THE DOUGLAS–RACHFORD ALGORITHM CONVERGES ONLY WEAKLY*

MINH N. BÙI[†] AND PATRICK L. COMBETTES[‡]

Abstract. We show that the weak convergence of the Douglas–Rachford algorithm for finding a zero of the sum of two maximally monotone operators cannot be improved to strong convergence. Likewise, we show that strong convergence can fail for the method of partial inverses.

Key words. Douglas–Rachford algorithm, method of partial inverses, monotone operator, operator splitting, strong convergence.

The original Douglas–Rachford splitting algorithm was designed to decompose positive systems of linear equations [3]. It evolved in [5] into a powerful method for finding a zero of the sum of two maximally monotone operators in Hilbert spaces, a problem which is ubiquitous in applied mathematics (see [1] for background on monotone operators). In this context, the Douglas–Rachford algorithm constitutes a prime decomposition method in areas such as control, partial differential equations, optimization, statistics, variational inequalities, mechanics, optimal transportation, machine learning, and signal processing. Its asymptotic behavior is described next.

THEOREM 1. *Let $\mathcal{H}$ be a real Hilbert space, and let A and B be set-valued maximally monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$ with resolvents $J_A = (\mathrm{Id} + A)^{-1}$ and $J_B = (\mathrm{Id} + B)^{-1}$. Suppose that $\operatorname{zer}(A + B) = \big\{x \in \mathcal{H} \;\big|\; 0 \in Ax + Bx\big\} \neq \varnothing$, let $y_0 \in \mathcal{H}$, and iterate*

$$(\forall n \in \mathbb{N}) \quad x_n = J_B y_n \quad \text{and} \quad y_{n+1} = y_n + J_A(2x_n - y_n) - x_n. \tag{1}$$

Then the following hold for some $(y, x) \in \operatorname{graph} J_B$:

(i) *$x = J_A(2x - y)$, $y_n \rightharpoonup y$, and $x \in \operatorname{zer}(A + B)$.*

(ii) *$x_n \rightharpoonup x$.*

Property (i) was established in [5]. Let us note that, since J_B is not weakly sequentially continuous in general, the weak convergence of $(y_n)_{n\in\mathbb{N}}$ in (i) does not imply (ii). The latter was first established in [7] (see also [1, Theorem 26.11(iii)] for an alternative proof). While various additional conditions on A and B have been proposed to ensure the strong convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ in (1) [1, 2, 5], it remains an open question whether it can fail in the general setting of Theorem 1. We show that this is indeed the case. Our argument relies on a result of Hundal [4] concerning the method of alternating projections.

COUNTEREXAMPLE 2. *In Theorem 1, suppose that $\mathcal{H}$ is infinite-dimensional and separable. Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$, let $V = \{e_0\}^{\perp}$, let $y_0 = e_2$, and let K be the smallest closed convex cone containing the set*

$$\bigg\{\exp\big(-100\xi^3\big)e_0 + \cos\Big(\frac{\pi}{2}(\xi - \lfloor\xi\rfloor)\Big)e_{\lfloor\xi\rfloor+1} + \sin\Big(\frac{\pi}{2}(\xi - \lfloor\xi\rfloor)\Big)e_{\lfloor\xi\rfloor+2} \;\Big|\; \xi \in [0, +\infty[\bigg\}, \tag{2}$$

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[†]North Carolina State University, Department of Mathematics, Raleigh, NC 27695-8205, USA (mnbui@ncsu.edu)

[‡]North Carolina State University, Department of Mathematics, Raleigh, NC 27695-8205, USA (plc@math.ncsu.edu)

where $\lfloor \xi \rfloor$ *denotes the integer part of* $\xi \in [0, +\infty[$. *Let* proj_V *and* proj_K *be the projection operators onto* V *and* K, *and set*

$$A\colon x \mapsto \begin{cases} V^{\perp}, & \text{if } x \in V;\\ \varnothing, & \text{if } x \notin V \end{cases} \quad \text{and} \quad B = \big(\mathrm{proj}_V \circ \mathrm{proj}_K \circ \mathrm{proj}_V\big)^{-1} - \mathrm{Id}. \tag{3}$$

Then A *and* B *are maximally monotone, and the sequence* $(x_n)_{\in\mathbb{N}}$ *constructed in Theorem 1 converges weakly, but not strongly, to a zero of* $A + B$.

Proof. We first note that A is maximally monotone by virtue of [1, Examples 6.43 and 20.26]. Now set $T = \mathrm{proj}_V \circ \mathrm{proj}_K \circ \mathrm{proj}_V$. Then it follows from [1, Example 4.14] that T is firmly nonexpansive, that is,

$$(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad \langle x - y \mid Tx - Ty\rangle \geqslant \|Tx - Ty\|^2. \tag{4}$$

In turn, we derive from [1, Proposition 23.10] that $B = T^{-1} - \mathrm{Id}$ is maximally monotone. Next, we observe that $0 \in \mathrm{zer}\, A$ and that, since K is a closed cone, $0 \in K$. Thus, $0 = (\mathrm{proj}_V \circ \mathrm{proj}_K \circ \mathrm{proj}_V)0$, which implies that $0 \in \mathrm{zer}\, B$. Hence,

$$0 \in \mathrm{zer}\,(A + B). \tag{5}$$

Now set

$$z_0 = \exp(-100)e_0 + e_2 \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad z_{n+1} = \mathrm{proj}_K\big(\mathrm{proj}_V z_n\big). \tag{6}$$

