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SYSTEMS OF STRUCTURED MONOTONE INCLUSIONS: DUALITY, ALGORITHMS, AND APPLICATIONS*

PATRICK L. COMBETTES†

Abstract. A general primal-dual splitting algorithm for solving systems of structured coupled monotone inclusions in Hilbert spaces is introduced and its asymptotic behavior is analyzed. Each inclusion in the primal system features compositions with linear operators, parallel sums, and Lipschitzian operators. All the operators involved in this structured model are used separately in the proposed algorithm, most steps of which can be executed in parallel. This provides a flexible solution method applicable to a variety of problems beyond the reach of the state-of-the-art. Several applications are discussed to illustrate this point.

Key words. convex minimization, coupled system, infimal convolution, monotone inclusion, monotone operator, operator splitting, parallel algorithm, structured minimization problem

AMS subject classifications. Primary, 47H05; Secondary, 65K05, 90C25

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1. Introduction.

Traditional monotone operator splitting techniques [8, 18, 24, 25, 29, 35, 37, 41, 43, 44] have their roots in matrix decomposition methods in numerical analysis [22, 45] and in nonlinear methods for solving optimization and variational inequality problems [7, 12, 31, 34, 40]. These methods are designed to solve inclusions of the type $0 \in B_1x + B_2x$, where B_1 and B_2 are maximally monotone operators acting on a Hilbert space $\mathcal{H}$. Extensions to sums of the type $0 \in \sum_{k=1}^{K} B_kx$ are typically handled via reformulations in product spaces [8, 41]. In recent years, new splitting algorithms have emerged for problems involving more complex models featuring compositions with linear operators [14] and parallel sums [20, 46]; we recall that the parallel sum of two set-valued operators B and D is

$$B \,\square\, D = \big(B^{-1} + D^{-1}\big)^{-1}. \tag{1.1}$$

These algorithms rely on reformulations of the inclusions as two-operator problems in a primal-dual space, in which the splitting is performed via an existing method. This construct makes it possible to separately activate each of the operators involved in the model, and it leads to flexible algorithms implementable on parallel architectures. In the present paper, we pursue this strategy toward more sophisticated models featuring systems of structured coupled inclusions in duality. The primal-dual problem under consideration is the following.

PROBLEM 1.1. *Let m and K be strictly positive integers, let $(\mathcal{H}_i)_{1\leqslant i\leqslant m}$ and $(\mathcal{G}_k)_{1\leqslant k\leqslant K}$ be real Hilbert spaces, let $(\mu_i)_{1\leqslant i\leqslant m} \in [0,+\infty[^m$, and let $(\nu_k)_{1\leqslant i\leqslant K} \in [0,+\infty[^K$. For every $i \in \{1,\ldots,m\}$ and $k \in \{1,\ldots,K\}$, let $C_i\colon \mathcal{H}_i \to \mathcal{H}_i$ be monotone and μ_i-Lipschitzian, let $A_i\colon \mathcal{H}_i \to 2^{\mathcal{H}_i}$ and $B_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ be maximally monotone, let $D_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ be maximally monotone and such that $D_k^{-1}\colon \mathcal{G}_k \to \mathcal{G}_k$ is*

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†UPMC Université Paris 06, Laboratoire Jacques-Louis Lions–UMR 7598, 75005 Paris, France (plc@math.jussieu.fr) and Department of Mathematics, King Abdulaziz University KAU, Jeddah, Saudi Arabia.

ν_k-Lipschitzian, let $z_i \in \mathcal{H}_i$, let $r_k \in \mathcal{G}_k$, and let $L_{ki} \in \mathcal{B}(\mathcal{H}_i, \mathcal{G}_k)$. It is assumed that

$$\beta = \max\Big\{\max_{1\leqslant i\leqslant m}\mu_i, \max_{1\leqslant k\leqslant K}\nu_k\Big\} + \sqrt{\lambda} > 0, \quad \text{where} \quad \lambda \in \Bigg[\sup_{\sum_{i=1}^m \|x_i\|^2\leqslant 1}\sum_{k=1}^K\Big\|\sum_{i=1}^m L_{ki}x_i\Big\|^2, +\infty\Bigg[, \tag{1.2}$$

and that the system of coupled inclusions

(1.3) *find* $\overline{x_1} \in \mathcal{H}_1, \ldots, \overline{x_m} \in \mathcal{H}_m$ *such that*

$$\begin{cases} z_1 \in A_1\overline{x_1} + \displaystyle\sum_{k=1}^K L_{k1}^*\Bigg((B_k \,\Box\, D_k)\Bigg(\sum_{i=1}^m L_{ki}\overline{x_i} - r_k\Bigg)\Bigg) + C_1\overline{x_1} \\ \quad\vdots \\ z_m \in A_m\overline{x_m} + \displaystyle\sum_{k=1}^K L_{km}^*\Bigg((B_k \,\Box\, D_k)\Bigg(\sum_{i=1}^m L_{ki}\overline{x_i} - r_k\Bigg)\Bigg) + C_m\overline{x_m} \end{cases}$$

possesses at least one solution. Solve (1.3) *together with the dual problem*

(1.4) *find* $\overline{v_1} \in \mathcal{G}_1, \ldots, \overline{v_K} \in \mathcal{G}_K$ *such that*

$$\begin{cases} -r_1 \in -\displaystyle\sum_{i=1}^m L_{1i}\big(A_i + C_i\big)^{-1}\Bigg(z_i - \sum_{k=1}^K L_{ki}^*\overline{v_k}\Bigg) + B_1^{-1}\overline{v_1} + D_1^{-1}\overline{v_1} \\ \quad\vdots \\ -r_K \in -\displaystyle\sum_{i=1}^m L_{Ki}\big(A_i + C_i\big)^{-1}\Bigg(z_i - \sum_{k=1}^K L_{ki}^*\overline{v_k}\Bigg) + B_K^{-1}\overline{v_K} + D_K^{-1}\overline{v_K}. \end{cases}$$

The primal system (1.3) captures a broad class of problems in nonlinear analysis in which m variables $x_1, \ldots, x_m$ interact. The ith inclusion in (1.3) features two operators A_i and C_i which model some abstract utility of the variable x_i, while the operators $(B_k)_{1\leqslant k\leqslant K}$, $(D_k)_{1\leqslant k\leqslant K}$, and $(L_{ki})_{\substack{1\leqslant i\leqslant m\\1\leqslant k\leqslant K}}$ model the interaction between x_i and the remaining variables. One of the simplest realizations of (1.3) is the problem considered in [10], namely,

$$\text{find } \overline{x_1} \in \mathcal{H}, \overline{x_2} \in \mathcal{H} \text{ such that } \begin{cases} 0 \in A_1\overline{x_1} + \overline{x_1} - \overline{x_2}, \\ 0 \in A_2\overline{x_2} - \overline{x_1} + \overline{x_2}, \end{cases} \tag{1.5}$$

where $(\mathcal{H}, \|\cdot\|)$ is a real Hilbert space, and where A_1 and A_2 are maximally monotone operators acting on $\mathcal{H}$. In particular, if $A_1 = \partial f_1$ and $A_2 = \partial f_2$ are the subdifferentials of proper lower semicontinuous convex functions f_1 and f_2 from $\mathcal{H}$ to $]-\infty, +\infty]$, (1.5) becomes

$$\underset{x_1\in\mathcal{H},\, x_2\in\mathcal{H}}{\text{minimize}} \; f_1(x_1) + f_2(x_2) + \frac{1}{2}\|x_1 - x_2\|^2. \tag{1.6}$$

This formulation arises in areas such as optimization [1], the cognitive sciences [5], image recovery [21], signal synthesis [30], best approximation [9], and mechanics [38]. In [3], we considered the extension of (1.6) which amounts to setting in Problem 1.1, for every $i \in \{1,\ldots,m\}$ and $k \in \{1,\ldots,K\}$, $A_i = \partial f_i$, $C_i = 0$, and $B_k = \nabla g_k$, where $f_i\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ is a proper lower semicontinuous convex function and $g_k\colon \mathcal{G}_k \to \mathbb{R}$ is convex and differentiable with a Lipschitzian gradient. This leads to the minimization problem

$$\underset{x_1\in\mathcal{H}_1,\ldots,\,x_m\in\mathcal{H}_m}{\text{minimize}} \quad \sum_{i=1}^m f_i(x_i) + \sum_{k=1}^K g_k\bigg(\sum_{i=1}^m L_{ki}x_i\bigg), \tag{1.7}$$

which has numerous applications in signal processing, machine learning, image recovery, partial differential equations, and game theory; see [2, 6, 13, 15, 26, 28, 42] and the references therein. This minimization problem arose in [3] as an instance of a multivariate inclusion problem which is a special case of (1.3) in which the operators $(C_i)_{1\leqslant i\leqslant m}$ and $(D_k^{-1})_{1\leqslant k\leqslant K}$ are zero, and the coupling operators $(B_k)_{1\leqslant k\leqslant K}$ are restricted to be single-valued and to jointly satisfy a cocoercivity property.

The goals of the present paper are to develop a flexible algorithm to solve Problem 1.1 without the restrictions imposed by current method, and to illustrate its flexibility by applying it to a variety of problems for which no solution method exists currently. Our setting places no additional hypotheses on the coupling operators $(B_k)_{1\leqslant k\leqslant K}$ and $(D_k)_{1\leqslant k\leqslant K}$, or on the number m of variables. In the proposed parallel splitting algorithm, the structure of the problem is fully exploited to the extent that the operators are all used individually, either explicitly if they are single-valued, or by means of their resolvent if they are set-valued. In the case when $m = 1$ in Problem 1.1, we obtain the univariate primal-dual problem investigated in [20] (see also [14, 46] for special cases), namely,

$$\text{find } \overline{x}\in\mathcal{H} \text{ such that } z \in A\overline{x} + \sum_{k=1}^K L_k^*\big((B_k \,\square\, S_k)(L_k\overline{x} - r_k)\big) + C\overline{x} \tag{1.8}$$

and

$$\text{find } \overline{v_1}\in\mathcal{G}_1,\ \ldots,\ \overline{v_K}\in\mathcal{G}_K \text{ such that} \tag{1.9}$$
$$(\forall k\in\{1,\ldots,K\})\quad -r_k \in -L_k\left((A+C)^{-1}\left(z - \sum_{l=1}^K L_l^*\overline{v_l}\right)\right) + B_k^{-1}\overline{v_k} + S_k^{-1}\overline{v_k}.$$

Conversely, formulating these inclusions in a suitable product space $\mathcal{H}$ formally leads to (1.3)–(1.4). However, transcribing the algorithm of [20] and its convergence analysis in such a product setting would lead to much weaker results than those to be presented in section 2, which will employ a finer analysis and leverage the properties of each of the operators involved in the model. Our problem formulation and its asymptotic analysis will enable us to extend existing results and solve much more complex problems than those afforded by the state-of-the-art. These advances are highlighted by the following applications of the results of section 2.

- In section 3, we revisit (1.8)–(1.9) and solve it without the restriction that the operators $(S_k^{-1})_{1\leqslant k\leqslant K}$ be Lipschitzian, as is required in [20]. This is achieved by showing that, through the introduction of auxiliary variables, the problem is reducible to an instance of (1.3).
- In section 4, we address the problem of approximating—by means of parallel sums—inconsistent common zero problems. By reformulating this univariate problem as an instance of Problem 1.1, we obtain a framework which allows

for relaxations with operators possessing strictly monotone inverses, while existing results [18] are limited to relaxations with multiples of identity.

- In section 5, we apply the results of section 2 to multivariate structured convex minimization problems, thus obtaining notable improvements over the results of [3, 13].
- In section 6, we use the results of section 3 to solve univariate minimization problems featuring infimal convolutions with general lower semicontinuous convex functions. In such models, the state-of-the-art is limited to strongly convex functions [20].
- Another application of the results of section 2 is that developed in [11] after the submission of the present paper. In this work, inclusions of the form

(1.10)
$$\text{find } \overline{x} \in \mathcal{H} \text{ such that } z \in A\overline{x} + \sum_{k=1}^{r} \big((L_k^* \circ B_k \circ L_k) \,\Box\, (M_k^* \circ D_k \circ M_k)\big)\overline{x} + C\overline{x}$$

together with their duals were considered for the first time. We reformulated this problem as a special case of (1.3) and showed that it captured variational formulations in the area of signal recovery for which no solution method was available until now.

Notation. We denote the scalar product of a Hilbert space by $\langle \cdot \mid \cdot \rangle$ and the associated norm by $\| \cdot \|$. The symbols $\rightharpoonup$ and $\to$ denote, respectively, weak and strong convergence, and Id denotes the identity operator. Let $\mathcal{H}$ and $\mathcal{G}$ be real Hilbert spaces and let $2^{\mathcal{H}}$ be the power set of $\mathcal{H}$. The space of bounded linear operators from $\mathcal{H}$ to $\mathcal{G}$ is denoted by $\mathcal{B}(\mathcal{H},\mathcal{G})$. Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$. We denote by $\operatorname{ran} A = \big\{u \in \mathcal{H} \;\big|\; (\exists\, x \in \mathcal{H})\; u \in Ax\big\}$ the range A, by $\operatorname{dom} A = \big\{x \in \mathcal{H} \;\big|\; Ax \neq \varnothing\big\}$ the domain of A, by $\operatorname{zer} A = \big\{x \in \mathcal{H} \;\big|\; 0 \in Ax\big\}$ the set of zeros of A, by $\operatorname{gra} A = \big\{(x,u) \in \mathcal{H} \times \mathcal{H} \;\big|\; u \in Ax\big\}$ the graph of A, and by A^{-1} the inverse of A, i.e., the operator with graph $\big\{(u,x) \in \mathcal{H} \times \mathcal{H} \;\big|\; u \in Ax\big\}$. The resolvent of A is $J_A = (\mathrm{Id} + A)^{-1}$. Moreover, A is declared monotone if

$$(\forall (x,u) \in \operatorname{gra} A)(\forall (y,v) \in \operatorname{gra} A) \quad \langle x - y \mid u - v \rangle \geqslant 0, \tag{1.11}$$

and maximally monotone if there exists no monotone operator $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ such that $\operatorname{gra} A \subset \operatorname{gra} B \neq \operatorname{gra} A$. In this case, J_A is a nonexpansive operator defined everywhere on $\mathcal{H}$. Furthermore, A is uniformly monotone at $x \in \operatorname{dom} A$ if there exists an increasing function $\phi\colon\, [0,+\infty[\, \to [0,+\infty]$ that vanishes only at 0 such that

$$(\forall u \in Ax)(\forall (y,v) \in \operatorname{gra} A) \quad \langle x - y \mid u - v \rangle \geqslant \phi(\|x - y\|), \tag{1.12}$$

and A is couniformly monotone at $u \in \operatorname{ran} A$ if A^{-1} is uniformly monotone at u. The parallel sum of A and $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ is $A \,\Box\, B = (A^{-1} + B^{-1})^{-1}$. The infimal convolution of two functions g and ℓ from $\mathcal{H}$ to $]-\infty,+\infty]$ is

$$g \,\Box\, \ell\colon \mathcal{H} \to [-\infty,+\infty] : x \mapsto \inf_{y \in \mathcal{H}} \big(g(y) + \ell(x-y)\big). \tag{1.13}$$

We denote by $\Gamma_0(\mathcal{H})$ the class of lower semicontinuous convex functions $f\colon \mathcal{H} \to\,]-\infty,+\infty]$ such that $\operatorname{dom} f = \big\{x \in \mathcal{H} \;\big|\; f(x) < +\infty\big\} \neq \varnothing$. Let $f \in \Gamma_0(\mathcal{H})$. The conjugate of f is $\Gamma_0(\mathcal{H}) \ni f^*\colon u \mapsto \sup_{x \in \mathcal{H}}(\langle x \mid u \rangle - f(x))$, and f is uniformly convex at

$x \in \operatorname{dom} f$ if there exists an increasing function $\phi\colon [0,+\infty[\to [0,+\infty]$ that vanishes only at 0 such that

$$(\forall y \in \operatorname{dom} f)(\forall \alpha \in]0,1[) \quad f(\alpha x+(1-\alpha)y)+\alpha(1-\alpha)\phi(\|x-y\|) \leqslant \alpha f(x)+(1-\alpha)f(y). \tag{1.14}$$

For every $x \in \mathcal{H}$, $f + \|x - \cdot\|^2/2$ possesses a unique minimizer, which is denoted by $\operatorname{prox}_f x$. We have

$$\operatorname{prox}_f = J_{\partial f}, \quad \text{where} \quad \partial f\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{u \in \mathcal{H} \;\big|\; (\forall y \in \mathcal{H})\;\; \langle y - x \mid u\rangle + f(x) \leqslant f(y)\big\} \tag{1.15}$$

is the subdifferential of f. Let C be a convex subset of $\mathcal{H}$. The indicator function of C is denoted by ι_C and the distance function to C by d_C. The relative interior (respectively, the strong relative interior) of C, i.e., the set of points $x \in C$ such that the cone generated by $-x + C$ is a vector subspace (respectively, closed vector subspace) of $\mathcal{H}$, by $\operatorname{ri} C$ (respectively, $\operatorname{sri} C$). See [8, 47] for background on convex analysis and monotone operators.

