Proximity Operators of Perspective Functions with Nonlinear Scaling*

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Abstract. A perspective function is a construction which combines a base function defined on a given space with a nonlinear scaling function defined on another space and which yields a lower semicontinuous convex function on the product space. Since perspective functions are typically nonsmooth, their use in first-order algorithms necessitates the computation of their proximity operator. This paper establishes closed-form expressions for the proximity operator of a perspective function defined on a Hilbert space in terms of a proximity operator involving its base function and one involving its scaling function.

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1 Introduction

Throughout, $\mathcal{H}$ and $\mathcal{G}$ are real Hilbert spaces and $\Gamma_0(\mathcal{H})$ is the class of proper lower semicontinuous convex functions from $\mathcal{H}$ to $]-\infty,+\infty]$. The focus of this paper is on the following construction which combines a base function defined on a given space with a nonlinear scaling function defined on another space and which yields a lower semicontinuous convex function on the product space (alternative constructions of nonlinearly scaled perspective functions in certain settings have been studied in [43, 44, 55]; see [14] for a discussion).

Definition 1.1 [14] The *preperspective* of a *base* function $\varphi\colon \mathcal{H} \to [-\infty,+\infty]$ with respect to a *scaling* function $s\colon \mathcal{G} \to [-\infty,+\infty]$ is

$$\begin{aligned} \varphi \ltimes s\colon\ \mathcal{H}\times\mathcal{G} &\to [-\infty,+\infty] \\ (x,y) &\mapsto \begin{cases} s(y)\varphi\Big(\dfrac{x}{s(y)}\Big), & \text{if } 0 < s(y) < +\infty; \\ +\infty, & \text{if } -\infty \leqslant s(y) \leqslant 0 \ \text{ or } \ s(y) = +\infty, \end{cases} \end{aligned} \tag{1.1}$$

and the *perspective* of φ with respect to s is the largest lower semicontinuous convex function $\varphi \mathbin{\blacktriangleright\!\!<} s$ minorizing $\varphi \ltimes s$.

If $\varphi \in \Gamma_0(\mathcal{H})$, $\mathcal{G} = \mathbb{R}$, and $s\colon y \mapsto y$ in Definition 1.1, it follows from [51, Theorem 3.E] that $\varphi \mathbin{\blacktriangleright\!\!<} s$ reduces to

$$\widetilde{\varphi}\colon \mathcal{H}\times\mathbb{R} \to]-\infty,+\infty] : (x,y) \mapsto \begin{cases} y\varphi\Big(\dfrac{x}{y}\Big), & \text{if } y > 0; \\ (\operatorname{rec}\varphi)(x), & \text{if } y = 0; \\ +\infty, & \text{if } y < 0, \end{cases} \tag{1.2}$$

where $\operatorname{rec}\varphi$ denotes the recession function of φ. This construction, which features a linear scaling function, corresponds to the classical notion of a perspective function. It was first considered in [51] and its properties have been further investigated in [24, 52]. On the application side, an early occurrence of (1.2) is found in statistical inference. In this context, a fundamental objective is the estimation of both the location x (i.e., the regression vector) and the scale y (e.g., the standard deviation of the noise or some other parameter) of the statistical model from the data. In robust statistics, the maximum likelihood-type estimator (M-estimator) for location with concomitant scale [35, p. 179] couples both parameters via a convex objective function which is precisely of the form (1.2). Other applications involving linearly scaled perspective functions can be found in [5, 9, 26, 27, 34, 37, 38].

In recent years, perspective functions with nonlinear scaling have appeared implicitly in several applications. Thus, in the classical dynamical formulation of optimal transport problems [9], x and y represent, respectively, the momentum and density variables, and the transport cost is written in terms of the classical perspective function with linear scaling (1.2). In order to model more complex phenomena nonlinear scaling functions have also been used. For instance, the transport cost used in [16, 32] involves the nonlinear scaling function

$$s\colon \mathbb{R} \to [-\infty,+\infty[\,:\, y \mapsto \begin{cases} y^q, & \text{if } y \geqslant 0; \\ -\infty, & \text{if } y < 0, \end{cases} \tag{1.3}$$

where $q \in \,]0,1[$. This modification makes it possible to take into account congestion effects during the transport of an initial distribution towards a target one. On the other hand, in multiphase optimal transport problems, the density is split in several components. For instance, in the case of two phases, the model of [12] hinges on the scaling function

$$s\colon \mathbb{R} \to [-\infty,+\infty[\; : y \mapsto \begin{cases} \left(\dfrac{\alpha}{y} + \dfrac{\beta}{1-y}\right)^{-1}, & \text{if } y \in \,]0,1[\,; \\ -\infty, & \text{otherwise}, \end{cases} \tag{1.4}$$

where α and β are strictly positive parameters representing the contribution of each phase to the kinetic energy of the system. In [20, 21], the scaling function

$$s\colon \mathbb{R} \to [-\infty,+\infty[\; : y \mapsto \begin{cases} y(1-y), & \text{if } y \in [0,1]; \\ -\infty, & \text{otherwise}, \end{cases} \tag{1.5}$$

is employed to model the mobility of particles via a nonlinear function of the density with saturation in order to avoid overcrowding; this model arises in chemotaxis, phase segregation, and thin liquid film problems. Definition 1.1 can also be found in the family of scaling functions considered in [41], which includes the logarithmic mean

$$s\colon \mathbb{R}^2 \to [-\infty,+\infty[\; : (y_1,y_2) \mapsto \begin{cases} 0, & \text{if } (y_1,y_2) \in (\{0\}\times[0,+\infty[) \cup (]0,+\infty[\times\{0\}); \\ y_1, & \text{if } y_1 = y_2 \in \,]0,+\infty[\,; \\ \dfrac{y_2-y_1}{\log(y_2)-\log(y_1)}, & \text{if } (y_1,y_2) \in \,]0,+\infty[\times]0,+\infty[\text{ and } y_1 \neq y_2; \\ -\infty, & \text{otherwise}, \end{cases} \tag{1.6}$$

as well as the geometric mean

$$s\colon \mathbb{R}^2 \to [-\infty,+\infty[\; : (y_1,y_2) \mapsto \begin{cases} \sqrt{y_1y_2}, & \text{if } (y_1,y_2) \in [0,+\infty[\times[0,+\infty[\,; \\ -\infty, & \text{otherwise}. \end{cases} \tag{1.7}$$

In [41], these scaling functions are used to define metrics on the space of probability measures on graphs and to describe the associated gradient flows. Nonlinear scaling functions also appear in mean field type control [2, 3], machine learning [7], physics [11], operator theory [18, 33], mathematical programming [36, 42], information theory [40, 57], kinetic theory [19], and economics [56].

A key tool in Hilbertian convex analysis to study variational problems and design solution algorithms for them is Moreau's proximity operator [46, 47]. Recall that, given $f \in \Gamma_0(\mathcal{H})$ and $x \in \mathcal{H}$,

$$\operatorname{prox}_f x \text{ is the unique minimizer over } \mathcal{H} \text{ of the function } y \mapsto f(y) + \frac{1}{2}\|x-y\|^2. \tag{1.8}$$

This process defines the proximity operator $\operatorname{prox}_f\colon \mathcal{H} \to \mathcal{H}$ of f, which is extensively discussed in [8]. From an algorithmic viewpoint, proximity operators play a critical role in first order methods as they constitute the main device to activate nonsmooth functions in convex minimization problems; see [8, 22, 25, 29] and the references therein. Since the nonlinearly scaled perspectives functions of Definition 1.1 are typically nonsmooth, knowledge of their proximity operator is required to solve the variational formulations in which they are present. Formulas for the proximity operator of the classical perspective function $\widetilde{\varphi}$ of (1.2) were derived in [26, 27] and they have been employed to solve minimization problems arising in areas such as statistical biosciences [28], information theory [34], signal recovery [37], and machine learning [54]. For nonlinear scales, an instance of a proximity operator of $\varphi\ltimes s$, where $\varphi = |\cdot|^2$ and s as in (1.5), is computed partially in [21].

