-dimensional output
- We get rid of assumption $B = I$

Example 4

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 13 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We get:

$$\widehat{Y} = \begin{pmatrix} 1.7400 & -15.6830 & \mathbf{0} \end{pmatrix}$$

$$\Longrightarrow \qquad \widetilde{K} = \begin{pmatrix} 1.7400 & -15.6830 \end{pmatrix}, \qquad \widetilde{C} = \begin{pmatrix} 0.4000 & 0.1527 & 0.0127 \\ 0.1527 & 0.8994 & 0.2368 \end{pmatrix}$$

$$\max_i \operatorname{Re} \lambda_i (A + B\widetilde{K}\widetilde{C}) = -0.0509$$

Let

$$\widehat{Y} = \begin{pmatrix} \mathbf{0} & -15.6830 & 0 \end{pmatrix} \quad \Longrightarrow \quad \widetilde{K} = -15.6830, \qquad \widetilde{C} = \begin{pmatrix} 0.1527 & 0.8994 & 0.2368 \end{pmatrix}$$

$$\max_i \operatorname{Re} \lambda_i (A + B\widetilde{K}\widetilde{C}) = -0.0609$$

Syrmos V.L., Abdallah C.T., Dorato P., Grigoriadis K. Static output feedback: a survey // Automatica. 1997. Vol. 33. P. 125–137.

Conclusions

- The technique of l_1-optimization works in optimal control.

- l_1-alternative to Kalman filter looks promising.

- We believe that the new approach — reduction of number of states, outputs or controls — has numerous applications.