2. General algorithm.

We start with three preliminary results. The first one is an error-tolerant version of a forward-backward-forward splitting algorithm originally proposed by Tseng [44, Theorem 3.4(b)].

LEMMA 2.1 (see [14, Theorem 2.5(i)–(ii)]). *Let $\boldsymbol{\mathcal{K}}$ be a real Hilbert space, let $\boldsymbol{P}\colon \boldsymbol{\mathcal{K}} \to 2^{\boldsymbol{\mathcal{K}}}$ be maximally monotone, and let $\boldsymbol{Q}\colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}}$ be monotone and χ-Lipschitzian for some $\chi \in]0,+\infty[$. Suppose that $\operatorname{zer}(\boldsymbol{P}+\boldsymbol{Q}) \neq \varnothing$. Let $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$, $(\boldsymbol{b}_n)_{n\in\mathbb{N}}$, and $(\boldsymbol{c}_n)_{n\in\mathbb{N}}$ be absolutely summable sequences in $\boldsymbol{\mathcal{K}}$, let $\boldsymbol{w}_0 \in \boldsymbol{\mathcal{K}}$, let $\varepsilon \in]0, 1/(\chi+1)[$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon,(1-\varepsilon)/\chi]$, and set*

$$\begin{array}{l} \text{for } n = 0,1,\ldots \\ \left\lfloor \begin{array}{l} \boldsymbol{s}_n = \boldsymbol{w}_n - \gamma_n(\boldsymbol{Q}\boldsymbol{w}_n + \boldsymbol{a}_n),\\ \boldsymbol{p}_n = J_{\gamma_n \boldsymbol{P}}\, \boldsymbol{s}_n + \boldsymbol{b}_n,\\ \boldsymbol{q}_n = \boldsymbol{p}_n - \gamma_n(\boldsymbol{Q}\boldsymbol{p}_n + \boldsymbol{c}_n),\\ \boldsymbol{w}_{n+1} = \boldsymbol{w}_n - \boldsymbol{s}_n + \boldsymbol{q}_n. \end{array}\right. \end{array} \tag{2.1}$$

Then $\sum_{n\in\mathbb{N}} \|\boldsymbol{w}_n - \boldsymbol{p}_n\|^2 < +\infty$ and there exists $\overline{\boldsymbol{w}} \in \operatorname{zer}(\boldsymbol{P}+\boldsymbol{Q})$ such that $\boldsymbol{w}_n \rightharpoonup \overline{\boldsymbol{w}}$ and $\boldsymbol{p}_n \rightharpoonup \overline{\boldsymbol{w}}$.

LEMMA 2.2. *Let $\mathcal{H}$ be a real Hilbert space, let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a maximally monotone operator, let $\gamma \in]0,+\infty[$, and let x and r be in $\mathcal{H}$. Then $J_{\gamma(r+A^{-1})}x = x - \gamma(r + J_{\gamma^{-1}A}(\gamma^{-1}x - r))$.*

Proof. It follows from [8, Proposition 23.15(ii)] that $J_{\gamma(r+A^{-1})}x = J_{\gamma r+\gamma A^{-1}}x = J_{\gamma A^{-1}}(x-\gamma r)$. On the other hand, we derive from [8, Proposition 23.18] that $(\forall y \in \mathcal{H})$ $J_{\gamma A^{-1}}y = y - \gamma J_{\gamma^{-1}A}(\gamma^{-1}y)$. Applying this identity to $y = x - \gamma r$ yields the result. ☐

LEMMA 2.3. [14, Proposition 2.8] *Let $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{G}}$ be two real Hilbert spaces, let $\boldsymbol{E}\colon \boldsymbol{\mathcal{H}} \to 2^{\boldsymbol{\mathcal{H}}}$ and $\boldsymbol{F}\colon \boldsymbol{\mathcal{G}} \to 2^{\boldsymbol{\mathcal{G}}}$ be maximally monotone, let $\boldsymbol{L} \in \mathcal{B}(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{G}})$, let $\boldsymbol{z} \in \boldsymbol{\mathcal{H}}$, and let $\boldsymbol{r} \in \boldsymbol{\mathcal{G}}$. Set $\boldsymbol{\mathcal{K}} = \boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}}$,*

$$\begin{cases} \boldsymbol{M}\colon \boldsymbol{\mathcal{K}} \to 2^{\boldsymbol{\mathcal{K}}}\colon (\boldsymbol{x},\boldsymbol{v}) \mapsto (-\boldsymbol{z} + \boldsymbol{E}\boldsymbol{x}) \times (\boldsymbol{r} + \boldsymbol{F}^{-1}\boldsymbol{v}),\\ \boldsymbol{S}\colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}}\colon (\boldsymbol{x},\boldsymbol{v}) \mapsto (\boldsymbol{L}^*\boldsymbol{v}, -\boldsymbol{L}\boldsymbol{x}), \end{cases} \tag{2.2}$$

and

$$\begin{cases}\mathfrak{P} = \big\{\boldsymbol{x} \in \boldsymbol{\mathcal{H}} \;\big|\; \boldsymbol{z} \in \boldsymbol{E}\boldsymbol{x} + \boldsymbol{L}^*(\boldsymbol{F}(\boldsymbol{L}\boldsymbol{x} - \boldsymbol{r}))\big\},\\ \mathfrak{D} = \big\{\boldsymbol{v} \in \boldsymbol{\mathcal{G}} \;\big|\; -\boldsymbol{r} \in -\boldsymbol{L}(\boldsymbol{E}^{-1}(\boldsymbol{z} - \boldsymbol{L}^*\boldsymbol{v})) + \boldsymbol{F}^{-1}\boldsymbol{v}\big\}.\end{cases} \tag{2.3}$$

Then $\operatorname{zer}(\boldsymbol{M}+\boldsymbol{S})$ *is a closed convex subset of* $\mathfrak{P}\times\mathfrak{D}$, *and* $\mathfrak{P} \neq \varnothing \Leftrightarrow \operatorname{zer}(\boldsymbol{M}+\boldsymbol{S}) \neq \varnothing \Leftrightarrow \mathfrak{D} \neq \varnothing$.

The following theorem contains our algorithm for solving Problem 1.1 and states its main asymptotic properties. In this primal-dual splitting algorithm, each single-valued operator is used explicitly, while each set-valued operator is activated via its resolvent. Approximations in the evaluations of the operators are tolerated and modeled by absolutely summable error sequences. The algorithm consists of three main loops, each of which can be implemented on a parallel architecture.

THEOREM 2.4. *Consider the setting of Problem* 1.1. *For every* $i \in \{1,\dots,m\}$, *let* $(a_{1,i,n})_{n\in\mathbb{N}}$, $(b_{1,i,n})_{n\in\mathbb{N}}$, *and* $(c_{1,i,n})_{n\in\mathbb{N}}$ *be absolutely summable sequences in* $\mathcal{H}_i$ *and, for every* $k \in \{1,\dots,K\}$, *let* $(a_{2,k,n})_{n\in\mathbb{N}}$, $(b_{2,k,n})_{n\in\mathbb{N}}$, *and* $(c_{2,k,n})_{n\in\mathbb{N}}$ *be absolutely summable sequences in* $\mathcal{G}_k$. *Let* $x_{1,0} \in \mathcal{H}_1, \dots, x_{m,0} \in \mathcal{H}_m$, $v_{1,0} \in \mathcal{G}_1, \dots, v_{K,0} \in \mathcal{G}_K$, *let* $\varepsilon \in \left]0, 1/(\beta+1)\right[$, *let* $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence in* $[\varepsilon, (1-\varepsilon)/\beta]$, *and set*

$$\begin{array}{l}\textit{for } n = 0,1,\dots\\ \left\lfloor\begin{array}{l}\textit{for } i = 1,\dots,m\\ \left\lfloor\begin{array}{l} s_{1,i,n} = x_{i,n} - \gamma_n\Big(C_i x_{i,n} + \sum_{k=1}^K L_{ki}^* v_{k,n} + a_{1,i,n}\Big),\\ p_{1,i,n} = J_{\gamma_n A_i}(s_{1,i,n} + \gamma_n z_i) + b_{1,i,n},\end{array}\right.\\ \textit{for } k = 1,\dots,K\\ \left\lfloor\begin{array}{l} s_{2,k,n} = v_{k,n} - \gamma_n\Big(D_k^{-1} v_{k,n} - \sum_{i=1}^m L_{ki} x_{i,n} + a_{2,k,n}\Big),\\ p_{2,k,n} = s_{2,k,n} - \gamma_n\big(r_k + J_{\gamma_n^{-1}B_k}(\gamma_n^{-1}s_{2,k,n} - r_k) + b_{2,k,n}\big),\\ q_{2,k,n} = p_{2,k,n} - \gamma_n\Big(D_k^{-1} p_{2,k,n} - \sum_{i=1}^m L_{ki} p_{1,i,n} + c_{2,k,n}\Big),\\ v_{k,n+1} = v_{k,n} - s_{2,k,n} + q_{2,k,n},\end{array}\right.\\ \textit{for } i = 1,\dots,m\\ \left\lfloor\begin{array}{l} q_{1,i,n} = p_{1,i,n} - \gamma_n\Big(C_i p_{1,i,n} + \sum_{k=1}^K L_{ki}^* p_{2,k,n} + c_{1,i,n}\Big),\\ x_{i,n+1} = x_{i,n} - s_{1,i,n} + q_{1,i,n}.\end{array}\right.\end{array}\right.\end{array} \tag{2.4}$$

Then the following hold:

(i) $(\forall i \in \{1,\dots,m\})\ \sum_{n\in\mathbb{N}} \|x_{i,n} - p_{1,i,n}\|^2 < +\infty$.

(ii) $(\forall k \in \{1,\dots,K\})\ \sum_{n\in\mathbb{N}} \|v_{k,n} - p_{2,k,n}\|^2 < +\infty$.

(iii) *There exist a solution* $(\overline{x_1},\dots,\overline{x_m})$ *to* (1.3) *and a solution* $(\overline{v_1},\dots,\overline{v_K})$ *to* (1.4) *such that the following hold:*

(a) $(\forall i \in \{1,\dots,m\})\ z_i - \sum_{k=1}^K L_{ki}^*\overline{v_k} \in A_i\overline{x_i} + C_i\overline{x_i}$.

(b) $(\forall k \in \{1,\dots,K\})\ \sum_{i=1}^m L_{ki}\overline{x_i} - r_k \in B_k^{-1}\overline{v_k} + D_k^{-1}\overline{v_k}$.

(c) $(\forall i \in \{1,\dots,m\})\ x_{i,n} \rightharpoonup \overline{x_i}$ *and* $p_{1,i,n} \rightharpoonup \overline{x_i}$.

(d) $(\forall k \in \{1,\dots,K\})\ v_{k,n} \rightharpoonup \overline{v_k}$ *and* $p_{2,k,n} \rightharpoonup \overline{v_k}$.

(e) *Suppose that, for some* $j \in \{1,\dots,m\}$, A_j *or* C_j *is uniformly monotone at* $\overline{x_j}$. *Then* $x_{j,n} \to \overline{x_j}$ *and* $p_{1,j,n} \to \overline{x_j}$.

(f) *Suppose that, for some* $l \in \{1,\dots,K\}$, B_l *or* D_l *is couniformly monotone at* $\overline{v_l}$. *Then* $v_{l,n} \to \overline{v_l}$ *and* $p_{2,l,n} \to \overline{v_l}$.

Proof. Let us introduce the Hilbert direct sums

$$\boldsymbol{\mathcal{H}} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m, \quad \boldsymbol{\mathcal{G}} = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_K, \quad \text{and} \quad \boldsymbol{\mathcal{K}} = \boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}}, \tag{2.5}$$

and let us denote by $\boldsymbol{x} = (x_i)_{1\leqslant i\leqslant m}$ and $\boldsymbol{v} = (v_k)_{1\leqslant k\leqslant K}$ generic elements in $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{G}}$, respectively. We also define

$$
\begin{cases}
\boldsymbol{A}\colon \boldsymbol{\mathcal{H}} \to 2^{\boldsymbol{\mathcal{H}}}\colon \boldsymbol{x} \mapsto \displaystyle\bigtimes_{i=1}^{m} A_i x_i,\\
\boldsymbol{C}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}\colon \boldsymbol{x} \mapsto (C_i x_i)_{1\leqslant i\leqslant m},\\
\boldsymbol{E} = \boldsymbol{A} + \boldsymbol{C},\\
\boldsymbol{L}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{G}}\colon \boldsymbol{x} \mapsto \left(\displaystyle\sum_{i=1}^{m} L_{ki} x_i\right)_{1\leqslant k\leqslant K},\\
\boldsymbol{z} = (z_i)_{1\leqslant i\leqslant m}
\end{cases}
\tag{2.6}
$$

$$
\text{and}\qquad
\begin{cases}
\boldsymbol{B}\colon \boldsymbol{\mathcal{G}} \to 2^{\boldsymbol{\mathcal{G}}}\colon \boldsymbol{v} \mapsto \displaystyle\bigtimes_{k=1}^{K} B_k v_k,\\
\boldsymbol{D}\colon \boldsymbol{\mathcal{G}} \to 2^{\boldsymbol{\mathcal{G}}}\colon \boldsymbol{v} \mapsto \displaystyle\bigtimes_{k=1}^{K} D_k v_k,\\
\boldsymbol{F} = \boldsymbol{B} \,\square\, \boldsymbol{D},\\
\boldsymbol{r} = (r_k)_{1\leqslant k\leqslant K}.
\end{cases}
$$

It follows from [8, Propositions 20.22 and 20.23, Corollaries 20.25 and 24.4(i)] that $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{E}$, and $\boldsymbol{F}$ are maximally monotone. Moreover, $\boldsymbol{L} \in \mathcal{B}(\boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{G}})$, $\boldsymbol{L}^*\colon \boldsymbol{\mathcal{G}} \to \boldsymbol{\mathcal{H}}\colon \boldsymbol{v} \mapsto (\sum_{k=1}^{K} L_{ki}^* v_k)_{1\leqslant i\leqslant m}$, and

$$
(\forall \boldsymbol{x} \in \boldsymbol{\mathcal{H}}) \quad \|\boldsymbol{L}\boldsymbol{x}\|^2 = \sum_{k=1}^{K} \left\| \sum_{i=1}^{m} L_{ki} x_i \right\|^2 \leqslant \lambda \|\boldsymbol{x}\|^2.
\tag{2.7}
$$

Next, we set

$$
\begin{cases}
\boldsymbol{M}\colon \boldsymbol{\mathcal{K}} \to 2^{\boldsymbol{\mathcal{K}}}\colon (\boldsymbol{x}, \boldsymbol{v}) \mapsto (-\boldsymbol{z} + \boldsymbol{E}\boldsymbol{x}) \times (\boldsymbol{r} + \boldsymbol{F}^{-1}\boldsymbol{v}),\\
\boldsymbol{P}\colon \boldsymbol{\mathcal{K}} \to 2^{\boldsymbol{\mathcal{K}}}\colon (\boldsymbol{x}, \boldsymbol{v}) \mapsto (-\boldsymbol{z} + \boldsymbol{A}\boldsymbol{x}) \times (\boldsymbol{r} + \boldsymbol{B}^{-1}\boldsymbol{v}),\\
\boldsymbol{Q}\colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}}\colon (\boldsymbol{x}, \boldsymbol{v}) \mapsto \big(\boldsymbol{C}\boldsymbol{x} + \boldsymbol{L}^*\boldsymbol{v}, \boldsymbol{D}^{-1}\boldsymbol{v} - \boldsymbol{L}\boldsymbol{x}\big),\\
\boldsymbol{R}\colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}}\colon (\boldsymbol{x}, \boldsymbol{v}) \mapsto (\boldsymbol{C}\boldsymbol{x}, \boldsymbol{D}^{-1}\boldsymbol{v}),\\
\boldsymbol{S}\colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}}\colon (\boldsymbol{x}, \boldsymbol{v}) \mapsto (\boldsymbol{L}^*\boldsymbol{v}, -\boldsymbol{L}\boldsymbol{x}).
\end{cases}
\tag{2.8}
$$

Note that

$$
\begin{aligned}
&\operatorname{zer}(\boldsymbol{P} + \boldsymbol{Q})\\
&\quad = \big\{(\boldsymbol{x}, \boldsymbol{v}) \in \boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}} \;\big|\; \boldsymbol{z} - \boldsymbol{L}^*\boldsymbol{v} \in \boldsymbol{A}\boldsymbol{x} + \boldsymbol{C}\boldsymbol{x} \quad \text{and} \quad \boldsymbol{L}\boldsymbol{x} - \boldsymbol{r} \in \boldsymbol{B}^{-1}\boldsymbol{v} + \boldsymbol{D}^{-1}\boldsymbol{v}\big\}.
\end{aligned}
\tag{2.9}
$$