It is the objective of the present paper to derive closed-form expressions for the proximity operator of the general nonlinearly scaled perspective functions of Definition 1.1. In turn, our results make minimization problems involving perspective functions with nonlinear scaling readily amenable to powerful proximal splitting solution algorithms. The closed-form expressions we shall obtain for $\mathrm{prox}_{\varphi\ltimes s}$ will be formulated in terms of a proximity operator involving the base function φ and one involving its scaling function s. For instance, Example 5.3 gives the formula for the proximity operator of the perspective function of [16, 32], which involves the scaling function (1.3).

In Section 2, we define our notation and provide the background necessary to our investigation. Section 3 is devoted to preliminary results. Closed-form expressions of $\mathrm{prox}_{\varphi\ltimes s}$ are established in Section 4 and Examples are provided in Section 5.

2 Notation and background

The scalar product of a Hilbert space is denoted by $\langle\cdot \mid \cdot\rangle$ and the associated norm by $\|\cdot\|$. The closed ball with center $x \in \mathcal{H}$ and radius $\rho \in \left]0,+\infty\right[$ is denoted by $B(x;\rho)$. The Hilbert direct sum of $\mathcal{H}$ and $\mathcal{G}$ is denoted by $\mathcal{H}\oplus\mathcal{G}$. Let $f\colon \mathcal{H} \to [-\infty,+\infty]$. Then $\mathrm{dom}\, f = \big\{x \in \mathcal{H} \;\big|\; f(x) < +\infty\big\}$ is the domain of f, $\mathrm{epi}\, f = \big\{(x,\xi) \in \mathcal{H}\times\mathbb{R} \;\big|\; f(x) \leqslant \xi\big\}$ is the epigraph of f,

$$f^*\colon \mathcal{H} \to [-\infty,+\infty] : x^* \mapsto \sup_{x\in\mathcal{H}}\big(\langle x \mid x^*\rangle - f(x)\big), \tag{2.1}$$

is the conjugate of f, and ∂f is the subdifferential of f. We declare f convex if $\mathrm{epi}\, f$ is convex, lower semicontinuous if $\mathrm{epi}\, f$ is closed, and proper if $-\infty \notin f(\mathcal{H}) \neq \{+\infty\}$. The recession of $f \in \Gamma_0(\mathcal{H})$ is

$$\mathrm{rec}\, f\colon \mathcal{H} \to [-\infty,+\infty] : x \mapsto \lim_{0<\lambda\to+\infty} \frac{f(z+\lambda x) - f(z)}{\lambda}, \tag{2.2}$$

where $z \in \mathrm{dom}\, f$ is arbitrary. Let C be a subset of $\mathcal{H}$. Then ι_C is the indicator function of C and $\sigma_C = \iota_C^*$ is the support function of C; if C is nonempty, closed, and convex, then $\mathrm{proj}_C = \mathrm{prox}_{\iota_C}$ is the projection operator onto C. See [8] for background on Hilbertian convex analysis and [52] for the Euclidean setting.

Definition 2.1 Let $f\colon \mathcal{H} \to [-\infty,+\infty]$. Then

$$(\forall \xi \in [0,+\infty[)\quad \xi\odot f = \begin{cases} \iota_{\overline{\mathrm{dom}}\, f}, & \text{if } \xi = 0;\\ \xi f, & \text{if } \xi > 0.\end{cases} \tag{2.3}$$

In addition,

$$f^{\vee}\colon \mathcal{H} \to \left]-\infty,+\infty\right] : x \mapsto \begin{cases} f(x), & \text{if } -\infty < f(x) < 0;\\ +\infty, & \text{otherwise}\end{cases} \tag{2.4}$$

and the $\blacktriangledown$ envelope of f is $f^{\blacktriangledown} = f^{\vee**}$. Furthermore,

$$f^{\wedge}\colon \mathcal{H} \to \left]-\infty,+\infty\right] : x \mapsto \begin{cases} f(x), & \text{if } 0 < f(x) < +\infty;\\ +\infty, & \text{otherwise}\end{cases} \tag{2.5}$$

and the ▴ envelope of f is $f^{\blacktriangle} = f^{\wedge **}$.

Let us record a few facts.

Lemma 2.2 [8, Proposition 13.15] *Let $f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ be proper, let $x \in \mathcal{H}$, and let $x^* \in \mathcal{H}$. Then $f(x) + f^*(x^*) \geqslant \langle x \mid x^* \rangle$.*

Lemma 2.3 [51, Theorem 3E] *Let $f \in \Gamma_0(\mathcal{H})$ and $\gamma \in \left[0,+\infty\right[$. Then the following hold:*

(i) $\gamma \odot f \in \Gamma_0(\mathcal{H})$.

(ii) $[\widetilde{f}(\cdot,\gamma)]^* = \gamma \odot f^*$ *and* $(\gamma \odot f)^* = \widetilde{f^*}(\cdot,\gamma)$.

Lemma 2.4 [14, Lemma 3.2] *Let $f \in \Gamma_0(\mathcal{H})$ be such that $f^{-1}(\left]-\infty,0\right[) \neq \varnothing$. Then the following hold:*

(i) $f^{\blacktriangledown} \in \Gamma_0(\mathcal{H})$.

(ii) $\operatorname{dom} f^{\blacktriangledown} = \overline{f^{-1}(\left]-\infty,0\right[)} = f^{-1}(\left]-\infty,0\right])$.

(iii) *Let $x \in \mathcal{H}$ be such that $f(x) \in \left]-\infty,0\right]$. Then $f^{\blacktriangledown}(x) = f(x)$.*

Lemma 2.5 *Let $f \in \Gamma_0(\mathcal{H})$ be such that $f^{-1}(\left]0,+\infty\right[) \neq \varnothing$. Then the following hold:*

(i) $f^{\blacktriangle} \in \Gamma_0(\mathcal{H})$.

(ii) $\operatorname{dom} f^{\blacktriangle} = \operatorname{dom} f \cap \overline{\operatorname{conv}}\, f^{-1}(\left]0,+\infty\right[)$.

(iii) $f^{\blacktriangle}(\operatorname{dom} f^{\blacktriangle}) \subset \left[0,+\infty\right[$.

(iv) *Let $x \in \mathcal{H}$ be such that $f(x) \in \left]0,+\infty\right[$. Then $f^{\blacktriangle}(x) = f(x)$.*

(v) $\overline{\operatorname{dom}}\, f^{\blacktriangle} = \overline{\operatorname{conv}}\, f^{-1}(\left]0,+\infty\right[)$.

Proof. (i)–(iv): See [14, Lemma 3.3].