Then, by nonexpansiveness of proj_K,

$$\begin{aligned}(\forall n \in \mathbb{N}) \quad \|z_{n+1}\|^2 &= \|\mathrm{proj}_K\big(\mathrm{proj}_V z_n\big) - \mathrm{proj}_K 0\|^2\\ &\leqslant \|\mathrm{proj}_V z_n\|^2\\ &= \|z_n\|^2 - \|\mathrm{proj}_V z_n - z_n\|^2\end{aligned} \tag{7}$$

and, therefore,

$$\mathrm{proj}_V z_n - z_n \to 0. \tag{8}$$

As shown in [4], we also have

$$z_n \rightharpoonup 0 \quad \text{and} \quad z_n \not\to 0. \tag{9}$$

On the other hand, we derive from (3) that

$$J_A = \mathrm{proj}_V \quad \text{and} \quad J_B = \mathrm{proj}_V \circ \mathrm{proj}_K \circ \mathrm{proj}_V, \tag{10}$$

and from (6) that $\mathrm{proj}_V z_0 = e_2 = y_0$. It thus follows from (1) and (6) that $x_0 = \mathrm{proj}_V(\mathrm{proj}_K(\mathrm{proj}_V y_0)) = \mathrm{proj}_V(\mathrm{proj}_K(\mathrm{proj}_V z_0)) = \mathrm{proj}_V z_1$. Now, assume that, for some $n \in \mathbb{N}$, $y_n = \mathrm{proj}_V z_n$ and $x_n = \mathrm{proj}_V z_{n+1}$. Since x_n and y_n lie in V, we derive from (1) and (10) that

$$y_{n+1} = y_n + \mathrm{proj}_V(2x_n - y_n) - x_n = x_n = \mathrm{proj}_V z_{n+1} \tag{11}$$

and hence that

$$\begin{aligned} x_{n+1} &= \big(\mathrm{proj}_V \circ \mathrm{proj}_K \circ \mathrm{proj}_V\big)\big(\mathrm{proj}_V z_{n+1}\big)\\ &= \mathrm{proj}_V\big(\mathrm{proj}_K\big(\mathrm{proj}_V z_{n+1}\big)\big)\\ &= \mathrm{proj}_V z_{n+2}. \end{aligned} \tag{12}$$

We have thus proven by induction that

$$(\forall n \in \mathbb{N}) \quad x_n = \mathrm{proj}_V z_{n+1}. \tag{13}$$

In view of (8), we obtain $x_n - z_{n+1} \to 0$ and therefore derive from (9) and (5) that $x_n \rightharpoonup 0 \in \mathrm{zer}\,(A+B)$ and $x_n \not\to 0$. ☐

Next, we settle a similar open question for Spingarn's method of partial inverses [6] by showing that its strong convergence can fail.

THEOREM 3 ([6]). *Let $\mathcal{H}$ be a real Hilbert space, let $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone, and let V be a closed vector subspace of $\mathcal{H}$. Suppose that the problem*

$$\text{find } x \in V \text{ and } u \in V^{\perp} \text{ such that } u \in Bx \tag{14}$$

has at least one solution. Let $x_0 \in V$, let $u_0 \in V^{\perp}$, and iterate

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = \mathrm{proj}_V\big(J_B(x_n+u_n)\big) \quad \text{and} \quad u_{n+1} = \mathrm{proj}_{V^{\perp}}\big(J_{B^{-1}}(x_n+u_n)\big). \tag{15}$$

Then $(x_n, u_n)_{n\in\mathbb{N}}$ converges weakly to a solution to (14).

COUNTEREXAMPLE 4. *Define $\mathcal{H}$, V, K, and B as in Counterexample 2, and set $x_0 = e_2$ and $u_0 = 0$. Then $(0,0)$ solves (14) and the sequence $(x_n, u_n)_{n\in\mathbb{N}}$ constructed in Theorem 3 converges weakly, but not strongly, to $(0,0)$.*

Proof. Since $J_B = \mathrm{proj}_V \circ \mathrm{proj}_K \circ \mathrm{proj}_V$ and $J_{B^{-1}} = \mathrm{Id} - J_B$, (15) implies that

$$(\forall n \in \mathbb{N}) \quad \begin{cases} x_{n+1} = \big(\mathrm{proj}_V \circ \mathrm{proj}_K \circ \mathrm{proj}_V\big)(x_n + u_n) \\ u_{n+1} = \mathrm{proj}_{V^{\perp}}\big(x_n + u_n - \big(\mathrm{proj}_V \circ \mathrm{proj}_K \circ \mathrm{proj}_V\big)(x_n + u_n)\big). \end{cases} \tag{16}$$

We therefore obtain inductively that

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = \mathrm{proj}_V\big(\mathrm{proj}_K x_n\big) \quad \text{and} \quad u_n = 0. \tag{17}$$

Now define $(z_n)_{n\in\mathbb{N}}$ as in (6). Then, by induction, $(\forall n \in \mathbb{N})$ $x_n = \mathrm{proj}_V z_n$. Hence, in view of (8) and (9), we conclude that $0 \not\leftarrow x_n \rightharpoonup 0$. ☐

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