Furthermore, in view of [8, Propositions 20.22 and 20.23], $\boldsymbol{P}$ is maximally monotone, and Lemma 2.2 and [8, Proposition 23.16] yield

$$
\begin{aligned}
&(\forall \gamma \in \left]0, +\infty\right[)(\forall \boldsymbol{x} \in \boldsymbol{\mathcal{H}})(\forall \boldsymbol{v} \in \boldsymbol{\mathcal{G}}) \quad J_{\gamma \boldsymbol{P}}(\boldsymbol{x}, \boldsymbol{v})\\
&\quad = \Big(J_{\gamma A_1}(x_1 + \gamma z_1), \ldots, J_{\gamma A_m}(x_m + \gamma z_m), v_1 - \gamma\big(r_1 + J_{\gamma^{-1} B_1}(\gamma^{-1} v_1 - r_1)\big),\\
&\qquad\qquad \ldots, v_K - \gamma\big(r_K + J_{\gamma^{-1} B_K}(\gamma^{-1} v_K - r_K)\big)\Big).
\end{aligned}
\tag{2.10}
$$

On the other hand, since $\boldsymbol{C}$ and $\boldsymbol{D}^{-1}$ are monotone and Lipschitzian with, respectively, constants $\mu = \max_{1\leqslant i\leqslant m} \mu_i$ and $\nu = \max_{1\leqslant k\leqslant K} \nu_k$, $\boldsymbol{R}$ is monotone and

Lipschitzian with constant $\max\{\mu,\nu\}$. In addition, it follows from [14, Proposition 2.7(ii)] and (2.7) that $\boldsymbol{S} \in \mathcal{B}(\boldsymbol{\mathcal{K}},\boldsymbol{\mathcal{K}})$ is a skew (hence monotone) operator with $\|\boldsymbol{S}\| = \|\boldsymbol{L}\| \leqslant \sqrt{\lambda}$. Altogether, since $\boldsymbol{Q} = \boldsymbol{R} + \boldsymbol{S}$, we derive from (1.2) that

$$\boldsymbol{P} \text{ is maximally monotone and } \boldsymbol{Q} \text{ is monotone and } \beta\text{-Lipschitzian.} \tag{2.11}$$

Let us call $\mathfrak{P}$ and $\mathfrak{D}$ the sets of solutions to (1.3) and (1.4), respectively. It follows from (2.6) that

$$\begin{cases} \mathfrak{P} = \big\{ \boldsymbol{x} \in \boldsymbol{\mathcal{H}} \;\big|\; \boldsymbol{z} \in \boldsymbol{E}\boldsymbol{x} + \boldsymbol{L}^*(\boldsymbol{F}(\boldsymbol{L}\boldsymbol{x} - \boldsymbol{r}))\big\}, \\ \mathfrak{D} = \big\{ \boldsymbol{v} \in \boldsymbol{\mathcal{G}} \;\big|\; -\boldsymbol{r} \in -\boldsymbol{L}(\boldsymbol{E}^{-1}(\boldsymbol{z} - \boldsymbol{L}^*\boldsymbol{v})) + \boldsymbol{F}^{-1}\boldsymbol{v}\big\}. \end{cases} \tag{2.12}$$

Hence, since $\mathfrak{P} \neq \varnothing$ by assumption, we deduce from Lemma 2.3 that

$$\varnothing \neq \operatorname{zer}(\boldsymbol{M} + \boldsymbol{S}) = \operatorname{zer}(\boldsymbol{P} + \boldsymbol{Q}) \subset \mathfrak{P} \times \mathfrak{D}. \tag{2.13}$$

Thus, to solve Problem 1.1, it is enough to find a zero of $\boldsymbol{P} + \boldsymbol{Q}$. For every $n \in \mathbb{N}$, let us set

$$\begin{cases} \boldsymbol{w}_n = (x_{1,n}, \ldots, x_{m,n}, v_{1,n}, \ldots, v_{K,n}), \\ \boldsymbol{s}_n = (s_{1,1,n}, \ldots, s_{1,m,n}, s_{2,1,n}, \ldots, s_{2,K,n}), \\ \boldsymbol{p}_n = (p_{1,1,n}, \ldots, p_{1,m,n}, p_{2,1,n}, \ldots, p_{2,K,n}), \\ \boldsymbol{q}_n = (q_{1,1,n}, \ldots, q_{1,m,n}, q_{2,1,n}, \ldots, q_{2,K,n}) \end{cases} \tag{2.14}$$

and

$$\begin{cases} \boldsymbol{a}_n = (a_{1,1,n}, \ldots, a_{1,m,n}, a_{2,1,n}, \ldots, a_{2,K,n}), \\ \boldsymbol{b}_n = (b_{1,1,n}, \ldots, b_{1,m,n}, -\gamma_n b_{2,1,n}, \ldots, -\gamma_n b_{2,K,n}), \\ \boldsymbol{c}_n = (c_{1,1,n}, \ldots, c_{1,m,n}, c_{2,1,n}, \ldots, c_{2,K,n}). \end{cases} \tag{2.15}$$

Then, using (2.6), (2.8), and (2.10), we see that (2.4) reduces to (2.1). Moreover, our assumptions and (2.5) imply that $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$, $(\boldsymbol{b}_n)_{n\in\mathbb{N}}$, and $(\boldsymbol{c}_n)_{n\in\mathbb{N}}$ are absolutely summable sequences in $\boldsymbol{\mathcal{K}}$. Hence, it follows from (2.11), (2.13), and Lemma 2.1 that $\sum_{n\in\mathbb{N}} \|\boldsymbol{w}_n - \boldsymbol{p}_n\|^2 < +\infty$ and that there exists $\overline{\boldsymbol{w}} \in \operatorname{zer}(\boldsymbol{P} + \boldsymbol{Q})$ such that $\boldsymbol{w}_n \rightharpoonup \overline{\boldsymbol{w}}$ and $\boldsymbol{p}_n \rightharpoonup \overline{\boldsymbol{w}}$. Upon setting $\overline{\boldsymbol{w}} = (\overline{x_1}, \ldots, \overline{x_m}, \overline{v_1}, \ldots, \overline{v_K})$ and appealing to (2.5) and (2.9), we thus obtain assertions (i), (ii), and (iii)(a)–(iii)(d).

(iii)(e) Let us introduce the variables

$$(\forall i \in \{1, \ldots, m\})(\forall n \in \mathbb{N}) \quad \begin{cases} \widetilde{s}_{1,i,n} = x_{i,n} - \gamma_n \bigg(C_i x_{i,n} + \displaystyle\sum_{k=1}^{K} L_{ki}^* v_{k,n} \bigg), \\ \widetilde{p}_{1,i,n} = J_{\gamma_n A_i}(\widetilde{s}_{1,i,n} + \gamma_n z_i) \end{cases} \tag{2.16}$$

and

$$(\forall k \in \{1, \ldots, K\})(\forall n \in \mathbb{N}) \quad \begin{cases} \widetilde{s}_{2,k,n} = v_{k,n} - \gamma_n \bigg(D_k^{-1} v_{k,n} - \displaystyle\sum_{i=1}^{m} L_{ki} x_{i,n} \bigg), \\ \widetilde{p}_{2,k,n} = \widetilde{s}_{2,k,n} - \gamma_n \Big(r_k + J_{\gamma_n^{-1} B_k}(\gamma_n^{-1} \widetilde{s}_{2,k,n} - r_k) \Big). \end{cases} \tag{2.17}$$

It follows from (2.4) that

$$(\forall i \in \{1,\ldots,m\})(\forall n \in \mathbb{N}) \quad \|s_{1,i,n} - \widetilde{s}_{1,i,n}\| = \gamma_n \|a_{1,i,n}\| \leqslant \beta^{-1}\|a_{1,i,n}\|. \tag{2.18}$$

Hence, by virtue of the nonexpansiveness of the resolvents [8, Proposition 23.7], we have

$$\begin{aligned}(\forall i \in \{1,\ldots,m\})(\forall n \in \mathbb{N}) \quad & \|p_{1,i,n} - \widetilde{p}_{1,i,n}\| \\ &= \|J_{\gamma_n A_i}(s_{1,i,n} + \gamma_n z_i) + b_{1,i,n} - J_{\gamma_n A_i}(\widetilde{s}_{1,i,n} + \gamma_n z_i)\| \\ &\leqslant \|s_{1,i,n} - \widetilde{s}_{1,i,n}\| + \|b_{1,i,n}\| \\ &\leqslant \beta^{-1}\|a_{1,i,n}\| + \|b_{1,i,n}\|. \end{aligned} \tag{2.19}$$

In turn, since, for every $i \in \{1,\ldots,m\}$, $(a_{1,i,n})_{n\in\mathbb{N}}$ and $(b_{1,i,n})_{n\in\mathbb{N}}$ are absolutely summable, we get

$$(\forall i \in \{1,\ldots,m\}) \quad s_{1,i,n} - \widetilde{s}_{1,i,n} \to 0 \quad \text{and} \quad p_{1,i,n} - \widetilde{p}_{1,i,n} \to 0. \tag{2.20}$$

Likewise, we derive from (2.4) and (2.17) that

$$(\forall k \in \{1,\ldots,K\}) \quad s_{2,k,n} - \widetilde{s}_{2,k,n} \to 0 \quad \text{and} \quad p_{2,k,n} - \widetilde{p}_{2,k,n} \to 0. \tag{2.21}$$

On the other hand, we deduce from (iii)(a) that

$$(\forall i \in \{1,\ldots,m\})(\exists\, u_i \in \mathcal{H}_i) \quad u_i \in A_i\overline{x_i} \quad \text{and} \quad z_i = u_i + \sum_{k=1}^{K} L_{ki}^*\overline{v_k} + C_i\overline{x_i}, \tag{2.22}$$

and from (iii)(b) that

$$(\forall k \in \{1,\ldots,K\}) \quad \overline{v_k} \in B_k\Bigg(\sum_{i=1}^{m} L_{ki}\overline{x_i} - r_k - D_k^{-1}\overline{v_k}\Bigg). \tag{2.23}$$

In addition, (2.16) yields

$$(\forall i \in \{1,\ldots,m\})(\forall n \in \mathbb{N}) \quad \frac{x_{i,n} - \widetilde{p}_{1,i,n}}{\gamma_n} - \sum_{k=1}^{K} L_{ki}^* v_{k,n} - C_i x_{i,n} + z_i \in A_i\widetilde{p}_{1,i,n}, \tag{2.24}$$

while (2.17) yields

$$\begin{aligned}&(\forall k \in \{1,\ldots,K\})(\forall n \in \mathbb{N}) \\ &\qquad \widetilde{p}_{2,k,n} \in B_k\left(\frac{v_{k,n} - \widetilde{p}_{2,k,n}}{\gamma_n} + \sum_{i=1}^{m} L_{ki}x_{i,n} - r_k - D_k^{-1}v_{k,n}\right).\end{aligned} \tag{2.25}$$

Now, for every $n \in \mathbb{N}$, let us set

$$\text{(2.26)}\quad \delta_n = \sum_{k=1}^{K} \left(\frac{1}{\varepsilon} + \nu_k\right) \|v_{k,n} - \widetilde{p}_{2,k,n}\| \, \|\widetilde{p}_{2,k,n} - \overline{v_k}\| \quad \text{and} \quad (\forall i \in \{1, \ldots, m\})$$
$$\alpha_{i,n} = \|\widetilde{p}_{1,i,n} - x_{i,n}\| \left(\frac{1}{\varepsilon}\|\widetilde{p}_{1,i,n} - \overline{x_i}\| + \mu_i \|x_{i,n} - \overline{x_i}\| + \sum_{k=1}^{K} \|L_{ki}\| \, \|v_{k,n} - \overline{v_k}\|\right).$$

It follows from (i), (ii), (iii)(c), (iii)(d), (2.20), and (2.21) that

$$\text{(2.27)}\qquad \delta_n \to 0 \quad \text{and} \quad (\forall i \in \{1, \ldots, m\}) \quad \alpha_{i,n} \to 0.$$

Using the Cauchy–Schwarz inequality, the Lipschitz-continuity and the monotonicity of the operators $(C_i)_{1 \leqslant i \leqslant m}$, (2.22), (2.24), and the monotonicity of the operators $(A_i)_{1 \leqslant i \leqslant m}$, we obtain

$$\begin{aligned}
&(\forall i \in \{1, \ldots, m\})(\forall n \in \mathbb{N}) \quad \alpha_{i,n} + \left\langle x_{i,n} - \overline{x_i} \,\middle|\, \sum_{k=1}^{K} L_{ki}^*(\overline{v_k} - v_{k,n})\right\rangle \\
&\quad \geqslant \|\widetilde{p}_{1,i,n} - x_{i,n}\| \big(\varepsilon^{-1} \|\widetilde{p}_{1,i,n} - \overline{x_i}\| + \|C_i x_{i,n} - C_i \overline{x_i}\|\big) \\
&\qquad + \left\langle \widetilde{p}_{1,i,n} - x_{i,n} \,\middle|\, \sum_{k=1}^{K} L_{ki}^*(\overline{v_k} - v_{k,n})\right\rangle + \left\langle x_{i,n} - \overline{x_i} \,\middle|\, \sum_{k=1}^{K} L_{ki}^*(\overline{v_k} - v_{k,n})\right\rangle \\
&\quad = \|\widetilde{p}_{1,i,n} - x_{i,n}\| \big(\varepsilon^{-1} \|\widetilde{p}_{1,i,n} - \overline{x_i}\| + \|C_i x_{i,n} - C_i \overline{x_i}\|\big) \\
&\qquad + \left\langle \widetilde{p}_{1,i,n} - \overline{x_i} \,\middle|\, \sum_{k=1}^{K} L_{ki}^*(\overline{v_k} - v_{k,n})\right\rangle \\
&\quad \geqslant \left\langle \widetilde{p}_{1,i,n} - \overline{x_i} \,\middle|\, \frac{x_{i,n} - \widetilde{p}_{1,i,n}}{\gamma_n} + \sum_{k=1}^{K} L_{ki}^*(\overline{v_k} - v_{k,n})\right\rangle \\
&\qquad + \langle \widetilde{p}_{1,i,n} - x_{i,n} \mid C_i \overline{x_i} - C_i x_{i,n}\rangle \\
&\quad = \left\langle \widetilde{p}_{1,i,n} - \overline{x_i} \,\middle|\, \frac{x_{i,n} - \widetilde{p}_{1,i,n}}{\gamma_n} - \sum_{k=1}^{K} L_{ki}^* v_{k,n} - C_i x_{i,n} + \sum_{k=1}^{K} L_{ki}^* \overline{v_k} + C_i \overline{x_i}\right\rangle \\
&\qquad + \langle x_{i,n} - \overline{x_i} \mid C_i x_{i,n} - C_i \overline{x_i}\rangle \\
&\quad = \left\langle \widetilde{p}_{1,i,n} - \overline{x_i} \,\middle|\, \frac{x_{i,n} - \widetilde{p}_{1,i,n}}{\gamma_n} - \sum_{k=1}^{K} L_{ki}^* v_{k,n} - C_i x_{i,n} + z_i - u_i\right\rangle \\
\text{(2.28)}&\qquad + \langle x_{i,n} - \overline{x_i} \mid C_i x_{i,n} - C_i \overline{x_i}\rangle \\
\text{(2.29)}&\quad \geqslant \left\langle \widetilde{p}_{1,i,n} - \overline{x_i} \,\middle|\, \left(\frac{x_{i,n} - \widetilde{p}_{1,i,n}}{\gamma_n} - \sum_{k=1}^{K} L_{ki}^* v_{k,n} - C_i x_{i,n} + z_i\right) - u_i\right\rangle \\
\text{(2.30)}&\quad \geqslant 0.
\end{aligned}$$

On the other hand, since the operators $(D_k^{-1})_{1 \leqslant k \leqslant K}$ are Lipschitzian and monotone, and since the operators $(B_k)_{1 \leqslant k \leqslant K}$ are monotone, we deduce from (2.26), (2.23), and

(2.25) that

$$(\forall l \in \{1.\dots,K\})(\forall n \in \mathbb{N}) \quad \delta_n + \sum_{i=1}^m \left\langle x_{i,n} - \overline{x_i} \,\middle|\, \sum_{k=1}^K L_{ki}^*(\widetilde{p}_{2,k,n} - \overline{v_k}) \right\rangle$$

$$\geqslant \sum_{k=1}^K \left\langle \frac{v_{k,n} - \widetilde{p}_{2,k,n}}{\gamma_n} + D_k^{-1}\widetilde{p}_{2,k,n} - D_k^{-1}v_{k,n} + \sum_{i=1}^m L_{ki}(x_{i,n} - \overline{x_i}) \,\middle|\, \widetilde{p}_{2,k,n} - \overline{v_k} \right\rangle$$

$$= \sum_{k=1}^K \left\langle \left(\frac{v_{k,n} - \widetilde{p}_{2,k,n}}{\gamma_n} + \sum_{i=1}^m L_{ki}x_{i,n} - r_k - D_k^{-1}v_{k,n} \right) - \left(\sum_{i=1}^m L_{ki}\overline{x_i} - r_k - D_k^{-1}\overline{v_k} \right) \,\middle|\, \widetilde{p}_{2,k,n} - \overline{v_k} \right\rangle$$

$$+ \sum_{k=1}^K \left\langle D_k^{-1}\widetilde{p}_{2,k,n} - D_k^{-1}\overline{v_k} \mid \widetilde{p}_{2,k,n} - \overline{v_k} \right\rangle \tag{2.31}$$

$$\geqslant \left\langle \left(\frac{v_{l,n} - \widetilde{p}_{2,l,n}}{\gamma_n} + \sum_{i=1}^m L_{li}x_{i,n} - r_l - D_l^{-1}v_{l,n} \right) - \left(\sum_{i=1}^m L_{li}\overline{x_i} - r_l - D_l^{-1}\overline{v_l} \right) \,\middle|\, \widetilde{p}_{2,l,n} - \overline{v_l} \right\rangle$$

$$+ \left\langle D_l^{-1}\widetilde{p}_{2,l,n} - D_l^{-1}\overline{v_l} \mid \widetilde{p}_{2,l,n} - \overline{v_l} \right\rangle \tag{2.32}$$

$$\geqslant \left\langle \left(\frac{v_{l,n} - \widetilde{p}_{2,l,n}}{\gamma_n} + \sum_{i=1}^m L_{li}x_{i,n} - r_l - D_l^{-1}v_{l,n} \right) - \left(\sum_{i=1}^m L_{li}\overline{x_i} - r_l - D_l^{-1}\overline{v_l} \right) \,\middle|\, \widetilde{p}_{2,l,n} - \overline{v_l} \right\rangle \tag{2.33}$$

$$\geqslant 0. \tag{2.34}$$

We consider two cases.