(v): By (iv), $f^{-1}(\left]0,+\infty\right[) \subset \operatorname{dom} f^{\blacktriangle}$. Hence, since $f^{\blacktriangle}$ is convex by (i), $\operatorname{dom} f^{\blacktriangle}$ is convex, and therefore $\overline{\operatorname{conv}}\, f^{-1}(\left]0,+\infty\right[) \subset \overline{\operatorname{dom}}\, f^{\blacktriangle}$. The assertion therefore follows from (ii). ☐

3 Preliminary results

We establish results on which the derivations of Section 4 will rest.

Lemma 3.1 *Let $f \in \Gamma_0(\mathcal{H})$, $x \in \mathcal{H}$, $p \in \mathcal{H}$, and $\gamma \in \left[0,+\infty\right[$. Then the following hold:*

(i) $\operatorname{prox}_{\gamma \odot f} = \begin{cases} \operatorname{proj}_{\overline{\operatorname{dom}}\, f}, & \text{if } \gamma = 0;\\ \operatorname{prox}_{\gamma f}, & \text{if } \gamma \in \left]0,+\infty\right[. \end{cases}$

(ii) $\operatorname{ran} \operatorname{prox}_{\gamma \odot f} \subset \operatorname{dom}(\gamma \odot f) \subset \overline{\operatorname{dom}}\, f$.

(iii) $p = \operatorname{prox}_{\gamma \odot f} x \Leftrightarrow (\forall y \in \mathcal{H})\ \langle y - p \mid x - p \rangle + (\gamma \odot f)(p) \leqslant (\gamma \odot f)(y)$.

(iv) $p = \operatorname{prox}_{\gamma \odot f} x \Leftrightarrow (\gamma \odot f)(p) + (\gamma \odot f)^*(x - p) = \langle p \mid x - p \rangle$.

(v) *Suppose that* $\gamma > 0$. *Then* $p = \operatorname{prox}_{\gamma f} x \Leftrightarrow f(p) + f^*((x-p)/\gamma) = \langle p \mid x - p\rangle/\gamma$.

(vi) *Suppose that* $\gamma > 0$. *Then* $x = \operatorname{prox}_{\gamma f} x + \gamma\, \operatorname{prox}_{f^*/\gamma}(x/\gamma)$.

Proof. Recall from Lemma 2.3(i) that $\gamma \odot f \in \Gamma_0(\mathcal{H})$.

(i): This follows from (2.3).

(ii): This follows from (1.8) and (2.3).

(iii): In view of (2.3), for $\gamma = 0$, this is the characterization of the projection of x onto the nonempty closed convex set $\overline{\operatorname{dom}}\, f$ [8, Theorem 3.16] while, for $\gamma > 0$, this is [8, Proposition 12.26].

(iv): By virtue of (ii), $\operatorname{dom}(\gamma \odot f) \subset \overline{\operatorname{dom}}\, f$. Hence, Lemma 2.2 and (2.1) yield

$$\langle p \mid x - p\rangle \leqslant (\gamma \odot f)(p) + (\gamma \odot f)^*(x - p) = \sup_{y \in \operatorname{dom} f} \big(\langle y \mid x - p\rangle + (\gamma \odot f)(p) - (\gamma \odot f)(y)\big). \tag{3.1}$$

On the other hand, we derive from (iii) that

$$p = \operatorname{prox}_{\gamma \odot f} x \quad \Leftrightarrow \quad \sup_{y \in \operatorname{dom} f} \big(\langle y \mid x - p\rangle + (\gamma \odot f)(p) - (\gamma \odot f)(y)\big) \leqslant \langle p \mid x - p\rangle. \tag{3.2}$$

Combining (3.1) and (3.2) furnishes the desired characterization.

(v): This follows from (iv) and Lemma 2.3(ii).

(vi): See [8, Proposition 14.3(ii)]. ☐

Lemma 3.2 *Let* $\gamma \in [0, +\infty[$, *let* $\phi \in \Gamma_0(\mathbb{R})$ *be even and such that* $0 \in \operatorname{int} \operatorname{dom} \phi$, *set* $\varphi = \phi \circ \|\cdot\|$, *and let* $x \in \mathcal{H}$. *Then* $\varphi \in \Gamma_0(\mathcal{H})$ *and the following hold:*

(i) $$\operatorname{prox}_{\gamma \odot \varphi} x = \begin{cases} \dfrac{\operatorname{prox}_{\gamma \odot \phi} \|x\|}{\|x\|} x, & \text{if } x \neq 0;\\ 0, & \text{if } x = 0.\end{cases}$$

(ii) $\varphi(\operatorname{prox}_{\gamma \odot \varphi} x) = \phi(\operatorname{prox}_{\gamma \odot \phi} \|x\|)$.

Proof. Since (ii) follows from (i), we prove the latter. We have $\varphi \in \Gamma_0(\mathcal{H})$. In addition, by [8, Propositions 16.17(ii) and 16.27], $\partial\phi(0)$ is a symmetric compact interval, say $\partial\phi(0) = [-\tau, \tau]$, where $\tau \in [0, +\infty[$. We also note that there exists $\rho \in \,]0, +\infty]$ such that

$$\overline{\operatorname{dom}}\, \phi = \begin{cases} [-\rho, \rho], & \text{if } \rho < +\infty;\\ \mathbb{R}, & \text{if } \rho = +\infty \end{cases} \quad \text{and} \quad \overline{\operatorname{dom}}\, \varphi = \begin{cases} B(0; \rho), & \text{if } \rho < +\infty;\\ \mathcal{H}, & \text{if } \rho = +\infty. \end{cases} \tag{3.3}$$

If $\rho < +\infty$, we derive from (3.3) and [8, Example 3.18] that

$$\operatorname{proj}_{\overline{\operatorname{dom}}\, \varphi} x = \frac{\rho x}{\max\{\|x\|, \rho\}} = \begin{cases} \dfrac{\rho\|x\|}{\max\{\|x\|, \rho\}\|x\|} x, & \text{if } x \neq 0;\\ 0, & \text{if } x = 0\end{cases} = \begin{cases} \dfrac{\operatorname{proj}_{\overline{\operatorname{dom}}\, \phi} \|x\|}{\|x\|} x, & \text{if } x \neq 0;\\ 0, & \text{if } x = 0\end{cases} \tag{3.4}$$

whereas, if $\rho = +\infty$, it is clear that $\operatorname{proj}_{\overline{\operatorname{dom}}\, \varphi} x$ coincides with the last term above. In view of Lemma 3.1(i), this establishes the claim for $\gamma = 0$. Now suppose that $\gamma > 0$. Then it follows from [13, Proposition 2.1] that

$$\operatorname{prox}_{\gamma\varphi} x = \begin{cases} \dfrac{\operatorname{prox}_{\gamma\phi} \|x\|}{\|x\|} x, & \text{if } \|x\| > \gamma\tau;\\ 0, & \text{if } \|x\| \leqslant \gamma\tau. \end{cases} \tag{3.5}$$

Moreover, since, in view of (1.8), $\|x\| \leqslant \gamma\tau \Leftrightarrow \|x\| \in \gamma\partial\phi(0) \Leftrightarrow \text{prox}_{\gamma\phi}\|x\| = 0$, (3.5) reduces to

$$\text{prox}_{\gamma\varphi}x = \begin{cases} \dfrac{\text{prox}_{\gamma\phi}\|x\|}{\|x\|}x, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0, \end{cases} \tag{3.6}$$

as required. ☐

Lemma 3.3 *Let $f \in \Gamma_0(\mathcal{H})$, let $x \in \mathcal{H}$, and set $\phi\colon [0,+\infty[\to]-\infty,+\infty] : \gamma \mapsto f(\text{prox}_{\gamma\odot f}x)$. Then the following hold:*

(i) *Let $\mu \in [0,+\infty[$ and $\gamma \in]\mu,+\infty[$. Then $\phi(\gamma) \leqslant \phi(\mu) - \|\text{prox}_{\mu\odot f}x - \text{prox}_{\gamma f}x\|^2/(\gamma-\mu)$.*

(ii) *ϕ is decreasing on $[0,+\infty[$.*

(iii) *ϕ is continuous.*

Proof. First note that Lemma 2.3(i) guarantees that $\text{prox}_{\gamma\odot f}$ and, therefore ϕ, are well defined.