- If A_j is uniformly monotone at $\overline{x_j}$, then, in view of (2.29), (2.22), (2.24), and (1.12), there exists an increasing function $\phi_{A_j}\colon\ [0,+\infty[\to [0,+\infty]$ that vanishes only at 0 such that

  $$(\forall n \in \mathbb{N}) \quad \alpha_{j,n} + \left\langle x_{j,n} - \overline{x_j} \,\middle|\, \sum_{k=1}^K L_{kj}^*(\overline{v_k} - v_{k,n}) \right\rangle \geqslant \phi_{A_j}(\|\widetilde{p}_{1,j,n} - \overline{x_j}\|). \tag{2.35}$$

  Combining (2.34), (2.30), and (2.35) yields

  $$(\forall n \in \mathbb{N}) \quad \delta_n + \sum_{i=1}^m \alpha_{i,n} + \sum_{i=1}^m \left\langle x_{i,n} - \overline{x_i} \,\middle|\, \sum_{k=1}^K L_{ki}^*(\widetilde{p}_{2,k,n} - v_{k,n}) \right\rangle \geqslant \phi_{A_j}(\|\widetilde{p}_{1,j,n} - \overline{x_j}\|). \tag{2.36}$$

  It follows from (2.27), (ii), (iii)(c), (2.21), and [8, Lemma 2.41(iii)] that $\phi_{A_j}(\|\widetilde{p}_{1,j,n} - \overline{x_j}\|) \to 0$ and, in turn, that $\widetilde{p}_{1,j,n} \to \overline{x_j}$. In view of (i) and (2.20), we get $p_{1,j,n} \to \overline{x_j}$ and $x_{j,n} \to \overline{x_j}$.

- If C_j is uniformly monotone at $\overline{x_j}$, then we derive from (2.34), (2.28), and (2.30) that there exists an increasing function $\phi_{C_j}\colon [0,+\infty[\,\to[0,+\infty]$ that vanishes only at 0 such that

$$(\forall n\in\mathbb{N})\quad \delta_n+\sum_{i=1}^m\alpha_{i,n}+\sum_{i=1}^m\left\langle x_{i,n}-\overline{x_i}\,\middle|\,\sum_{k=1}^K L_{ki}^*(\widetilde{p}_{2,k,n}-v_{k,n})\right\rangle \geqslant \phi_{C_j}(\|x_{j,n}-\overline{x_j}\|). \tag{2.37}$$

This implies that $\phi_{C_j}(\|x_{j,n}-\overline{x_j}\|)\to 0$ and hence that $x_{j,n}\to\overline{x_j}$. Finally, (i) yields $p_{1,j,n}\to\overline{x_j}$.

(iii)(f) We consider two cases.

- If B_l is couniformly monotone at $\overline{v_l}$, then (2.33), (2.23), and (2.25) imply that there exists an increasing function $\phi_{B_l^{-1}}\colon [0,+\infty[\,\to[0,+\infty]$ that vanishes only at 0 such that

$$\begin{aligned}(\forall n\in\mathbb{N})\quad \delta_n+\sum_{i=1}^m\left\langle x_{i,n}-\overline{x_i}\,\middle|\,\sum_{k=1}^K L_{ki}^*(\widetilde{p}_{2,k,n}-\overline{v_k})\right\rangle&\\ \geqslant\Bigg\langle\bigg(\frac{v_{l,n}-\widetilde{p}_{2,l,n}}{\gamma_n}+\sum_{i=1}^m L_{li}x_{i,n}-r_l-D_l^{-1}v_{l,n}\bigg)&\\ -\bigg(\sum_{i=1}^m L_{li}\overline{x_i}-r_l-D_l^{-1}\overline{v_l}\bigg)\,\bigg|\,\widetilde{p}_{2,l,n}-\overline{v_l}\Bigg\rangle&\\ \geqslant\phi_{B_l^{-1}}(\|\widetilde{p}_{2,l,n}-\overline{v_l}\|).&\end{aligned}\tag{2.38}$$

Combining this with (2.30) yields

$$(\forall n\in\mathbb{N})\quad \delta_n+\sum_{i=1}^m\alpha_{i,n}+\sum_{i=1}^m\left\langle x_{i,n}-\overline{x_i}\,\middle|\,\sum_{k=1}^K L_{ki}^*(\widetilde{p}_{2,k,n}-v_{k,n})\right\rangle \geqslant \phi_{B_l^{-1}}(\|\widetilde{p}_{2,l,n}-\overline{v_l}\|). \tag{2.39}$$

Hence, using (2.27), (ii), (iii)(c), (2.21), and [8, Lemma 2.41(iii)], we get $\phi_{B_l^{-1}}(\|\widetilde{p}_{2,l,n}-\overline{v_l}\|)\to 0$ and, in turn, $\widetilde{p}_{2,l,n}\to\overline{v_l}$. Using (2.21) and (ii), we conclude that $p_{2,l,n}\to\overline{v_l}$ and $v_{l,n}\to\overline{v_l}$.

- If D_l is couniformly monotone at $\overline{v_l}$, then it follows from (2.32) and (2.34) that there exists an increasing function $\phi_{D_l^{-1}}\colon [0,+\infty[\,\to[0,+\infty]$ that vanishes only at 0 such that

$$\begin{aligned}(\forall n\in\mathbb{N})\quad \delta_n+\sum_{i=1}^m\left\langle x_{i,n}-\overline{x_i}\,\middle|\,\sum_{k=1}^K L_{ki}^*(\widetilde{p}_{2,k,n}-\overline{v_k})\right\rangle&\\ \geqslant\left\langle D_l^{-1}\widetilde{p}_{2,l,n}-D_l^{-1}\overline{v_l}\mid\widetilde{p}_{2,l,n}-\overline{v_l}\right\rangle&\\ \geqslant\phi_{D_l^{-1}}(\|\widetilde{p}_{2,l,n}-\overline{v_l}\|).&\end{aligned}\tag{2.40}$$

Thus, (2.30) yields

$$(\forall n\in\mathbb{N})\quad \delta_n+\sum_{i=1}^m\alpha_{i,n}+\sum_{i=1}^m\left\langle x_{i,n}-\overline{x_i}\,\middle|\,\sum_{k=1}^K L_{ki}^*(\widetilde{p}_{2,k,n}-v_{k,n})\right\rangle \geqslant \phi_{D_l^{-1}}(\|\widetilde{p}_{2,l,n}-\overline{v_l}\|), \tag{2.41}$$

and we conclude as above. ∎

Remark 2.5. When $m = 1$, (1.3)–(1.4) assume the form of (1.8)–(1.9), and Theorem 2.4 specializes to [20, Theorem 3.1]. Our proof of Theorem 2.4(i)–(iii)(d) hinges on a self-contained application of Lemmas 2.1 and 2.3 in the primal-dual product space $\boldsymbol{\mathcal{K}}$ of (2.5). Alternatively, these results could be obtained as an application of [20, Theorem 3.1] using the product space $\boldsymbol{\mathcal{H}}$ of (2.5) as a primal space. This strategy, however, would not enable us to recover the strong convergence results of Theorem 2.4(iii)(e) since [20, Theorem 3.1] would impose uniform monotonicity properties on the product operators $\boldsymbol{A}$ or $\boldsymbol{C}$ of (2.6), which, in general, do not translate easily into properties of the individual operators $(A_i)_{1\leqslant i\leqslant m}$ and $(C_i)_{1\leqslant i\leqslant m}$. By contrast, our framework exploits properties of each operator individually without imposing a global uniform monotonicity property on their product.

Remark 2.6. It follows from the Cauchy–Schwarz inequality that, for every $(x_i)_{1\leqslant i\leqslant m} \in \bigoplus_{i=1}^m \mathcal{H}_i$,

(2.42)
$$\sum_{k=1}^K \left\| \sum_{i=1}^m L_{ki} x_i \right\|^2 \leqslant \sum_{k=1}^K \left(\sum_{i=1}^m \|L_{ki}\|\,\|x_i\| \right)^2 \leqslant \sum_{k=1}^K \left(\sum_{i=1}^m \|L_{ki}\|^2 \right) \left(\sum_{i=1}^m \|x_i\|^2 \right).$$

Hence, in general, one can use $\lambda = \sum_{k=1}^K \sum_{i=1}^m \|L_{ki}\|^2$ in (1.2). However, as will be seen in subsequent sections, this bound can be improved when the operator $\boldsymbol{L}$ of (2.6) has a special structure.

In the remainder the paper, we highlight a few instantiations of Theorem 2.4 that illustrate the variety of problems to which it can be applied and which are not explicitly solvable via existing techniques. (See also [11] for additional applications.)

3. Inclusions involving general parallel sums. The first special case of Problem 1.1 that we feature is an extension of a univariate inclusion problem investigated in [20], which involves parallel sums with monotone operators admitting Lipschitzian inverses. In the following formulation, we lift this restriction.

PROBLEM 3.1. *Let $\mathcal{H}$ be a real Hilbert space, let K_1, K_2, and K be integers such that $0 \leqslant K_1 \leqslant K_2 \leqslant K \geqslant 1$, let $z \in \mathcal{H}$, let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone, and let $C\colon \mathcal{H} \to \mathcal{H}$ be monotone and μ-Lipschitzian for some $\mu \in [0,+\infty[$. For every integer $k \in \{1,\ldots,K\}$, let $\mathcal{G}_k$ be a real Hilbert space, let $r_k \in \mathcal{G}_k$, let $B_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ and $S_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ be maximally monotone, and let $L_k \in \mathcal{B}(\mathcal{H},\mathcal{G}_k)$; moreover, if $K_1+1 \leqslant k \leqslant K_2$, $S_k\colon \mathcal{G}_k \to \mathcal{G}_k$ is β_k-Lipschitzian for some $\beta_k \in [0,+\infty[$, and if $K_2+1 \leqslant k \leqslant K$, $S_k^{-1}\colon \mathcal{G}_k \to \mathcal{G}_k$ is β_k-Lipschitzian for some $\beta_k \in [0,+\infty[$. It is assumed that*

$$\beta = \max\{\mu, \beta_{K_1+1}, \ldots, \beta_K\} + \sqrt{1 + \sum_{k=1}^K \|L_k\|^2} > 0 \tag{3.1}$$

and that the inclusion

$$\text{find } \overline{x} \in \mathcal{H} \text{ such that } z \in A\overline{x} + \sum_{k=1}^K L_k^*\big((B_k \,\square\, S_k)(L_k\overline{x} - r_k)\big) + C\overline{x} \tag{3.2}$$

possesses at least one solution. Solve (3.2) *together with the dual problem*

(3.3) *find* $\overline{v_1} \in \mathcal{G}_1,\ \ldots,\ \overline{v_K} \in \mathcal{G}_K$ *such that*

$$(\forall k \in \{1,\ldots,K\}) \quad -r_k \in -L_k\left((A+C)^{-1}\left(z - \sum_{l=1}^K L_l^*\overline{v_l}\right)\right) + B_k^{-1}\overline{v_k} + S_k^{-1}\overline{v_k}.$$

PROPOSITION 3.2. *Consider the setting of Problem* 3.1. *Let* $(a_{1,1,n})_{n\in\mathbb{N}}$, $(b_{1,1,n})_{n\in\mathbb{N}}$, *and* $(c_{1,1,n})_{n\in\mathbb{N}}$ *be absolutely summable sequences in* $\mathcal{H}$. *For every integer* $k\in\{1,\ldots,K\}$, *let* $(a_{2,k,n})_{n\in\mathbb{N}}$, $(b_{2,k,n})_{n\in\mathbb{N}}$, *and* $(c_{2,k,n})_{n\in\mathbb{N}}$ *be absolutely summable sequences in* $\mathcal{G}_k$; *moreover, if* $1\leqslant k\leqslant K_1$, *let* $(b_{1,k+1,n})_{n\in\mathbb{N}}$ *be an absolutely summable sequence in* $\mathcal{G}_k$, *and if* $K_1+1\leqslant k\leqslant K_2$, *let* $(a_{1,k+1,n})_{n\in\mathbb{N}}$ *and* $(c_{1,k+1,n})_{n\in\mathbb{N}}$ *be absolutely summable sequences in* $\mathcal{G}_k$. *Let* $x_0\in\mathcal{H}$, $y_{1,0}\in\mathcal{G}_1$, $\ldots$, $y_{K_2,0}\in\mathcal{G}_{K_2}$, $v_{1,0}\in\mathcal{G}_1$, $\ldots$, *and* $v_{K,0}\in\mathcal{G}_K$, *let* $\varepsilon\in\,]0,1/(\beta+1)[$, *let* $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence in* $[\varepsilon,(1-\varepsilon)/\beta]$, *and set*

$$
\begin{array}{l}
\textit{for } n=0,1,\ldots\\
\left\lfloor
\begin{array}{l}
s_{1,1,n}=x_n-\gamma_n\big(Cx_n+\sum_{k=1}^K L_k^*v_{k,n}+a_{1,1,n}\big),\\
p_{1,1,n}=J_{\gamma_n A}(s_{1,1,n}+\gamma_n z)+b_{1,1,n},\\
\textit{if } K_1\neq 0,\ \textit{for } k=1,\ldots,K_1\\
\left\lfloor
\begin{array}{l}
s_{1,k+1,n}=y_{k,n}+\gamma_n v_{k,n},\\
p_{1,k+1,n}=J_{\gamma_n S_k}s_{1,k+1,n}+b_{1,k+1,n},\\
s_{2,k,n}=v_{k,n}-\gamma_n\big(y_{k,n}-L_kx_n+a_{2,k,n}\big),\\
p_{2,k,n}=s_{2,k,n}-\gamma_n\big(r_k+J_{\gamma_n^{-1}B_k}(\gamma_n^{-1}s_{2,k,n}-r_k)+b_{2,k,n}\big),\\
q_{2,k,n}=p_{2,k,n}-\gamma_n\big(p_{1,k+1,n}-L_kp_{1,1,n}+c_{2,k,n}\big),\\
v_{k,n+1}=v_{k,n}-s_{2,k,n}+q_{2,k,n},
\end{array}
\right.\\
\textit{if } K_1\neq K_2,\ \textit{for } k=K_1+1,\ldots,K_2\\
\left\lfloor
\begin{array}{l}
s_{1,k+1,n}=y_{k,n}-\gamma_n\big(S_ky_{k,n}-v_{k,n}+a_{1,k+1,n}\big),\\
p_{1,k+1,n}=s_{1,k+1,n},\\
s_{2,k,n}=v_{k,n}-\gamma_n\big(y_{k,n}-L_kx_n+a_{2,k,n}\big),\\
p_{2,k,n}=s_{2,k,n}-\gamma_n\big(r_k+J_{\gamma_n^{-1}B_k}(\gamma_n^{-1}s_{2,k,n}-r_k)+b_{2,k,n}\big),\\
q_{2,k,n}=p_{2,k,n}-\gamma_n\big(p_{1,k+1,n}-L_kp_{1,1,n}+c_{2,k,n}\big),\\
v_{k,n+1}=v_{k,n}-s_{2,k,n}+q_{2,k,n},
\end{array}
\right.\\
\textit{if } K_2\neq K,\ \textit{for } k=K_2+1,\ldots,K\\
\left\lfloor
\begin{array}{l}
s_{2,k,n}=v_{k,n}-\gamma_n\big(S_k^{-1}v_{k,n}-L_kx_n+a_{2,k,n}\big),\\
p_{2,k,n}=s_{2,k,n}-\gamma_n\big(r_k+J_{\gamma_n^{-1}B_k}(\gamma_n^{-1}s_{2,k,n}-r_k)+b_{2,k,n}\big),\\
q_{2,k,n}=p_{2,k,n}-\gamma_n\big(S_k^{-1}p_{2,k,n}-L_kp_{1,1,n}+c_{2,k,n}\big),\\
v_{k,n+1}=v_{k,n}-s_{2,k,n}+q_{2,k,n},
\end{array}
\right.\\
q_{1,1,n}=p_{1,1,n}-\gamma_n\big(Cp_{1,1,n}+\sum_{k=1}^K L_k^*p_{2,k,n}+c_{1,1,n}\big),\\
x_{n+1}=x_n-s_{1,1,n}+q_{1,1,n},\\
\textit{if } K_1\neq 0,\ \textit{for } k=1,\ldots,K_1\\
\left\lfloor
\begin{array}{l}
q_{1,k+1,n}=p_{1,k+1,n}+\gamma_n p_{2,k,n},\\
y_{k,n+1}=y_{k,n}-s_{1,k+1,n}+q_{1,k+1,n},
\end{array}
\right.\\
\textit{if } K_1\neq K_2,\ \textit{for } k=K_1+1,\ldots,K_2\\
\left\lfloor
\begin{array}{l}
q_{1,k+1,n}=p_{1,k+1,n}-\gamma_n\big(S_kp_{1,k+1,n}-p_{2,k,n}+c_{1,k+1,n}\big),\\
Sy_{k,n+1}=y_{k,n}-s_{1,k+1,n}+q_{1,k+1,n}.
\end{array}
\right.
\end{array}
\right.
\end{array}
\tag{3.4}
$$

Then the following hold for some solution $\overline{x}$ *to* (3.2) *and some solution* $(\overline{v_1},\ldots,\overline{v_K})$ *to* (3.3):

(i) $x_n\rightharpoonup\overline{x}$ *and* $(\forall k\in\{1,\ldots,K\})$ $v_{k,n}\rightharpoonup\overline{v_k}$.

(ii) *Suppose that* A *or* C *is uniformly monotone at* $\overline{x}$. *Then* $x_n\to\overline{x}$.

(iii) *Suppose that, for some* $l\in\{1,\ldots,K\}$, B_l *is couniformly monotone at* $\overline{v_l}$. *Then* $v_{l,n}\to\overline{v_l}$.

(iv) *Suppose that $K_2 \neq K$ and that, for some $l \in \{K_2+1,\dots,K\}$, S_l is couniformly monotone at $\overline{v_l}$. Then $v_{l,n} \to \overline{v_l}$.*

Proof. We assume that $K_2 \neq 0$ and consider the auxiliary problem

(3.5) find $\overline{x} \in \mathcal{H}$, $\overline{y_1} \in \mathcal{G}_1, \dots, \overline{y_{K_2}} \in \mathcal{G}_{K_2}$ such that

$$\begin{cases} z \in A\overline{x} + \displaystyle\sum_{k=1}^{K_2} L_k^*\big(B_k(L_k\overline{x} - \overline{y_k} - r_k)\big) + \sum_{k=K_2+1}^{K} L_k^*\big((B_k \,\Box\, S_k)(L_k\overline{x} - r_k)\big) + C\overline{x}, \\ 0 \in S_1\overline{y_1} - B_1(L_1\overline{x} - \overline{y_1} - r_1) \\ \quad\vdots \\ 0 \in S_{K_2}\overline{y_{K_2}} - B_{K_2}(L_{K_2}\overline{x} - \overline{y_{K_2}} - r_{K_2}) \end{cases}$$

together with the dual problem (3.3) (if $K_2 = 0$, (3.5) should be replaced by (3.2) and the resulting simplifications in the proof are straightforward). Let us show that solving the primal-dual problem (3.5)/(3.3) is a special case of Problem 1.1 with

(3.6)
$$\begin{cases} m = K_2 + 1 \\ \mathcal{H}_1 = \mathcal{H} \\ A_1 = A \\ C_1 = C \\ \mu_1 = \mu \\ \overline{x_1} = \overline{x} \\ z_1 = z, \end{cases} \qquad (\forall k \in \{1,\dots,K_2\}) \quad \begin{cases} \mathcal{H}_{k+1} = \mathcal{G}_k, \\ A_{k+1} = \begin{cases} S_k & \text{if } 1 \leqslant k \leqslant K_1, \\ 0 & \text{if } K_1+1 \leqslant k \leqslant K_2, \end{cases} \\ C_{k+1} = \begin{cases} 0 & \text{if } 1 \leqslant k \leqslant K_1, \\ S_k & \text{if } K_1+1 \leqslant k \leqslant K_2, \end{cases} \\ \mu_{k+1} = \begin{cases} 0 & \text{if } 1 \leqslant k \leqslant K_1, \\ \beta_k & \text{if } K_1+1 \leqslant k \leqslant K_2, \end{cases} \\ \overline{x_{k+1}} = \overline{y_k}, \\ z_{k+1} = 0, \end{cases}$$

and

$$(3.7) \quad (\forall k \in \{1,\dots,K\}) \quad \begin{cases} D_k = \begin{cases} \{0\}^{-1} & \text{if } 1 \leqslant k \leqslant K_2, \\ S_k & \text{if } K_2+1 \leqslant k \leqslant K, \end{cases} \\ \nu_{k+1} = \begin{cases} 0 & \text{if } 1 \leqslant k \leqslant K_2, \\ \beta_k & \text{if } K_2+1 \leqslant k \leqslant K, \end{cases} \\ L_{k1} = L_k, \\ (\forall i \in \{2,\dots,K_2+1\}) \;\; L_{ki} = \begin{cases} -\mathrm{Id} & \text{if } i = k+1, \\ 0 & \text{otherwise.} \end{cases} \end{cases}$$

First, we note that, in this setting, (1.3) reduces to (3.5), and (1.4) to (3.3). Now define $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{G}}$ as in (2.5), let $x \in \mathcal{H}$, let $(y_k)_{1 \leqslant k \leqslant K_2} \in \bigoplus_{k=1}^{K_2} \mathcal{G}_k$, set $(x_i)_{1 \leqslant i \leqslant m} = (x, y_1, \dots, y_{K_2}) \in \boldsymbol{\mathcal{H}}$, set $\boldsymbol{y} = (y_1, \dots, y_{K_2}, 0, \dots, 0) \in \boldsymbol{\mathcal{G}}$, and set $\lambda = 1 + \sum_{k=1}^{K_2} \|L_k\|^2$.

Then, using the Cauchy–Schwarz inequality in $\mathbb{R}^2$,

$$\sum_{k=1}^{K}\Bigg\|\sum_{i=1}^{m}L_{ki}x_i\Bigg\|^2 = \|(L_kx)_{1\leqslant k\leqslant K_2}-\boldsymbol{y}\|^2 \leqslant \big(\|\boldsymbol{y}\|+\|(L_kx)_{1\leqslant k\leqslant K_2}\|\big)^2$$
$$\leqslant \left(\|\boldsymbol{y}\|+\sqrt{\sum_{k=1}^{K_2}\|L_k\|^2}\,\|x\|\right)^2$$
$$\leqslant \left(1+\sum_{k=1}^{K_2}\|L_k\|^2\right)\big(\|\boldsymbol{y}\|^2+\|x\|^2\big) = \lambda\sum_{i=1}^{m}\|x_i\|^2. \tag{3.8}$$

Thus, (3.1) is a special case of (1.2). On the other hand, by assumption, (3.2) has a solution, say, x. Therefore, there exist $v_1 \in \mathcal{G}_1, \ldots, v_{K_2} \in \mathcal{G}_{K_2}$ such that

$$\begin{cases} z \in Ax + \displaystyle\sum_{k=1}^{K_2} L_k^* v_k + \sum_{k=K_2+1}^{K} L_k^*\big((B_k \,\Box\, S_k)(L_kx - r_k)\big) + Cx, \\ (\forall k \in \{1,\ldots,K_2\}) \quad v_k \in (B_k \,\Box\, S_k)(L_kx - r_k). \end{cases} \tag{3.9}$$

Therefore, in view of (1.1), there exist $y_1 \in \mathcal{G}_1, \ldots, y_{K_2} \in \mathcal{G}_{K_2}$ such that

$$\begin{cases} z \in Ax + \displaystyle\sum_{k=1}^{K_2} L_k^* v_k + \sum_{k=K_2+1}^{K} L_k^*\big((B_k \,\Box\, S_k)(L_kx - r_k)\big) + Cx, \\ (\forall k \in \{1,\ldots,K_2\}) \quad y_k \in S_k^{-1}v_k \quad \text{and} \quad L_kx - y_k - r_k \in B_k^{-1}v_k, \end{cases} \tag{3.10}$$

which implies that

$$\begin{cases} z \in Ax + \displaystyle\sum_{k=1}^{K_2} L_k^* v_k + \sum_{k=K_2+1}^{K} L_k^*\big((B_k \,\Box\, S_k)(L_kx - r_k)\big) + Cx, \\ (\forall k \in \{1,\ldots,K_2\}) \quad v_k \in S_ky_k \quad \text{and} \quad v_k \in B_k(L_kx - y_k - r_k), \end{cases} \tag{3.11}$$

and therefore that

$$\begin{cases} z \in Ax + \displaystyle\sum_{k=1}^{K_2} L_k^*\big(B_k(L_kx - y_k - r_k)\big) + \sum_{k=K_2+1}^{K} L_k^*\big((B_k \,\Box\, S_k)(L_kx - r_k)\big) + Cx, \\ (\forall k \in \{1,\ldots,K_2\}) \quad 0 \in S_ky_k - B_k(L_kx - y_k - r_k). \end{cases} \tag{3.12}$$

This shows that (3.5) possesses a solution. Next, upon defining

$$(\forall n \in \mathbb{N}) \quad x_{1,n} = x_n \quad \text{and} \tag{3.13}$$
$$(\forall k \in \{1,\ldots,K_2\}) \quad \begin{cases} x_{k+1,n} = y_{k,n}, \\ a_{1,k+1,n} = 0 & \text{if } 1 \leqslant k \leqslant K_1, \\ b_{1,k+1,n} = 0 & \text{if } K_1+1 \leqslant k \leqslant K_2, \\ c_{1,k+1,n} = 0 & \text{if } 1 \leqslant k \leqslant K_1, \end{cases}$$

we see that (2.4) specializes to (3.4). Hence, in view of (3.6)–(3.7) and Theorem 2.4(iii)(a)–(iii)(d), there exist a solution $(\overline{x}, \overline{y_1}, \ldots, \overline{y_{K_2}})$ to (3.5) and a solution $(\overline{v_1}, \ldots, \overline{v_K})$ to (3.3) such that

$$x_n \rightharpoonup \overline{x} \quad \text{and} \quad (\forall k \in \{1,\ldots,K\}) \quad v_{k,n} \rightharpoonup \overline{v_k}, \tag{3.14}$$

with

(3.15)
$$z-\sum_{k=1}^K L_k^*\overline{v_k}\in A\overline{x}+C\overline{x},\quad (\forall k\in\{1,\ldots,K_2\})\quad \begin{cases}L_k\overline{x}-\overline{y_k}-r_k\in B_k^{-1}\overline{v_k},\\ \overline{v_k}\in S_k\overline{y_k},\end{cases}$$
$$\text{and}\quad (\forall k\in\{K_2+1,\ldots,K\})\quad L_k\overline{x}-r_k\in B_k^{-1}\overline{v_k}+S_k^{-1}\overline{v_k}.$$

Since the strong convergence claims (ii)–(iv) are immediate consequences of Theorem 2.4(iii)(e)–(iii)(f), it remains to show that $\overline{x}$ solves (3.2). We derive from (3.15) that, for every $k\in\{1,\ldots,K_2\}$, $L_k\overline{x}-\overline{y_k}-r_k\in B_k^{-1}\overline{v_k}$ and $\overline{y_k}\in S_k^{-1}\overline{v_k}$, and, for every $k\in\{K_2+1,\ldots,K\}$, $L_k\overline{x}-r_k\in B_k^{-1}\overline{v_k}+S_k^{-1}\overline{v_k}$. Altogether,

$$(\forall k\in\{1,\ldots,K\})\quad L_k\overline{x}-r_k\in\big(B_k^{-1}+S_k^{-1}\big)\overline{v_k} \tag{3.16}$$

and, therefore,

$$\sum_{k=1}^K L_k^*\overline{v_k}\in\sum_{k=1}^K L_k^*\Big(\big(B_k^{-1}+S_k^{-1}\big)^{-1}(L_k\overline{x}-r_k)\Big)=\sum_{k=1}^K L_k^*\big((B_k\,\Box\,S_k)(L_k\overline{x}-r_k)\big). \tag{3.17}$$

Thus, since (3.15) also asserts that $z-\sum_{k=1}^K L_k^*\overline{v_k}\in A\overline{x}+C\overline{x}$, we conclude that $\overline{x}$ solves (3.2). ☐

Remark 3.3. Problem 3.1 encompasses more general scenarios than that of [20], which corresponds to the case when $K_1=K_2=0$, i.e., when all the operators $(D_k^{-1})_{1\leqslant k\leqslant K}$ are restricted to be Lipschitzian. This extension has been made possible by reformulating the original primal problem (3.2), which involves only one variable, as the extended primal problem (3.5), in which we added K_2 auxiliary variables. We also note that Algorithm (3.4) uses all the single-valued operators present in Problem 3.1, including $(S_k)_{K_1+1\leqslant k\leqslant K_2}$ and $(S_k^{-1})_{K_2+1\leqslant k\leqslant K}$, through explicit steps.

4. Relaxation of inconsistent common zero problems.

A standard problem in nonlinear analysis is to find a common zero of maximally monotone operators A and $(B_k)_{1\leqslant k\leqslant K}$ acting on a real Hilbert space $\mathcal{H}$ [17, 23, 33], i.e.,

$$\text{find}\ \ \overline{x}\in\mathcal{H}\ \ \text{such that}\ \ 0\in A\overline{x}\cap\bigcap_{k=1}^K B_k\overline{x}. \tag{4.1}$$

In many situations, this problem may be inconsistent (see [19] and the references therein) and must be approximated. We study the following relaxation of (4.1), together with its dual problem.

Problem 4.1. *Let $\mathcal{H}$ be a real Hilbert space, let K be a strictly positive integer, and let $A\colon\mathcal{H}\to2^{\mathcal{H}}$ be maximally monotone. For every $k\in\{1,\ldots,K\}$, let $B_k\colon\mathcal{H}\to2^{\mathcal{H}}$ be maximally monotone and let $S_k\colon\mathcal{H}\to2^{\mathcal{H}}$ be maximally monotone and such that S_k^{-1} is at most single-valued and strictly monotone with $S_k^{-1}0=\{0\}$. It is assumed that the inclusion*

$$\textit{find}\ \ \overline{x}\in\mathcal{H}\ \ \textit{such that}\ \ 0\in A\overline{x}+\sum_{k=1}^K(B_k\,\Box\,S_k)\overline{x} \tag{4.2}$$

possesses at least one solution. Solve (4.2) *together with the dual problem*

(4.3) *find* $\overline{v_1} \in \mathcal{H}, \ldots, \overline{v_K} \in \mathcal{H}$ *such that*

$$(\forall k \in \{1,\ldots,K\}) \quad 0 \in -A^{-1}\left(-\sum_{l=1}^{K}\overline{v_l}\right) + B_k^{-1}\overline{v_k} + S_k^{-1}\overline{v_k}.$$

First, we justify the fact that (4.2) is indeed a relaxation of (4.1).

PROPOSITION 4.2. *In the setting of Problem* 4.1, *set* $Z = (\operatorname{zer} A) \cap \bigcap_{k=1}^{K} \operatorname{zer} B_k$ *and suppose that* $Z \neq \varnothing$. *Then the set of solutions to the primal problem* (4.2) *is* Z.