(i): Set $p = \text{prox}_{\mu\odot f}x$ and $q = \text{prox}_{\gamma f}x$, and note that (1.8) implies that $q \in \text{dom}\, f$. If $\mu = 0$, we assume that $p = \text{proj}_{\overline{\text{dom}}\, f}x \in \text{dom}\, f$ since, otherwise, $\phi(\mu) = +\infty$ and the inequality holds trivially. By Lemma 3.1(iii), $\langle q-p \mid x-p\rangle \leqslant \mu(f(q)-f(p))$ and $\langle p-q \mid x-q\rangle \leqslant \gamma(f(p)-f(q))$. Adding these inequalities yields

$$\|p-q\|^2 \leqslant (\gamma-\mu)\big(f(p)-f(q)\big) = (\gamma-\mu)\big(\phi(\mu)-\phi(\gamma)\big), \tag{3.7}$$

which is equivalent to the announced inequality.

(ii): Clear from (i).

(iii): Set $T\colon [0,+\infty[\to \mathcal{H}\colon \gamma \mapsto \text{prox}_{\gamma\odot f}x$. It follows from [8, Proposition 23.31(iii)] applied to the maximally monotone operator ∂f that T is continuous on $]0,+\infty[$ and from [8, Theorem 23.48] that it is right-continuous at 0. Now suppose that $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $]0,+\infty[$ such that $\gamma_n \to \mu \in [0,+\infty[$. Then $T(\gamma_n) \to T(\mu)$. If $\mu = 0$, by invoking the lower semicontinuity of f and (ii), we get

$$\phi(0) = f\big(T(\mu)\big) \leqslant \underline{\lim}\, f\big(T(\gamma_n)\big) = \underline{\lim}\, \phi(\gamma_n) \leqslant \overline{\lim}\, \phi(\gamma_n) \leqslant \phi(0) \tag{3.8}$$

and therefore $\phi(\gamma_n) \to \phi(0)$. If $\mu > 0$, the continuity of ϕ at μ is established in [6, Lemma 3.27]. ☐

The following proposition provides explicit expressions for the perspective function of Definition 1.1 as well as conditions that guarantee that the perspective $\varphi \ltimes s$ is in $\Gamma_0(\mathcal{H}\oplus\mathcal{G})$, and hence that $\text{prox}_{\varphi\ltimes s}$ is well defined.

Proposition 3.4 *Let $\varphi \in \Gamma_0(\mathcal{H})$ and let $s\colon \mathcal{G} \to [-\infty,+\infty]$ be such that $S = s^{-1}(]0,+\infty[) \neq \varnothing$. Let $x \in \mathcal{H}$ and $y \in \mathcal{G}$. Then the following hold:*

(i) *Suppose that $\varphi^*(\mathcal{H}) \subset [0,+\infty]$, $(\varphi^*)^{-1}(]0,+\infty[) \neq \varnothing$, and $-s \in \Gamma_0(\mathcal{G})$. Then*

$$(\varphi\ltimes s)(x,y) = \begin{cases} s(y)\varphi\left(\dfrac{x}{s(y)}\right), & \text{if } 0 < s(y) < +\infty; \\ (\text{rec}\,\varphi)(x), & \text{if } s(y) = 0; \\ +\infty, & \text{otherwise}. \end{cases} \tag{3.9}$$

(ii) *Suppose that $\varphi^*(\mathcal{H}) \subset \{0,+\infty\}$. Then $(\varphi\ltimes s)(x,y) = \varphi(x) + \iota_{\overline{\text{conv}}\, S}(y)$.*

(iii) *Suppose that* $\varphi^*(\mathcal{H}) \subset]-\infty, 0] \cup \{+\infty\}$, $(\varphi^*)^{-1}(]-\infty, 0[) \neq \varnothing$, *and* $s \in \Gamma_0(\mathcal{G})$. *Then*

$$(\varphi \ltimes s)(x, y) = \begin{cases} s(y)\varphi\left(\dfrac{x}{s(y)}\right), & \text{if } 0 < s(y) < +\infty; \\ (\operatorname{rec}\varphi)(x), & \text{if } y \in \overline{\operatorname{conv}}\, S \text{ and } s(y) \leqslant 0; \\ +\infty, & \text{otherwise}. \end{cases} \tag{3.10}$$

Additionally, in each case, $\varphi \ltimes s \in \Gamma_0(\mathcal{H} \oplus \mathcal{G})$.

Proof. (i): Since $\varphi \in \Gamma_0(\mathcal{H})$, (2.1) yields

$$\varphi(0) = \varphi^{**}(0) = \sup_{x^* \in \mathcal{H}} -\varphi^*(x^*) = -\inf_{x^* \in \mathcal{H}} \varphi^*(x^*). \tag{3.11}$$

Hence,

$$\varphi(0) \leqslant 0 \quad \Leftrightarrow \quad \varphi^*(\operatorname{dom}\varphi^*) \subset [0, +\infty[\,. \tag{3.12}$$

The result therefore follows from [14, Lemma 2.5(iii)] and [14, Corollary 5.3(iii)].

(ii): This follows from [14, Lemma 2.5(iii)] and [14, Corollary 5.3(ii)].

(iii): Since $\varphi \in \Gamma_0(\mathcal{H})$, [14, Lemma 2.6] yields

$$\operatorname{rec}\varphi \leqslant \varphi \quad \Leftrightarrow \quad \varphi^*(\operatorname{dom}\varphi^*) \subset]-\infty, 0]\,. \tag{3.13}$$

Hence, the result follows from [14, Lemma 2.5(iii)] and [14, Corollary 5.3(i)]. ☐

4 Computation of the proximity operator

Our strategy to compute the proximity operator of perspective functions is to use the following results based on the Fenchel–Young identity.

Proposition 4.1 *Suppose that* $\varphi \in \Gamma_0(\mathcal{H})$ *and* $s\colon \mathcal{G} \to [-\infty, +\infty]$ *satisfy the conditions of Proposition 3.4. Let* $x \in \mathcal{H}$, $y \in \mathcal{G}$, *and* $\gamma \in]0, +\infty[$. *Set* $(p, q) = \operatorname{prox}_{\gamma(\varphi \ltimes s)}(x, y)$. *Then the following hold:*

(i) $(\varphi \ltimes s)(p, q) + (\varphi \ltimes s)^*\left(\dfrac{x - p}{\gamma}, \dfrac{y - q}{\gamma}\right) = \left\langle p \,\middle|\, \dfrac{x - p}{\gamma}\right\rangle + \left\langle q \,\middle|\, \dfrac{y - q}{\gamma}\right\rangle$.