Proof. It is clear that every point in Z solves (4.2). Conversely, let x be a solution to (4.2) and let $z \in Z$. We first note that the operators $(B_k \,\Box\, S_k)_{1\leqslant k\leqslant K}$ are at most single-valued. Indeed, let $k \in \{1,\ldots,K\}$ and let (y,p) and (y,q) be in $\operatorname{gra}(B_k \,\Box\, S_k)$. Then we must show that $p = q$. We have $p = (B_k \,\Box\, S_k)y \Leftrightarrow y \in B_k^{-1}p + S_k^{-1}p \Leftrightarrow y - S_k^{-1}p \in B_k^{-1}p$. Likewise, $y - S_k^{-1}q \in B_k^{-1}q$ and, by monotonicity of B_k, $-\left\langle p-q \mid S_k^{-1}p - S_k^{-1}q\right\rangle = \left\langle p-q \mid (y-S_k^{-1}p)-(y-S_k^{-1}q)\right\rangle \geqslant 0$. Consequently, by strict monotonicity of S_k^{-1}, $\left\langle p-q \mid S_k^{-1}p - S_k^{-1}q\right\rangle = 0$ and $p=q$. Hence, since x solves (4.2), there exists $(p_k)_{0\leqslant k\leqslant K} \in \mathcal{H}^{K+1}$ such that

$$\sum_{k=0}^{K} p_k = 0, \quad p_0 \in Ax, \quad \text{and} \quad (\forall k \in \{1,\ldots,K\}) \quad p_k = (B_k \,\Box\, S_k)x. \tag{4.4}$$

Therefore, we have

$$p_0 \in Ax,\ 0 \in Az, \quad \text{and} \quad (\forall k \in \{1,\ldots,K\}) \quad p_k \in B_k\big(x - S_k^{-1}p_k\big) \quad \text{and} \quad 0 \in B_k z, \tag{4.5}$$

and, by monotonicity of the operators A and $(B_k)_{1\leqslant k\leqslant K}$,

$$\langle x - z \mid p_0\rangle \geqslant 0 \quad \text{and} \quad (\forall k \in \{1,\ldots,K\}) \quad \left\langle x - S_k^{-1}p_k - z \mid p_k\right\rangle \geqslant 0. \tag{4.6}$$

Hence, since $\sum_{k=0}^{K} p_k = 0$, it follows from the monotonicity of the operators $(S_k^{-1})_{1\leqslant k\leqslant K}$ that

$$\begin{aligned} 0 &\geqslant -\sum_{k=1}^{K}\left\langle p_k - 0 \mid S_k^{-1}p_k - S_k^{-1}0\right\rangle \\ &= \sum_{k=0}^{K}\langle x-z \mid p_k\rangle - \sum_{k=1}^{K}\left\langle S_k^{-1}p_k \mid p_k\right\rangle \\ &= \langle x - z \mid p_0\rangle + \sum_{k=1}^{K}\left\langle x - S_k^{-1}p_k - z \mid p_k\right\rangle \\ &\geqslant 0. \end{aligned} \tag{4.7}$$

Thus, $\sum_{k=1}^{K}\left\langle p_k - 0 \mid S_k^{-1}p_k - S_k^{-1}0\right\rangle = 0$ and, therefore,

$$(\forall k \in \{1,\ldots,K\}) \quad \left\langle p_k - 0 \mid S_k^{-1}p_k - S_k^{-1}0\right\rangle = 0. \tag{4.8}$$

The strict monotonicity of the operators $(S_k^{-1})_{1\leqslant k\leqslant K}$ implies that for every $k \in \{1,\ldots,K\}$ $p_k = 0$, i.e., $x \in B_k^{-1}p_k + S_k^{-1}p_k = B_k^{-1}0 + S_k^{-1}0 = B_k^{-1}0$. In turn, $p_0 = -\sum_{k=1}^{K} p_k = 0$, i.e., $x \in A^{-1}0$. Altogether, $x \in Z$. ∎

Remark 4.3. Suppose that in Problem 4.1 we set, for every $k \in \{1,\dots,K\}$, $S_k = \gamma_k^{-1}\mathrm{Id}$, where $\gamma_k \in]0,+\infty[$, i.e., $B_k \,\square\, S_k = {}^{\gamma_k}B_k$ is the Yosida approximation of B_k of index γ_k [8, Proposition 23.6(ii)]. Then (4.2) reduces to the setting investigated in [18, section 6.3], namely,

$$\text{find } \overline{x} \in \mathcal{H} \text{ such that } 0 \in A\overline{x} + \sum_{k=1}^{K} {}^{\gamma_k}B_k\overline{x}, \tag{4.9}$$

which itself covers the frameworks of [10, 19, 36, 38] and the references therein. In this case, Proposition 4.2 specializes to [18, Proposition 6.10]. Now let us further specialize to the case when $\mathcal{H} = \mathbb{R}^N$, $A = 0$, and

$$(\forall k \in \{1,\dots,K\}) \quad \begin{cases} \gamma_k = 1, \\ B_k \colon x \mapsto \begin{cases} \operatorname{span}\{u_k\} & \text{if } \langle x \mid u_k\rangle = \rho_k, \\ \varnothing & \text{if } \langle x \mid u_k\rangle \neq \rho_k, \end{cases} \end{cases} \quad \text{where} \quad \begin{cases} u_k \in \mathbb{R}^N, \\ \|u_k\| = 1, \\ \rho_k \in \mathbb{R}. \end{cases} \tag{4.10}$$

Then (4.1) amounts to solving the system of linear equalities

$$\text{find } \overline{x} \in \mathbb{R}^N \text{ such that } (\forall k \in \{1,\dots,K\}) \quad \langle \overline{x} \mid u_k\rangle = \rho_k, \tag{4.11}$$

whereas (4.2) amounts to solving the least-squares problem

$$\underset{x\in\mathbb{R}^N}{\text{minimize}} \;\; \sum_{k=1}^{m} |\langle x \mid u_k\rangle - \rho_k|^2. \tag{4.12}$$

The idea of relaxing (4.11) to (4.12) is due to Legendre [32] and Gauss [27].

To solve Problem 4.1, we use Proposition 3.2 to derive the following algorithm.

PROPOSITION 4.4. *Consider the setting of Problem* 4.1. *Let* $(b_{1,1,n})_{n\in\mathbb{N}}$ *and, for every* $k \in \{1,\dots,K\}$, *let* $(b_{1,k+1,n})_{n\in\mathbb{N}}$ *and* $(b_{2,k,n})_{n\in\mathbb{N}}$ *be absolutely summable sequences in* $\mathcal{H}$. *Let* $x_0 \in \mathcal{H}$, $(y_{k,0})_{1\leqslant k\leqslant K} \in \mathcal{H}^K$, $(v_{k,0})_{1\leqslant k\leqslant K} \in \mathcal{H}^K$, *and* $\varepsilon \in]0, 1/(\sqrt{K+1}+1)[$, *let* $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence in* $[\varepsilon, (1-\varepsilon)/\sqrt{K+1}]$, *and set*

$$\begin{array}{l}
\text{for } n = 0,1,\dots \\
\left\lfloor \begin{array}{l}
p_{1,1,n} = J_{\gamma_n A}\big(x_n - \gamma_n \sum_{k=1}^{K} v_{k,n}\big) + b_{1,1,n}, \\
\text{for } k = 1,\dots,K \\
\left\lfloor \begin{array}{l}
p_{1,k+1,n} = J_{\gamma_n S_k}(y_{k,n} + \gamma_n v_{k,n}) + b_{1,k+1,n}, \\
s_{2,k,n} = v_{k,n} - \gamma_n(y_{k,n} - x_n), \\
p_{2,k,n} = s_{2,k,n} - \gamma_n\big(J_{\gamma_n^{-1}B_k}(\gamma_n^{-1}s_{2,k,n}) + b_{2,k,n}\big), \\
v_{k,n+1} = v_{k,n} - s_{2,k,n} + p_{2,k,n} - \gamma_n\big(p_{1,k+1,n} - p_{1,1,n}\big),
\end{array}\right. \\
x_{n+1} = p_{1,1,n} + \gamma_n \sum_{k=1}^{K}(v_{k,n} - p_{2,k,n}), \\
\text{for } k = 1,\dots,K \\
\lfloor \; y_{k,n+1} = p_{1,k+1,n} + \gamma_n(p_{2,k,n} - v_{k,n}).
\end{array}\right.
\end{array} \tag{4.13}$$

Then the following hold for some solution $\overline{x}$ *to* (4.2) *and some solution* $(\overline{v_1},\dots,\overline{v_K})$ *to* (4.3):

(i) $x_n \rightharpoonup \overline{x}$ *and* $(\forall k \in \{1,\dots,K\})$ $v_{k,n} \rightharpoonup \overline{v_k}$.

(ii) *Suppose that* A *is uniformly monotone at* $\overline{x}$. *Then* $x_n \to \overline{x}$.

(iii) *Suppose that, for some* $l \in \{1,\dots,K\}$, B_l *is couniformly monotone at* $\overline{v_l}$. *Then* $v_{l,n} \to \overline{v_l}$.

Proof. Problem 4.1 is a special case of Problem 3.1 with $K_1 = K_2 = K$, $z = 0$, $C = 0$, $\mu = 0$, $\beta = \sqrt{K+1}$, and $(\forall k \in \{1,\ldots,K\})$ $\mathcal{G}_k = \mathcal{H}$, $L_k = \mathrm{Id}$, and $r_k = 0$. In this context, (3.4) can be reduced to (4.13), and the claims therefore follow from Proposition 3.2. ∎

Remark 4.5. For brevity, we have presented an algorithm for solving Problem 4.1 in its general form. However, if some of the operators $(S_k)_{1\leqslant k\leqslant K}$ or their inverses are Lipschitzian, we can apply Proposition 3.2 with $K_1 \neq K$ and/or $K_2 \neq K$ to obtain a more efficient algorithm in which each Lipschitzian operator is used through an explicit step, rather than through its resolvent.

5. Multivariate structured convex minimization problems. We derive from Theorem 2.4 a primal-dual minimization algorithm for multivariate convex minimization problems involving infimal convolutions and composite functions.

PROBLEM 5.1. *Let m and K be strictly positive integers, let $(\mathcal{H}_i)_{1\leqslant i\leqslant m}$ and $(\mathcal{G}_k)_{1\leqslant k\leqslant K}$ be real Hilbert spaces, let $(\mu_i)_{1\leqslant i\leqslant m} \in [0,+\infty[^m$, and let $(\nu_k)_{1\leqslant i\leqslant K} \in \left]0,+\infty\right[^K$. For every $i \in \{1,\ldots,m\}$ and $k \in \{1,\ldots,K\}$, let $h_i\colon \mathcal{H}_i \to \mathbb{R}$ be convex and differentiable and such that ∇h_i is μ_i-Lipschitzian, let $f_i \in \Gamma_0(\mathcal{H}_i)$, let $g_k \in \Gamma_0(\mathcal{G}_k)$, let $\ell_k \in \Gamma_0(\mathcal{G}_k)$ be $1/\nu_k$-strongly convex, let $z_i \in \mathcal{H}_i$, let $r_k \in \mathcal{G}_k$, and let $L_{ki} \in \mathcal{B}(\mathcal{H}_i,\mathcal{G}_k)$. Set $\beta = \max\{\max_{1\leqslant i\leqslant m}\mu_i, \max_{1\leqslant k\leqslant K}\nu_k\} + \sqrt{\lambda} > 0$, where $\lambda \in [\sup_{\sum_{i=1}^m \|x_i\|^2\leqslant 1} \sum_{k=1}^K \|\sum_{i=1}^m L_{ki}x_i\|^2, +\infty[$, and assume that*

(5.1)
$$(\forall i \in \{1,\ldots,m\}) \quad z_i \in \mathrm{ran}\left(\partial f_i + \sum_{k=1}^K L_{ki}^* \circ (\partial g_k \,\Box\, \partial \ell_k) \circ \left(\sum_{j=1}^m L_{kj}\cdot - r_k\right) + \nabla h_i\right).$$

Solve the primal problem

(5.2)
$$\underset{x_1\in\mathcal{H}_1,\ldots,x_m\in\mathcal{H}_m}{\textit{minimize}} \quad \sum_{i=1}^m f_i(x_i) + \sum_{k=1}^K (g_k \,\Box\, \ell_k)\left(\sum_{i=1}^m L_{ki}x_i - r_k\right) + \sum_{i=1}^m \big(h_i(x_i) - \langle x_i \mid z_i\rangle\big),$$

together with the dual problem

(5.3)
$$\underset{v_1\in\mathcal{G}_1,\ldots,v_K\in\mathcal{G}_K}{\textit{minimize}} \quad \sum_{i=1}^m \big(f_i^* \,\Box\, h_i^*\big)\left(z_i - \sum_{k=1}^K L_{ki}^* v_k\right) + \sum_{k=1}^K \big(g_k^*(v_k) + \ell_k^*(v_k) + \langle v_k \mid r_k\rangle\big).$$

Remark 5.2. Problem 5.1 extends significantly the multivariate minimization framework of [3, 13]. The minimization problem under consideration there was the following specialization of (5.2)

$$\underset{x_1\in\mathcal{H}_1,\ldots,x_m\in\mathcal{H}_m}{\text{minimize}} \quad \sum_{i=1}^m f_i(x_i) + \sum_{k=1}^K g_k\left(\sum_{i=1}^m L_{ki}x_i\right), \tag{5.4}$$

where, in addition, the functions $(g_k)_{1\leqslant k\leqslant K}$ were required to be differentiable everywhere with a Lipschitzian gradient. Furthermore, no dual problem was considered.

PROPOSITION 5.3. *Consider the setting of Problem* 5.1. *Suppose that* (5.2) *has a solution and set*

(5.5)
$$E = \left\{ \left(\sum_{i=1}^m L_{ki}x_i - y_k\right)_{1\leqslant k\leqslant K} \;\middle|\; \begin{cases} (\forall i \in \{1,\ldots,m\})\ x_i \in \mathrm{dom}\, f_i, \\ (\forall k \in \{1,\ldots,K\})\ y_k \in \mathrm{dom}\, g_k + \mathrm{dom}\, \ell_k \end{cases} \right\}.$$

Then (5.1) *is satisfied in each of the following cases:*

(i) $(r_k)_{1\leqslant k\leqslant K} \in \operatorname{sri} E$.

(ii) $E - (r_k)_{1\leqslant k\leqslant K}$ *is a closed vector subspace.*

(iii) *For every* $i \in \{1,\ldots,m\}$, f_i *is real-valued and, for every* $k \in \{1,\ldots,K\}$, *the operator* $\bigoplus_{j=1}^m \mathcal{H}_j \to \mathcal{G}_k\colon (x_j)_{1\leqslant j\leqslant m} \mapsto \sum_{j=1}^m L_{kj}x_j$ *is surjective.*

(iv) *For every* $k \in \{1,\ldots,K\}$, g_k *or* ℓ_k *is real-valued.*

(v) $(\mathcal{H}_i)_{1\leqslant i\leqslant m}$ *and* $(\mathcal{G}_k)_{1\leqslant k\leqslant K}$ *are finite-dimensional, and* $(\forall i \in \{1,\ldots,m\})(\exists\, x_i \in \operatorname{ri}\operatorname{dom} f_i)(\forall k \in \{1,\ldots,K\})\ \sum_{i=1}^m L_{ki}x_i - r_k \in \operatorname{ri}\operatorname{dom} g_k + \operatorname{ri}\operatorname{dom}\ell_k$.

Proof. Define $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{G}}$ as in (2.5) and $\boldsymbol{L}$, $\boldsymbol{z}$, and $\boldsymbol{r}$ as in (2.6). Set

$$\begin{cases}\boldsymbol{f}\colon \boldsymbol{\mathcal{H}} \to \left]-\infty,+\infty\right] : \boldsymbol{x} \mapsto \sum_{i=1}^m f_i(x_i),\\ \boldsymbol{g}\colon \boldsymbol{\mathcal{G}} \to \left]-\infty,+\infty\right] : \boldsymbol{y} \mapsto \sum_{k=1}^K g_k(y_k)\end{cases} \quad\text{and}\quad \begin{cases}\boldsymbol{h}\colon \boldsymbol{\mathcal{H}} \to \mathbb{R}\colon \boldsymbol{x} \mapsto \sum_{i=1}^m h_i(x_i),\\ \boldsymbol{\ell}\colon \boldsymbol{\mathcal{G}} \to \left]-\infty,+\infty\right] : \boldsymbol{y} \mapsto \sum_{k=1}^K \ell_k(y_k).\end{cases} \tag{5.6}$$

Then (5.5) and [8, Proposition 12.6(ii)] yield

$$\begin{aligned} E &= \big\{\boldsymbol{L}\boldsymbol{x} - \boldsymbol{y} \;\big|\; \boldsymbol{x} \in \operatorname{dom}\boldsymbol{f} \text{ and } \boldsymbol{y} \in \operatorname{dom}\boldsymbol{g} + \operatorname{dom}\boldsymbol{\ell}\big\}\\ &= \boldsymbol{L}\big(\operatorname{dom}\boldsymbol{f}\big) - \big(\operatorname{dom}\boldsymbol{g} + \operatorname{dom}\boldsymbol{\ell}\big) \qquad (5.7)\\ &= \boldsymbol{L}\big(\operatorname{dom}(\boldsymbol{f} + \boldsymbol{h} - \langle\cdot \mid \boldsymbol{z}\rangle)\big) - \operatorname{dom}\big(\boldsymbol{g}\,\Box\,\boldsymbol{\ell}\big). \qquad (5.8)\end{aligned}$$

(i) Since the functions $(\ell_k)_{1\leqslant k\leqslant K}$ are strongly convex, so is $\boldsymbol{\ell}$. Hence, $\operatorname{dom}\boldsymbol{\ell}^* = \boldsymbol{\mathcal{G}}$ [8, Propositions 11.16 and 14.15] and therefore [8, Propositions 15.7(iv) and 24.27] imply that $\partial\boldsymbol{g}\,\Box\,\partial\boldsymbol{\ell} = \partial(\boldsymbol{g}\,\Box\,\boldsymbol{\ell})$ and $\boldsymbol{g}\,\Box\,\boldsymbol{\ell} \in \Gamma_0(\boldsymbol{\mathcal{G}})$. On the other hand, (5.8) yields $\boldsymbol{0} \in \operatorname{sri}(\boldsymbol{L}(\operatorname{dom}(\boldsymbol{f} + \boldsymbol{h} - \langle\cdot \mid \boldsymbol{z}\rangle)) - \operatorname{dom}(\boldsymbol{g}\,\Box\,\boldsymbol{\ell})(\cdot - \boldsymbol{r}))$. Thus, we derive from [8, Theorem 16.37(i)] that

$$\begin{aligned} &\partial\boldsymbol{f} + \boldsymbol{L}^*\circ(\partial\boldsymbol{g}\,\Box\,\partial\boldsymbol{\ell}_k)\circ(\boldsymbol{L}\cdot - \boldsymbol{r}) + \nabla\boldsymbol{h} - \boldsymbol{z}\\ &\qquad = \partial\big(\boldsymbol{f} + \boldsymbol{h} - \langle\cdot \mid \boldsymbol{z}\rangle\big) + \boldsymbol{L}^*\circ\partial(\boldsymbol{g}\,\Box\,\boldsymbol{\ell})\circ(\boldsymbol{L}\cdot - \boldsymbol{r})\\ &\qquad = \partial\big(\boldsymbol{f} + \boldsymbol{h} - \langle\cdot \mid \boldsymbol{z}\rangle + (\boldsymbol{g}\,\Box\,\boldsymbol{\ell})\circ(\boldsymbol{L}\cdot - \boldsymbol{r})\big). \qquad (5.9)\end{aligned}$$

Since (5.2) has a solution and is equivalent to minimizing $\boldsymbol{f}+\boldsymbol{h}-\langle\cdot \mid \boldsymbol{z}\rangle+(\boldsymbol{g}\,\Box\,\boldsymbol{\ell})\circ(\boldsymbol{L}\cdot - r)$ over $\boldsymbol{\mathcal{H}}$, Fermat's rule [8, Theorem 16.2] implies that $\boldsymbol{0} \in \operatorname{ran}\partial(\boldsymbol{f}+\boldsymbol{h}-\langle\cdot \mid \boldsymbol{z}\rangle+(\boldsymbol{g}\,\Box\,\boldsymbol{\ell})\circ(\boldsymbol{L}\cdot - \boldsymbol{r}))$. Hence (5.9) yields $\boldsymbol{z} \in \operatorname{ran}(\partial\boldsymbol{f} + \boldsymbol{L}^*\circ(\partial\boldsymbol{g}\,\Box\,\partial\boldsymbol{\ell}_k)\circ(\boldsymbol{L}\cdot - \boldsymbol{r}) + \nabla\boldsymbol{h})$ and we conclude that (5.1) is satisfied.

(ii)⇒(i) [8, Proposition 6.19(i)].

(iii)⇒(i) We have $\boldsymbol{L}(\operatorname{dom}\boldsymbol{f}) = \boldsymbol{L}(\boldsymbol{\mathcal{H}}) = \boldsymbol{\mathcal{G}}$. Hence, (5.7) yields $E = \boldsymbol{\mathcal{G}}$.

(iv)⇒(i) We have $\operatorname{dom}\boldsymbol{g} + \operatorname{dom}\boldsymbol{\ell} = \boldsymbol{\mathcal{G}}$. Hence, (5.7) yields $E = \boldsymbol{\mathcal{G}}$.

(v)⇒(i) Since $\dim\boldsymbol{\mathcal{G}} < +\infty$, $\operatorname{sri} E = \operatorname{ri} E$. On the other hand, by (5.7) and [8, Corollary 6.15],

$$\operatorname{ri} E = \operatorname{ri}\big(\boldsymbol{L}\big(\operatorname{dom}\boldsymbol{f}\big) - \operatorname{dom}\boldsymbol{g} - \operatorname{dom}\boldsymbol{\ell}\big) = \boldsymbol{L}\big(\operatorname{ri}\operatorname{dom}\boldsymbol{f}\big) - \operatorname{ri}\operatorname{dom}\boldsymbol{g} - \operatorname{ri}\operatorname{dom}\boldsymbol{\ell}. \tag{5.10}$$

Thus, $\boldsymbol{r} \in \operatorname{sri} E \Leftrightarrow (\exists\,\boldsymbol{x} \in \operatorname{ri}\operatorname{dom}\boldsymbol{f} = \bigtimes_{i=1}^m \operatorname{ri}\operatorname{dom} f_i)\ \boldsymbol{L}\boldsymbol{x} - \boldsymbol{r} \in \operatorname{ri}\operatorname{dom}\boldsymbol{g} + \operatorname{ri}\operatorname{dom}\boldsymbol{\ell} = \bigtimes_{k=1}^K(\operatorname{ri}\operatorname{dom} g_k + \operatorname{ri}\operatorname{dom}\ell_k)$. ◻

PROPOSITION 5.4. *Consider the setting of Problem* 5.1. *For every* $i \in \{1,\ldots,m\}$, *let* $(a_{1,i,n})_{n\in\mathbb{N}}$, $(b_{1,i,n})_{n\in\mathbb{N}}$, *and* $(c_{1,i,n})_{n\in\mathbb{N}}$ *be absolutely summable sequences in* $\mathcal{H}_i$ *and, for every* $k \in \{1,\ldots,K\}$, *let* $(a_{2,k,n})_{n\in\mathbb{N}}$, $(b_{2,k,n})_{n\in\mathbb{N}}$, *and* $(c_{2,k,n})_{n\in\mathbb{N}}$ *be absolutely summable sequences in* $\mathcal{G}_k$. *Furthermore, let* $x_{1,0} \in \mathcal{H}_1$, $\ldots$, $x_{m,0} \in \mathcal{H}_m$,

$v_{1,0} \in \mathcal{G}_1, \ \ldots, \ v_{K,0} \in \mathcal{G}_K$, *let* $\varepsilon \in \left]0, 1/(\beta+1)\right[$, *let* $(\gamma_n)_{n\in\mathbb{N}}$ *be a sequence in* $[\varepsilon, (1-\varepsilon)/\beta]$, *and set*

$$
\begin{array}{l}
\text{for } n = 0, 1, \ldots \\
\left\lfloor
\begin{array}{l}
\text{for } i = 1, \ldots, m \\
\left\lfloor
\begin{array}{l}
s_{1,i,n} = x_{i,n} - \gamma_n\big(\nabla h_i(x_{i,n}) + \sum_{k=1}^K L_{ki}^* v_{k,n} + a_{1,i,n}\big), \\
p_{1,i,n} = \operatorname{prox}_{\gamma_n f_i}(s_{1,i,n} + \gamma_n z_i) + b_{1,i,n},
\end{array}
\right. \\
\text{for } k = 1, \ldots, K \\
\left\lfloor
\begin{array}{l}
s_{2,k,n} = v_{k,n} - \gamma_n\big(\nabla \ell_k^*(v_{k,n}) - \sum_{i=1}^m L_{ki} x_{i,n} + a_{2,k,n}\big), \\
p_{2,k,n} = s_{2,k,n} - \gamma_n\big(r_k + \operatorname{prox}_{\gamma_n^{-1} g_k}(\gamma_n^{-1} s_{2,k,n} - r_k) + b_{2,k,n}\big), \\
q_{2,k,n} = p_{2,k,n} - \gamma_n\big(\nabla \ell_k^*(p_{2,k,n}) - \sum_{i=1}^m L_{ki} p_{1,i,n} + c_{2,k,n}\big), \\
v_{k,n+1} = v_{k,n} - s_{2,k,n} + q_{2,k,n},
\end{array}
\right. \\
\text{for } i = 1, \ldots, m \\
\left\lfloor
\begin{array}{l}
q_{1,i,n} = p_{1,i,n} - \gamma_n\big(\nabla h_i(p_{1,i,n}) + \sum_{k=1}^K L_{ki}^* p_{2,k,n} + c_{1,i,n}\big), \\
x_{i,n+1} = x_{i,n} - s_{1,i,n} + q_{1,i,n}.
\end{array}
\right.
\end{array}
\right.
\end{array}
\tag{5.11}
$$

Then the following hold:

(i) $(\forall i \in \{1, \ldots, m\})$ $\sum_{n\in\mathbb{N}} \|x_{i,n} - p_{1,i,n}\|^2 < +\infty$, *and* $(\forall k \in \{1, \ldots, K\})$ $\sum_{n\in\mathbb{N}} \|v_{k,n} - p_{2,k,n}\|^2 < +\infty$.

(ii) *There exist a solution* $(\overline{x_1}, \ldots, \overline{x_m})$ *to* (5.2) *and a solution* $(\overline{v_1}, \ldots, \overline{v_K})$ *to* (5.3) *such that the following hold:*

(a) $(\forall i \in \{1, \ldots, m\})$ $x_{i,n} \rightharpoonup \overline{x_i}$ *and* $z_i - \sum_{k=1}^K L_{ki}^* \overline{v_k} \in \partial f_i(\overline{x_i}) + \nabla h_i(\overline{x_i})$.

(b) $(\forall k \in \{1, \ldots, K\})$ $v_{k,n} \rightharpoonup \overline{v_k}$ *and* $\sum_{i=1}^m L_{ki} \overline{x_i} - r_k \in \partial g_k^*(\overline{v_k}) + \nabla \ell_k^*(\overline{v_k})$.

(c) *Suppose that, for some* $j \in \{1, \ldots, m\}$, f_j *or* h_j *is uniformly convex at* $\overline{x_j}$. *Then* $x_{j,n} \to \overline{x_j}$.

(d) *Suppose that, for some* $l \in \{1, \ldots, K\}$, g_l^* *or* ℓ_l^* *is uniformly convex at* $\overline{v_l}$. *Then* $v_{l,n} \to \overline{v_l}$.

Proof. Set

$$
\begin{cases}
(\forall i \in \{1, \ldots, m\}) & A_i = \partial f_i \quad \text{and} \quad C_i = \nabla h_i, \\
(\forall k \in \{1, \ldots, K\}) & B_k = \partial g_k \quad \text{and} \quad D_k = \partial \ell_k.
\end{cases}
\tag{5.12}
$$

It follows from [8, Proposition 17.10] that the operators $(C_i)_{1\leqslant i\leqslant m}$ are monotone, and from [8, Theorem 20.40] that the operators $(A_i)_{1\leqslant i\leqslant m}$, $(B_k)_{1\leqslant k\leqslant m}$, and $(D_k)_{1\leqslant k\leqslant K}$ are maximally monotone. Moreover, for every $k \in \{1, \ldots, K\}$, we derive from [8, Corollary 13.33 and Theorem 18.15] that ℓ_k^* is Fréchet differentiable on $\mathcal{G}_k$ and $\nabla \ell_k^*$ is ν_k-Lipschitzian, and from [8, Corollary 16.24 and Proposition 17.26(i)] that $D_k^{-1} = (\partial \ell_k)^{-1} = \partial \ell_k^* = \{\nabla \ell_k^*\}$. On the other hand, (5.1) implies that (1.3) possesses a solution, and (1.15) implies that (5.11) is a special case of (2.4). We also recall that the uniform convexity of a function $\varphi \in \Gamma_0(\mathcal{H})$ at $x \in \operatorname{dom} \partial\varphi$ implies the uniform monotonicity of $\partial\varphi$ at x [47, section 3.4]. Altogether, the claims will follow at once from Theorem 2.4 provided we show that, in the setting of (5.1) and (5.12), (1.3) becomes (5.2) and (1.4) becomes (5.3). To this end, let us first observe that since for every $k \in \{1, \ldots, K\}$, dom $\ell_k^* = \mathcal{G}_k$, [8, Proposition 24.27] yields

$$
(\forall k \in \{1, \ldots, K\}) \quad B_k \,\square\, D_k = \partial g_k \,\square\, \partial \ell_k = \partial(g_k \,\square\, \ell_k),
\tag{5.13}
$$

while [8, Corollaries 16.24 and 16.38(iii)] yield

$$
(\forall k \in \{1, \ldots, K\}) \quad B_k^{-1} + D_k^{-1} = \partial g_k^* + \{\nabla \ell_k^*\} = \partial\big(g_k^* + \ell_k^*\big).
\tag{5.14}
$$

Likewise, using [8, Theorem 15.3], we obtain

$$
\begin{aligned}
(5.15)\quad (\forall i \in \{1,\ldots,m\})\quad (A_i + C_i)^{-1} &= (\partial f_i + \nabla h_i)^{-1} \\
&= \big(\partial(f_i + h_i)\big)^{-1} = \partial(f_i + h_i)^* = \partial(f_i^* \,\square\, h_i^*).
\end{aligned}
$$

Now let us define $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{G}}$ as in (2.5), $\boldsymbol{L}$, $\boldsymbol{z}$, and $\boldsymbol{r}$ as in (2.6), and $\boldsymbol{f}$, $\boldsymbol{h}$, $\boldsymbol{g}$, and $\boldsymbol{\ell}$ as in (5.6). We derive from (5.12), (5.13), [8, Corollary 16.38(iii), Propositions 16.5(ii), 16.8, and 17.26(i)], and Fermat's rule [8, Theorem 16.2] that for every $\boldsymbol{x} = (x_i)_{1\leqslant i\leqslant m} \in \boldsymbol{\mathcal{H}}$,

$$
\begin{aligned}
\boldsymbol{x} \text{ solves (1.3)} &\Leftrightarrow (\forall i \in \{1,\ldots,m\})\quad 0 \in \partial f_i(x_i) \\
&\qquad + \sum_{k=1}^{K} L_{ki}^*\bigg(\partial(g_k \,\square\, \ell_k)\bigg(\sum_{j=1}^{m} L_{kj}x_j - r_k\bigg)\bigg) + \nabla h_i(x_i) - z_i \\
&\Leftrightarrow \boldsymbol{0} \in \partial \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{L}^*\big(\partial(\boldsymbol{g} \,\square\, \boldsymbol{\ell})(\boldsymbol{L}\boldsymbol{x} - \boldsymbol{r})\big) + \nabla(\boldsymbol{h} - \langle \cdot \mid \boldsymbol{z}\rangle)(\boldsymbol{x}) \\
&\Rightarrow \boldsymbol{0} \in \partial\Big(\boldsymbol{f} + (\boldsymbol{g} \,\square\, \boldsymbol{\ell}) \circ \big(\boldsymbol{L}\cdot - \boldsymbol{r}\big)\big) + \boldsymbol{h} - \langle \cdot \mid \boldsymbol{z}\rangle\Big)(\boldsymbol{x}) \\
(5.16)\qquad &\Leftrightarrow \boldsymbol{x} \text{ solves (5.2).}
\end{aligned}
$$

Next, let $\boldsymbol{v} = (v_k)_{1\leqslant k\leqslant K} \in \boldsymbol{\mathcal{G}}$. Then we derive from (5.14), (5.15), and the same subdifferential calculus rules as above that

$$
\begin{aligned}
\boldsymbol{v} \text{ solves (1.4)} &\Leftrightarrow (\forall k \in \{1,\ldots,K\})\quad 0 \in -\sum_{i=1}^{m} L_{ki}\bigg(\partial(f_i^* \,\square\, h_i^*)\bigg(z_i - \sum_{l=1}^{K} L_{li}^* v_l\bigg)\bigg) \\
&\qquad + \partial\big(g_k^* + \ell_k^* + \langle \cdot \mid r_k\rangle\big)(v_k) \\
&\Leftrightarrow \boldsymbol{0} \in -\boldsymbol{L}\big(\partial(\boldsymbol{f}^* \,\square\, \boldsymbol{h}^*)(\boldsymbol{z} - \boldsymbol{L}^*\boldsymbol{v})\big) + \partial\big(\boldsymbol{g}^* + \boldsymbol{\ell}^* + \langle \cdot \mid \boldsymbol{r}\rangle\big)(\boldsymbol{v}) \\
&\Rightarrow \boldsymbol{0} \in \partial\Big((\boldsymbol{f}^* \,\square\, \boldsymbol{h}^*) \circ (\boldsymbol{z} - \boldsymbol{L}^*\cdot) + \boldsymbol{g}^* + \boldsymbol{\ell}^* + \langle \cdot \mid \boldsymbol{r}\rangle\Big)(\boldsymbol{v}) \\
(5.17)\qquad &\Leftrightarrow \boldsymbol{v} \text{ solves (5.3),}
\end{aligned}
$$

which completes the proof. ∎

Remark 5.5. Proposition 5.4 provides a framework that captures and suggests extensions of multivariate or infimal convolution variational formulations found in areas such as partial differential equations [4], machine learning [6], and image recovery [15, 16, 39].