(ii) $(p, q) \in \operatorname{dom}(\varphi \ltimes s)$.

(iii) $\gamma^{-1}(x - p, y - q) \in \operatorname{dom}(\varphi \ltimes s)^*$.

Proof. We note that (p, q) is well defined by virtue of Proposition 3.4.

(i): This follows from Lemma 3.1(v).

(ii)–(iii): These follow from (i). ☐

To implement the above strategy, explicit expressions are required for $(\varphi \ltimes s)^*$.

Proposition 4.2 [14, Theorem 4.5] *Let* $\varphi\colon \mathcal{H} \to]-\infty, +\infty]$ *be proper, let* $s\colon \mathcal{G} \to [-\infty, +\infty]$ *be such that* $S = s^{-1}(]0, +\infty[) \neq \varnothing$, *let* $x^* \in \mathcal{H}$, *and let* $y^* \in \mathcal{G}$. *Then the following hold:*

(i) *Suppose that* $\varphi^*(\mathcal{H}) \subset [0,+\infty]$ *and* $(\varphi^*)^{-1}(]0,+\infty[) \neq \varnothing$. *Then*

$$(\varphi \ltimes s)^*(x^*,y^*) = \begin{cases} \varphi^*(x^*)\,(-s)^{\blacktriangledown *}\left(\dfrac{y^*}{\varphi^*(x^*)}\right), & \text{if } 0 < \varphi^*(x^*) < +\infty;\\ \sigma_{\overline{\operatorname{conv}}\,S}(y^*), & \text{if } \varphi^*(x^*) = 0;\\ +\infty, & \text{if } \varphi^*(x^*) = +\infty. \end{cases} \tag{4.1}$$

(ii) *Suppose that* $\varphi^*(\mathcal{H}) \subset \{0,+\infty\}$. *Then*

$$\big(\varphi \ltimes s\big)^*(x^*,y^*) = \iota_{(\varphi^*)^{-1}(\{0\})}(x^*) + \sigma_{\overline{\operatorname{conv}}\,S}(y^*). \tag{4.2}$$

(iii) *Suppose that* $\varphi^*(\mathcal{H}) \subset\,]-\infty,0] \cup \{+\infty\}$ *and* $(\varphi^*)^{-1}(]-\infty,0[) \neq \varnothing$. *Then*

$$\big(\varphi \ltimes s\big)^*(x^*,y^*) = \begin{cases} -\varphi^*(x^*)\, s^{\blacktriangle *}\left(\dfrac{y^*}{-\varphi^*(x^*)}\right), & \text{if } -\infty < \varphi^*(x^*) < 0;\\ \sigma_{\overline{\operatorname{conv}}\,S}(y^*), & \text{if } \varphi^*(x^*) = 0;\\ +\infty, & \text{if } \varphi^*(x^*) = +\infty. \end{cases} \tag{4.3}$$

We are now ready to present our main result.

Theorem 4.3 *Let* $\varphi \in \Gamma_0(\mathcal{H})$ *and let* $s\colon \mathcal{G} \to [-\infty,+\infty]$ *be such that* $S = s^{-1}(]0,+\infty[) \neq \varnothing$. *Let* $x \in \mathcal{H}$, $y \in \mathcal{G}$, *and* $\gamma \in\,]0,+\infty[$. *Then the following hold:*

(i) *Suppose that* $\varphi^*(\mathcal{H}) \subset [0,+\infty]$, $(\varphi^*)^{-1}(]0,+\infty[) \neq \varnothing$, *and* $-s \in \Gamma_0(\mathcal{G})$. *Then there exists a unique* $\eta \in [0,+\infty[$ *such that*

$$(-s)^{\blacktriangledown}\left(\operatorname{prox}_{\gamma\varphi^*\left(\operatorname{prox}_{\frac{\eta}{\gamma}\odot\varphi^*}\left(\frac{x}{\gamma}\right)\right)\odot(-s)^{\blacktriangledown}}\, y\right) + \eta = 0. \tag{4.4}$$

Furthermore,

$$\operatorname{prox}_{\gamma(\varphi\ltimes s)}(x,y) = \left(x - \gamma\operatorname{prox}_{\frac{\eta}{\gamma}\odot\varphi^*}\left(\frac{x}{\gamma}\right), \operatorname{prox}_{\gamma\varphi^*\left(\operatorname{prox}_{\frac{\eta}{\gamma}\odot\varphi^*}\left(\frac{x}{\gamma}\right)\right)\odot(-s)^{\blacktriangledown}}\, y\right). \tag{4.5}$$

(ii) *Suppose that* $\varphi^*(\mathcal{H}) \subset \{0,+\infty\}$. *Then* $\operatorname{prox}_{\gamma(\varphi\ltimes s)}(x,y) = (\operatorname{prox}_{\gamma\varphi} x, \operatorname{proj}_{\overline{\operatorname{conv}}\,S}\, y)$.

(iii) *Suppose that* $\varphi^*(\mathcal{H}) \subset\,]-\infty,0] \cup \{+\infty\}$, $(\varphi^*)^{-1}(]-\infty,0[) \neq \varnothing$, *and* $s \in \Gamma_0(\mathcal{G})$. *Then there exists a unique* $\eta \in [0,+\infty[$ *such that*

$$\varphi^*\left(\operatorname{prox}_{\frac{1}{\gamma}s^{\blacktriangle}(\operatorname{prox}_{\gamma\eta\odot s^{\blacktriangle}} y)\odot\varphi^*}\left(\frac{x}{\gamma}\right)\right) + \eta = 0. \tag{4.6}$$

Furthermore,

$$\operatorname{prox}_{\gamma(\varphi\ltimes s)}(x,y) = \left(x - \gamma\operatorname{prox}_{\frac{1}{\gamma}s^{\blacktriangle}(\operatorname{prox}_{\gamma\eta\odot s^{\blacktriangle}} y)\odot\varphi^*}\left(\frac{x}{\gamma}\right), \operatorname{prox}_{\gamma\eta\odot s^{\blacktriangle}} y\right). \tag{4.7}$$

Proof. Set $(p,q) = \operatorname{prox}_{\gamma(\varphi\ltimes s)}(x,y)$.

(i): We deduce from Proposition 4.1(iii) and Proposition 4.2(i) that

$$\varphi^*\left(\frac{x-p}{\gamma}\right) \in \left[0,+\infty\right[\tag{4.8}$$

and from Proposition 4.1(ii) and Proposition 3.4(i) that

$$s(q) \in \left[0,+\infty\right[\,. \tag{4.9}$$

Since $\varphi \in \Gamma_0(\mathcal{H})$, we have $\varphi^{**} = \varphi$ [8, Corollary 13.38]. Hence, it follows from Proposition 3.4(i), (1.2), Lemma 2.3(ii), and (4.9) that

$$(\varphi\ltimes s)(p,q) = \widetilde{\varphi}\big(p,s(q)\big) = \big(s(q)\odot\varphi^*\big)^*(p). \tag{4.10}$$

Next, since Lemma 2.4(i) asserts that $(-s)^{\blacktriangledown} \in \Gamma_0(\mathcal{G})$, we have $(-s)^{\blacktriangledown*} \in \Gamma_0(\mathcal{G})$ and hence deduce from Lemma 2.4(ii) and [8, Proposition 13.49] that

$$\sigma_{\overline{\operatorname{conv}}\,S} = \sigma_{\overline{S}} = \sigma_{\operatorname{dom}(-s)^{\blacktriangledown}} = \operatorname{rec}(-s)^{\blacktriangledown*}. \tag{4.11}$$

Thus, it follows from (4.8), Proposition 4.2(i), (1.2), and Lemma 2.3(ii) that

$$\begin{aligned}
(\varphi\ltimes s)^*&\left(\frac{x-p}{\gamma},\frac{y-q}{\gamma}\right)\\
&= \begin{cases}
\varphi^*\left(\dfrac{x-p}{\gamma}\right)(-s)^{\blacktriangledown*}\left(\dfrac{(y-q)/\gamma}{\varphi^*\big((x-p)/\gamma\big)}\right), & \text{if } 0<\varphi^*\left(\dfrac{x-p}{\gamma}\right)<+\infty;\\
\Big(\operatorname{rec}(-s)^{\blacktriangledown*}\Big)\left(\dfrac{y-q}{\gamma}\right), & \text{if } \varphi^*\left(\dfrac{x-p}{\gamma}\right)=0
\end{cases} \qquad (4.12)\\
&= \widetilde{(-s)^{\blacktriangledown*}}\left(\frac{y-q}{\gamma},\varphi^*\left(\frac{x-p}{\gamma}\right)\right)\\
&= \left(\varphi^*\left(\frac{x-p}{\gamma}\right)\odot(-s)^{\blacktriangledown}\right)^*\left(\frac{y-q}{\gamma}\right). \qquad (4.13)
\end{aligned}$$

On the other hand, (4.8) and (4.9) yield $(x-p)/\gamma \in (\varphi^*)^{-1}(\left[0,+\infty\right[)$ and $q \in \operatorname{dom} s$, respectively. Therefore, since (4.9) and Lemma 2.4(iii) yield

$$0 \leqslant s(q) = -(-s)^{\blacktriangledown}(q), \tag{4.14}$$

we deduce from (2.3) that

$$\big(s(q)\odot\varphi^*\big)\left(\frac{x-p}{\gamma}\right) + \left(\varphi^*\left(\frac{x-p}{\gamma}\right)\odot(-s)^{\blacktriangledown}\right)(q) = 0. \tag{4.15}$$

Consequently, it results from (4.10), (4.13), and Proposition 4.1(i) that

$$\begin{aligned}
\big(s(q)\odot\varphi^*\big)^*(p) + \big(s(q)\odot\varphi^*\big)\left(\frac{x-p}{\gamma}\right) + \left(\varphi^*\left(\frac{x-p}{\gamma}\right)\odot(-s)^{\blacktriangledown}\right)(q) + \left(\varphi^*\left(\frac{x-p}{\gamma}\right)\odot(-s)^{\blacktriangledown}\right)^*\left(\frac{y-q}{\gamma}\right)&\\
= \big(s(q)\odot\varphi^*\big)^*(p) + \left(\varphi^*\left(\frac{x-p}{\gamma}\right)\odot(-s)^{\blacktriangledown}\right)^*\left(\frac{y-q}{\gamma}\right)&\\
= (\varphi\ltimes s)(p,q) + (\varphi\ltimes s)^*\left(\frac{x-p}{\gamma},\frac{y-q}{\gamma}\right)&\\
= \left\langle p \,\middle|\, \frac{x-p}{\gamma}\right\rangle + \left\langle q \,\middle|\, \frac{y-q}{\gamma}\right\rangle.& \qquad (4.16)
\end{aligned}$$

We therefore derive from Lemma 2.2 and Lemma 2.3(i) that

$$\big(s(q)\odot\varphi^*\big)^*(p)+\big(s(q)\odot\varphi^*\big)^{**}\left(\frac{x-p}{\gamma}\right)=\left\langle p \,\middle|\, \frac{x-p}{\gamma}\right\rangle \tag{4.17}$$

and

$$\left(\varphi^*\left(\frac{x-p}{\gamma}\right)\odot(-s)^{\blacktriangledown}\right)(q)+\left(\varphi^*\left(\frac{x-p}{\gamma}\right)\odot(-s)^{\blacktriangledown}\right)^*\left(\frac{y-q}{\gamma}\right)=\left\langle q \,\middle|\, \frac{y-q}{\gamma}\right\rangle. \tag{4.18}$$

In turn, (4.17) and Lemma 3.1(v)–(vi) yield

$$p=\operatorname{prox}_{\gamma(s(q)\odot\varphi^*)^*}x=x-\gamma\operatorname{prox}_{\frac{s(q)}{\gamma}\odot\varphi^*}\left(\frac{x}{\gamma}\right), \tag{4.19}$$

while (4.18) and Lemma 3.1(v) yield

$$q=\operatorname{prox}_{\gamma\varphi^*((x-p)/\gamma)\odot(-s)^{\blacktriangledown}}y. \tag{4.20}$$

Upon combining (4.19) and (4.20), we obtain

$$q=\operatorname{prox}_{\gamma\varphi^*\left(\operatorname{prox}_{\frac{s(q)}{\gamma}\odot\varphi^*}\left(\frac{x}{\gamma}\right)\right)\odot(-s)^{\blacktriangledown}}y. \tag{4.21}$$

Consequently, we deduce from (4.14) that $\eta=s(q)\in[0,+\infty[$ solves (4.4), from which (4.5) follows. To establish the uniqueness of the solution to (4.4), define

$$\begin{cases}\phi_1\colon\ [0,+\infty[\,\to[0,+\infty]:\eta\mapsto\varphi^*\left(\operatorname{prox}_{\frac{\eta}{\gamma}\odot\varphi^*}\left(\frac{x}{\gamma}\right)\right)\\[2mm] \phi_2\colon\ [0,+\infty]\to[-\infty,0]:\mu\mapsto\begin{cases}(-s)^{\blacktriangledown}\left(\operatorname{prox}_{\gamma\mu\odot(-s)^{\blacktriangledown}}y\right), & \text{if } \mu<+\infty;\\ \inf\,(-s)^{\blacktriangledown}, & \text{if } \mu=+\infty,\end{cases}\end{cases} \tag{4.22}$$

and note that ϕ_2 is well defined, since Lemma 3.1(ii) and Lemma 2.4(ii) yield, for every $\mu\in[0,+\infty[$, $\operatorname{prox}_{\gamma\mu\odot(-s)^{\blacktriangledown}}y\in\overline{\operatorname{dom}}\,(-s)^{\blacktriangledown}=\operatorname{dom}\,(-s)^{\blacktriangledown}$. Since (4.4) is equivalent to $\psi(\eta)=0$, where $\psi=\phi_2\circ\phi_1+\mathrm{Id}\colon\ [0,+\infty[\,\to\,]-\infty,0]$, let us prove that ψ is continuous and strictly increasing in $[0,+\infty[$. Indeed, if $\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}(x/\gamma)\in\operatorname{dom}\varphi^*$, Lemma 3.1(i) yields $\phi_1([0,+\infty[)\subset[0,+\infty[$ and Lemma 3.3(ii)–(iii) implies that ψ is strictly increasing and continuous. On the other hand, if $\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}(x/\gamma)\notin\operatorname{dom}\varphi^*$, it follows from Lemma 3.3(iii) that

$$\phi_1(\eta)=\varphi^*\left(\operatorname{prox}_{\frac{\eta}{\gamma}\odot\varphi^*}\left(\frac{x}{\gamma}\right)\right)\uparrow\phi_1(0)=+\infty\quad\text{as}\quad\eta\downarrow 0 \tag{4.23}$$

and Lemma 3.3(ii) and [8, Proposition 12.33(i)] yield

$$\phi_2\big(\phi_1(\eta)\big)=(-s)^{\blacktriangledown}\left(\operatorname{prox}_{\gamma\varphi^*\left(\operatorname{prox}_{\frac{\eta}{\gamma}\odot\varphi^*}\left(\frac{x}{\gamma}\right)\right)\odot(-s)^{\blacktriangledown}}y\right)\downarrow\inf\,(-s)^{\blacktriangledown}=\phi_2\big(\phi_1(0)\big)\quad\text{as}\quad\eta\downarrow 0. \tag{4.24}$$

Moreover, since $S\neq\varnothing$, we have $\inf\,(-s)^{\blacktriangledown}\in[-\infty,0[$ and deduce from (4.24) that any solution to (4.4) is strictly positive. Altogether, (4.4) has at most one solution in $[0,+\infty[$.