6. Univariate structured convex minimization problems. Minimization problems involving a single primal variable can be obtained by setting $m = 1$ in Problem 5.1. However, this approach imposes that infimal convolutions be performed exclusively with strongly convex functions. We use a different strategy relying on Proposition 3.2, which leads to a formulation allowing for infimal convolutions with general lower semicontinuous convex functions.

PROBLEM 6.1. *Let $\mathcal{H}$ be a real Hilbert space, let K_1, K_2, and K be integers such that $0 \leqslant K_1 \leqslant K_2 \leqslant K \geqslant 1$, let $z \in \mathcal{H}$, let $f \in \Gamma_0(\mathcal{H})$, and let $h\colon \mathcal{H} \to \mathbb{R}$ be convex and differentiable and such that ∇h is μ-Lipschitzian for some $\mu \in [0, +\infty[$. For every* *integer $k \in \{1,\ldots,K\}$, let $\mathcal{G}_k$ be a real Hilbert space, let $r_k \in \mathcal{G}_k$, let $g_k \in \Gamma_0(\mathcal{G}_k)$, let $\varphi_k \in \Gamma_0(\mathcal{G}_k)$, and let $L_k \in \mathcal{B}(\mathcal{H},\mathcal{G}_k)$; moreover, if $K_1+1 \leqslant k \leqslant K_2$, φ_k is differentiable on $\mathcal{G}_k$ and such that $\nabla\varphi_k$ is β_k-Lipschitzian for some $\beta_k \in [0,+\infty[$, and, if $K_2+1 \leqslant k \leqslant K$, φ_k is $1/\beta_k$-strongly convex for some $\beta_k \in \,]0,+\infty[$. Set $\beta = \max\{\mu, \beta_{K_1+1},\ldots,\beta_K\} + \sqrt{1+\sum_{k=1}^K \|L_k\|^2}$ and assume that*

$$z \in \operatorname{ran}\left(\partial f + \sum_{k=1}^K L_k^* \circ (\partial g_k \,\Box\, \partial\varphi_k) \circ \big(L_k \cdot - r_k\big) + \nabla h\right) \tag{6.1}$$

and

$$(\forall k \in \{1,\ldots,K_2\}) \quad 0 \in \operatorname{sri}(\operatorname{dom} g_k^* - \operatorname{dom}\varphi_k^*). \tag{6.2}$$

Solve the primal problem

$$\underset{x\in\mathcal{H}}{\text{minimize}} \;\; f(x) + \sum_{k=1}^K (g_k \,\Box\, \varphi_k)(L_k x - r_k) + h(x) - \langle x \mid z\rangle, \tag{6.3}$$

together with the dual problem

$$\underset{v_1\in\mathcal{G}_1,\ldots,v_K\in\mathcal{G}_K}{\text{minimize}} \;\; \big(f^* \,\Box\, h^*\big)\left(z - \sum_{k=1}^K L_k^* v_k\right) + \sum_{k=1}^m \big(g_k^*(v_k) + \varphi_k^*(v_k) + \langle v_k \mid r_k\rangle\big). \tag{6.4}$$

Remark 6.2. It follows from (6.2) and [8, Propositions 11.16, 14.15, 15.7(i), and 24.27] that

$$(\forall k \in \{1,\ldots,K\}) \quad g_k \,\Box\, \varphi_k \in \Gamma_0(\mathcal{G}_k) \quad \text{and} \quad \partial g_k \,\Box\, \partial\varphi_k = \partial(g_k \,\Box\, \varphi_k). \tag{6.5}$$

Hence, using the same type of arguments as in the proof of Proposition 5.3, we can deduce similar conditions for (6.1) to hold, e.g., that (6.3) have a solution and that $(r_k)_{1\leqslant k\leqslant K}$ lie in the strong relative interior of

$$\big\{(L_k x - y_k)_{1\leqslant k\leqslant K} \;\big|\; x \in \operatorname{dom} f \text{ and } (\forall k \in \{1,\ldots,K\})\; y_k \in \operatorname{dom} g_k + \operatorname{dom}\varphi_k\big\}. \tag{6.6}$$

PROPOSITION 6.3. *Consider the setting of Problem 6.1. Let $(a_{1,1,n})_{n\in\mathbb{N}}$, $(b_{1,1,n})_{n\in\mathbb{N}}$, and $(c_{1,1,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{H}$. For every integer $k \in \{1,\ldots,K\}$, let $(a_{2,k,n})_{n\in\mathbb{N}}$, $(b_{2,k,n})_{n\in\mathbb{N}}$, and $(c_{2,k,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{G}_k$; moreover, if $1 \leqslant k \leqslant K_1$, let $(b_{1,k+1,n})_{n\in\mathbb{N}}$ be an absolutely summable sequence in $\mathcal{G}_k$, and, if $K_1+1 \leqslant k \leqslant K_2$, let $(a_{1,k+1,n})_{n\in\mathbb{N}}$ and $(c_{1,k+1,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{G}_k$. Let $x_0 \in \mathcal{H}$, $y_{1,0} \in \mathcal{G}_1$, $\ldots$, $y_{K_2,0} \in \mathcal{G}_{K_2}$, $v_{1,0} \in \mathcal{G}_1$, $\ldots$, and $v_{K,0} \in \mathcal{G}_K$, let $\varepsilon \in \,]0, 1/(\beta+1)[$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (1-\varepsilon)/\beta]$,* *and set*

$$
\begin{array}{l}
\text{for } n = 0,1,\ldots\\
\left\lfloor
\begin{array}{l}
s_{1,1,n} = x_n - \gamma_n\big(\nabla h(x_n) + \sum_{k=1}^K L_k^* v_{k,n} + a_{1,1,n}\big),\\
p_{1,1,n} = \operatorname{prox}_{\gamma_n f}(s_{1,1,n} + \gamma_n z) + b_{1,1,n},\\
\text{if } K_1 \neq 0,\ \text{for } k = 1,\ldots,K_1\\
\left\lfloor
\begin{array}{l}
s_{1,k+1,n} = y_{k,n} + \gamma_n v_{k,n},\\
p_{1,k+1,n} = \operatorname{prox}_{\gamma_n \varphi_k} s_{1,k+1,n} + b_{1,k+1,n},\\
s_{2,k,n} = v_{k,n} - \gamma_n\big(y_{k,n} - L_k x_n + a_{2,k,n}\big),\\
p_{2,k,n} = s_{2,k,n} - \gamma_n\big(r_k + \operatorname{prox}_{\gamma_n^{-1} g_k}(\gamma_n^{-1} s_{2,k,n} - r_k) + b_{2,k,n}\big),\\
q_{2,k,n} = p_{2,k,n} - \gamma_n\big(p_{1,k+1,n} - L_k p_{1,1,n} + c_{2,k,n}\big),\\
v_{k,n+1} = v_{k,n} - s_{2,k,n} + q_{2,k,n},
\end{array}
\right.\\
\text{if } K_1 \neq K_2,\ \text{for } k = K_1+1,\ldots,K_2\\
\left\lfloor
\begin{array}{l}
s_{1,k+1,n} = y_{k,n} - \gamma_n\big(\nabla\varphi_k(y_{k,n}) - v_{k,n} + a_{1,k+1,n}\big),\\
p_{1,k+1,n} = s_{1,k+1,n},\\
s_{2,k,n} = v_{k,n} - \gamma_n\big(y_{k,n} - L_k x_n + a_{2,k,n}\big),\\
p_{2,k,n} = s_{2,k,n} - \gamma_n\big(r_k + \operatorname{prox}_{\gamma_n^{-1} g_k}(\gamma_n^{-1} s_{2,k,n} - r_k) + b_{2,k,n}\big),\\
q_{2,k,n} = p_{2,k,n} - \gamma_n\big(p_{1,k+1,n} - L_k p_{1,1,n} + c_{2,k,n}\big),\\
v_{k,n+1} = v_{k,n} - s_{2,k,n} + q_{2,k,n},
\end{array}
\right.\\
\text{if } K_2 \neq K,\ \text{for } k = K_2+1,\ldots,K\\
\left\lfloor
\begin{array}{l}
s_{2,k,n} = v_{k,n} - \gamma_n\big(\nabla\varphi_k^*(v_{k,n}) - L_k x_n + a_{2,k,n}\big),\\
p_{2,k,n} = s_{2,k,n} - \gamma_n\big(r_k + \operatorname{prox}_{\gamma_n^{-1} g_k}(\gamma_n^{-1} s_{2,k,n} - r_k) + b_{2,k,n}\big),\\
q_{2,k,n} = p_{2,k,n} - \gamma_n\big(\nabla\varphi_k^*(p_{2,k,n}) - L_k p_{1,1,n} + c_{2,k,n}\big),\\
v_{k,n+1} = v_{k,n} - s_{2,k,n} + q_{2,k,n},
\end{array}
\right.\\
q_{1,1,n} = p_{1,1,n} - \gamma_n\big(\nabla h(p_{1,1,n}) + \sum_{k=1}^K L_k^* p_{2,k,n} + c_{1,1,n}\big),\\
x_{n+1} = x_n - s_{1,1,n} + q_{1,1,n},\\
\text{if } K_1 \neq 0,\ \text{for } k = 1,\ldots,K_1\\
\left\lfloor
\begin{array}{l}
q_{1,k+1,n} = p_{1,k+1,n} + \gamma_n p_{2,k,n},\\
y_{k,n+1} = y_{k,n} - s_{1,k+1,n} + q_{1,k+1,n},
\end{array}
\right.\\
\text{if } K_1 \neq K_2,\ \text{for } k = K_1+1,\ldots,K_2\\
\left\lfloor
\begin{array}{l}
q_{1,k+1,n} = p_{1,k+1,n} - \gamma_n\big(\nabla\varphi_k(p_{1,k+1,n}) - p_{2,k,n} + c_{1,k+1,n}\big),\\
y_{k,n+1} = y_{k,n} - s_{1,k+1,n} + q_{1,k+1,n}.
\end{array}
\right.
\end{array}
\right.
\end{array}
\tag{6.7}
$$

Then the following hold for some solution $\overline{x}$ *to* (6.3) *and some solution* $(\overline{v_1},\ldots,\overline{v_K})$ *to* (6.4):

(i) $x_n \rightharpoonup \overline{x}$ *and* $(\forall k \in \{1,\ldots,K\})\ v_{k,n} \rightharpoonup \overline{v_k}$.

(ii) *Suppose that* f *or* h *is uniformly convex at* $\overline{x}$. *Then* $x_n \to \overline{x}$.

(iii) *Suppose that, for some* $l \in \{1,\ldots,K\}$, g_l^* *is uniformly convex at* $\overline{v_l}$. *Then* $v_{l,n} \to \overline{v_l}$.

(iv) *Suppose that* $K_2 \neq K$ *and that, for some* $l \in \{K_2+1,\ldots,K\}$, φ_l^* *is uniformly convex at* $\overline{v_l}$. *Then* $v_{l,n} \to \overline{v_l}$.

Proof. Using (6.5) and the same arguments as in the proof of Proposition 5.4, we first identify Problem 6.1 as a special case of Problem 3.1 with $A = \partial f$, $C = \nabla h$, and $(\forall k \in \{1,\ldots,K\})\ B_k = \partial g_k$ and $S_k = \partial\varphi_k$. Using (1.15), we then deduce the results from Proposition 3.2. ∎

We conclude this section with an application to the approximation of inconsistent convex feasibility problems where, for the sake of brevity, we discuss only the primal problem.

Example 6.4. In Problem 6.1, set $K_1 = K_2 = K$, $z = 0$, $h = 0$, $f = 0$, and, for every $k \in \{1,\ldots,K\}$ $r_k = 0$ and $g_k = \iota_{C_k}$, where C_k is a nonempty closed convex subset of $\mathcal{G}_k$ with projection operator P_k. In addition, suppose that

$$(\forall k \in \{1,\ldots,K\}) \quad \operatorname{Argmin} \varphi_k = \{0\},\ \varphi_k(0) = 0, \text{ and } 0 \in \operatorname{sri}(\operatorname{dom} \iota^*_{C_k} - \operatorname{dom} \varphi^*_k). \tag{6.8}$$

It follows from [8, Proposition 15.7(i)] that the infimal convolutions $(\iota_{C_k} \,\square\, \varphi_k)_{1\leqslant k\leqslant K}$ are exact. Hence, (6.3) becomes

$$\underset{x\in\mathcal{H}}{\text{minimize}} \ \sum_{k=1}^{K} \min_{y_k \in C_k} \varphi_k(L_k x - y_k), \tag{6.9}$$

and it is assumed to have at least one solution. We can interpret (6.9) as a relaxation of the (possibly inconsistent) convex feasibility problem

$$\text{find } \overline{x} \in \mathcal{H} \text{ such that } (\forall k \in \{1,\ldots,K\}) \quad L_k\overline{x} \in C_k. \tag{6.10}$$

Indeed, it follows from (6.8) that, if (6.10) is consistent, then its solutions coincide with those of (6.9). Furthermore, in view of (1.15), Algorithm (6.7) can be written as

$$\begin{array}{l}
\text{for } n = 0,1,\ldots \\
\left\lfloor
\begin{array}{l}
p_{1,1,n} = x_n - \gamma_n\big(\sum_{k=1}^K L_k^* v_{k,n} + a_{1,1,n}\big),\\
\text{for } k = 1,\ldots,K\\
\left\lfloor
\begin{array}{l}
s_{1,k+1,n} = y_{k,n} + \gamma_n v_{k,n},\\
p_{1,k+1,n} = \operatorname{prox}_{\gamma_n\varphi_k} s_{1,k+1,n} + b_{1,k+1,n},\\
s_{2,k,n} = v_{k,n} - \gamma_n\big(y_{k,n} - L_k x_n + a_{2,k,n}\big),\\
p_{2,k,n} = s_{2,k,n} - \gamma_n\big(P_k(\gamma_n^{-1} s_{2,k,n}) + b_{2,k,n}\big),\\
q_{2,k,n} = p_{2,k,n} - \gamma_n\big(p_{1,k+1,n} - L_k p_{1,1,n} + c_{2,k,n}\big),\\
v_{k,n+1} = v_{k,n} - s_{2,k,n} + q_{2,k,n},
\end{array}\right.\\
q_{1,1,n} = p_{1,1,n} - \gamma_n\big(\sum_{k=1}^K L_k^* p_{2,k,n} + c_{1,1,n}\big),\\
x_{n+1} = x_n - p_{1,1,n} + q_{1,1,n},\\
\text{for } k = 1,\ldots,K\\
\left\lfloor
\begin{array}{l}
q_{1,k+1,n} = p_{1,k+1,n} + \gamma_n p_{2,k,n},\\
y_{k,n+1} = y_{k,n} - s_{1,k+1,n} + q_{1,k+1,n}.
\end{array}\right.
\end{array}\right.
\end{array} \tag{6.11}$$

By Proposition 6.3(i), $(x_n)_{n\in\mathbb{N}}$ converges weakly to a solution to (6.9) if $\inf_{n\in\mathbb{N}} \gamma_n > 0$ and $\sup_{n\in\mathbb{N}} \gamma_n < \big(1 + \sum_{k=1}^K \|L_k\|^2\big)^{-1/2}$. Now suppose that, for every $k \in \{1,\ldots,K\}$, $\mathcal{G}_k = \mathcal{H}$, $L_k = \mathrm{Id}$, $\varphi_k = \iota_{\{0\}}$ if $k = 1$, and $\varphi_k = \omega_k\|\cdot\|^2$, where $\omega_k \in\,]0,+\infty[$, if $k \neq 1$. Then (6.10) reduces to the feasibility problem of finding $\overline{x} \in \bigcap_{k=1}^K C_k$ and (6.9) reduces to the constrained least-squares relaxation studied in [19], namely, $\text{minimize}_{x\in C_1} \sum_{k=2}^K \omega_k d^2_{C_k}(x)$.

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