(ii): This follows from Proposition 3.4(ii) and [8, Proposition 24.11].

(iii): Lemma 3.1(vi) asserts that

$$(p,q) = (x,y) - \gamma\,\operatorname{prox}_{(\varphi \ltimes s)^*/\gamma}\left(\frac{x}{\gamma}, \frac{y}{\gamma}\right). \tag{4.25}$$

On the other hand, Lemma 2.5(v) yields $\overline{\operatorname{conv}}\,S = \overline{\operatorname{dom}}\,s^{\blacktriangle}$. It therefore follows from Lemma 2.5(i) and [8, Proposition 13.49] that

$$\operatorname{rec}\left(s^{\blacktriangle *}\right) = \sigma_{\operatorname{dom} s^{\blacktriangle}} = \sigma_{\overline{\operatorname{dom}}\, s^{\blacktriangle}} = \sigma_{\overline{\operatorname{conv}}\, S}. \tag{4.26}$$

Moreover, items (i), (iii), and (iv) in Lemma 2.5 yield

$$0 \leqslant s^{\blacktriangle} \in \Gamma_0(\mathcal{G}) \quad \text{and} \quad (s^{\blacktriangle})^{-1}(]0,+\infty[) = S \neq \varnothing, \tag{4.27}$$

while Lemma 2.4(iii) yields

$$\varphi^{*\blacktriangledown} = \varphi^* \in \Gamma_0(\mathcal{H}) \quad \text{and} \quad (\varphi^*)^{-1}(]-\infty,0[) \neq \varnothing. \tag{4.28}$$

In turn, by virtue of Proposition 4.2(iii), (4.26), and Proposition 3.4(i), we obtain

$$(\varphi \ltimes s)^* \colon (x^*, y^*) \mapsto \left(s^{\blacktriangle *} \ltimes (-\varphi^*)\right)(y^*, x^*). \tag{4.29}$$

Now set $(r,t) = \operatorname{prox}_{(\varphi\ltimes s)^*/\gamma}(x/\gamma, y/\gamma)$. Then we derive from (4.29) and [8, Proposition 24.8(iv)] that

$$(t,r) = \operatorname{prox}_{(s^{\blacktriangle *} \ltimes (-\varphi^*))/\gamma}\left(\frac{y}{\gamma}, \frac{x}{\gamma}\right). \tag{4.30}$$

Therefore, (4.27), (4.28), and (i) imply that

$$(t,r) = \left(\frac{y}{\gamma} - \frac{1}{\gamma}\operatorname{prox}_{\gamma\eta \odot s^{\blacktriangle}} y, \operatorname{prox}_{\frac{1}{\gamma} s^{\blacktriangle}(\operatorname{prox}_{\gamma\eta\odot s^{\blacktriangle}} y) \odot \varphi^*}\left(\frac{x}{\gamma}\right)\right), \tag{4.31}$$

where η is the unique solution in $[0,+\infty[$ to (4.6). The conclusion then comes from (4.25). ☐

Next, we provide explicit formulas for $\operatorname{prox}_{\gamma(\varphi\ltimes s)}(x,y)$ in items (i) and (iii) of Theorem 4.3 (item (ii) is already explicit).

Proposition 4.4 *Consider the assumptions and notation of Theorem 4.3(i), and set*

$$\begin{cases} \Omega_1 = \left\{(u,v) \in \mathcal{H}\times\mathcal{G} \mid \varphi^*\left(\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}\left(\frac{u}{\gamma}\right)\right) = 0 \;\; \textit{and} \;\; s(\operatorname{proj}_{\overline{S}} v) = 0\right\} \\ \Omega_2 = \left\{(u,v) \in \mathcal{H}\times\mathcal{G} \mid \varphi^*\left(\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}\left(\frac{u}{\gamma}\right)\right) \in \left]0,+\infty\right[\;\; \textit{and} \;\; s\left(\operatorname{prox}_{\gamma\varphi^*\left(\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}\left(\frac{u}{\gamma}\right)\right)(-s)^{\blacktriangledown}} v\right) = 0\right\} \\ \Omega_3 = \left\{(u,v) \in \mathcal{H}\times\mathcal{G} \mid \varphi^*\left(\operatorname{prox}_{\frac{s(\operatorname{proj}_{\overline{S}} v)}{\gamma}\varphi^*}\left(\frac{u}{\gamma}\right)\right) = 0 \;\; \textit{and} \;\; s(\operatorname{proj}_{\overline{S}} v) \in \left]0,+\infty\right[\right\} \\ \Omega_4 = (\mathcal{H}\times\mathcal{G}) \smallsetminus (\Omega_1 \cup \Omega_2 \cup \Omega_3). \end{cases} \tag{4.32}$$

Then exactly one of the following holds:

(i) $(x,y) \in \Omega_1$, $\eta = 0$, *and* $\operatorname{prox}_{\gamma(\varphi\ltimes s)}(x,y) = \left(x - \gamma\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}(x/\gamma), \operatorname{proj}_{\overline{S}} y\right)$.

(ii) $(x,y)\in\Omega_2$, $\eta=0$, *and* $\operatorname{prox}_{\gamma(\varphi\ltimes s)}(x,y)=\Big(x-\gamma\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}(x/\gamma),\operatorname{prox}_{\gamma\varphi^*(\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}(x/\gamma))(-s)^{\blacktriangledown}}y\Big)$.

(iii) $(x,y)\in\Omega_3$, $\eta=s(\operatorname{proj}_{\overline{S}}y)>0$, *and* $\operatorname{prox}_{\gamma(\varphi\ltimes s)}(x,y)=\Big(x-\gamma\operatorname{prox}_{\frac{s(\operatorname{proj}_{\overline{S}}y)}{\gamma}\varphi^*}(x/\gamma),\operatorname{proj}_{\overline{S}}y\Big)$.

(iv) $(x,y)\in\Omega_4$, $\eta>0$ *solves*

$$\eta=s\Big(\operatorname{prox}_{\gamma\varphi^*\left(\operatorname{prox}_{\frac{\eta}{\gamma}\varphi^*}\left(\frac{x}{\gamma}\right)\right)(-s)^{\blacktriangledown}}y\Big),\tag{4.33}$$

and

$$\operatorname{prox}_{\gamma(\varphi\ltimes s)}(x,y)=\Big(x-\gamma\operatorname{prox}_{\frac{\eta}{\gamma}\varphi^*}\Big(\frac{x}{\gamma}\Big),\operatorname{prox}_{\gamma\varphi^*\left(\operatorname{prox}_{\frac{\eta}{\gamma}\varphi^*}\left(\frac{x}{\gamma}\right)\right)(-s)^{\blacktriangledown}}y\Big).\tag{4.34}$$

Proof. Lemma 2.4(ii) yields

$$\operatorname{dom}(-s)^{\blacktriangledown}=\overline{(-s)^{-1}(]-\infty,0[)}=\overline{S}=s^{-1}([0,+\infty[).\tag{4.35}$$

Hence, it follows from Lemma 3.1(ii) that

$$(\forall\mu\in[0,+\infty[)\quad\operatorname{prox}_{\mu\odot(-s)^{\blacktriangledown}}y\in s^{-1}([0,+\infty[).\tag{4.36}$$

Therefore, Lemma 2.4(iii) implies that (4.4) is equivalent to

$$\eta=s\Big(\operatorname{prox}_{\gamma\varphi^*\left(\operatorname{prox}_{\frac{\eta}{\gamma}\odot\varphi^*}\left(\frac{x}{\gamma}\right)\right)\odot(-s)^{\blacktriangledown}}y\Big).\tag{4.37}$$

(i): Since Lemma 3.1(i) and (4.35) yield

$$s\Big(\operatorname{prox}_{\gamma\varphi^*\left(\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}\left(\frac{x}{\gamma}\right)\right)\odot(-s)^{\blacktriangledown}}y\Big)=s\big(\operatorname{proj}_{\overline{S}}y\big)=0,\tag{4.38}$$

we deduce from (4.37) and Lemma 3.1(i) that $\eta=0$. The claim therefore follows from (4.5).

(ii): Lemma 3.1(i) yields

$$s\Big(\operatorname{prox}_{\gamma\varphi^*\left(\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}\left(\frac{x}{\gamma}\right)\right)\odot(-s)^{\blacktriangledown}}y\Big)=s\Big(\operatorname{prox}_{\gamma\varphi^*\left(\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}\left(\frac{x}{\gamma}\right)\right)(-s)^{\blacktriangledown}}y\Big)=0,\tag{4.39}$$

and we deduce from (4.37) and Lemma 3.1(i) that $\eta=0$. Therefore, the claim follows from (4.5).

(iii): Since $s(\operatorname{proj}_{\overline{S}}y)>0$, Lemma 3.1(i) and (4.35) yield

$$\begin{aligned}s\Big(\operatorname{prox}_{\gamma\varphi^*\left(\operatorname{prox}_{\frac{s(\operatorname{proj}_{\overline{S}}y)}{\gamma}\odot\varphi^*}\left(\frac{x}{\gamma}\right)\right)\odot(-s)^{\blacktriangledown}}y\Big)&=s\Big(\operatorname{prox}_{\gamma\varphi^*\left(\operatorname{prox}_{\frac{s(\operatorname{proj}_{\overline{S}}y)}{\gamma}\varphi^*}\left(\frac{x}{\gamma}\right)\right)\odot(-s)^{\blacktriangledown}}y\Big)\\&=s\big(\operatorname{proj}_{\overline{S}}y\big),\end{aligned}\tag{4.40}$$

and we deduce from (4.37) that $\eta=s(\operatorname{proj}_{\overline{S}}y)>0$. Therefore, the claim follows from (4.5).

(iv): Suppose that $\eta=0$, hence $\varphi^*(\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}(x/\gamma))<+\infty$. Then it follows from (4.37) that

$$0=s\Big(\operatorname{prox}_{\gamma\varphi^*\left(\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}\left(\frac{x}{\gamma}\right)\right)\odot(-s)^{\blacktriangledown}}y\Big).\tag{4.41}$$

Therefore, if $\varphi^*(\text{proj}_{\overline{\text{dom}}\,\varphi^*}(x/\gamma)) = 0$, then (4.41) yields $0 = s(\text{proj}_{\overline{S}}y)$, which implies that $(x,y) \in \Omega_1$. On the other hand, if $\varphi^*(\text{proj}_{\overline{\text{dom}}\,\varphi^*}(x/\gamma)) \in \,]0,+\infty[$, then (4.41) yields

$$0 = s\Big(\text{prox}_{\gamma\varphi^*\left(\text{proj}_{\overline{\text{dom}}\,\varphi^*}\left(\frac{x}{\gamma}\right)\right)(-s)^{\blacktriangledown}}\, y\Big) \tag{4.42}$$

and thus $(x,y) \in \Omega_2$. However, since $(x,y) \in \Omega_4$, we have $(x,y) \notin \Omega_1 \cup \Omega_2$ and obtain a contradiction. This shows that $\eta > 0$. In turn, (4.37) reduces to

$$\eta = s\Big(\text{prox}_{\gamma\varphi^*\left(\text{prox}_{\frac{\eta}{\gamma}\varphi^*}\left(\frac{x}{\gamma}\right)\right)\odot(-s)^{\blacktriangledown}}\, y\Big). \tag{4.43}$$

Hence, if $\varphi^*(\text{prox}_{\frac{\eta}{\gamma}\varphi^*}(x/\gamma)) = 0$, we deduce from (4.35) that $0 < \eta = s(\text{proj}_{\overline{S}}y)$, which yields $(x,y) \in \Omega_3$. However, since $(x,y) \in \Omega_4$, we have $\varphi^*(\text{prox}_{\frac{\eta}{\gamma}\varphi^*}(x/\gamma)) \in \,]0,+\infty[$. Consequently, the claim follows from Lemma 3.1(i).

Finally, it is clear from (4.32) that $\Omega_1 \cap \Omega_2 = \varnothing$ and $\Omega_1 \cap \Omega_3 = \varnothing$. Moreover, we infer from (ii) and (iii) that $\Omega_2 \cap \Omega_3 = \varnothing$. Altogether, $(\Omega_i)_{1\leqslant i\leqslant 4}$ is a partition of $\mathcal{H} \times \mathcal{G}$ and the proof is complete. ☐

Proposition 4.5 *Consider the assumptions and notation of Theorem 4.3(iii), and set*

$$\begin{cases}
\Xi_1 = \Big\{(u,v) \in \mathcal{H} \times \mathcal{G} \;\Big|\; s^{\blacktriangle}(\text{proj}_{\overline{\text{conv}}\,S}v) = 0 \;\; \textit{and} \;\; \varphi^*\Big(\text{proj}_{\overline{\text{dom}}\,\varphi^*}\Big(\dfrac{u}{\gamma}\Big)\Big) = 0\Big\}\\
\Xi_2 = \Big\{(u,v) \in \mathcal{H} \times \mathcal{G} \;\Big|\; s^{\blacktriangle}(\text{proj}_{\overline{\text{conv}}\,S}v) \in \,]0,+\infty[\;\; \textit{and}\; \varphi^*\Big(\text{prox}_{\frac{1}{\gamma}s^{\blacktriangle}(\text{proj}_{\overline{\text{conv}}\,S}v)\varphi^*}\Big(\dfrac{u}{\gamma}\Big)\Big) = 0\Big\}\\
\Xi_3 = \Big\{(u,v) \in \mathcal{H} \times \mathcal{G} \;\Big|\; s^{\blacktriangle}\Big(\text{prox}_{\gamma\left(-\varphi^*\left(\text{proj}_{\overline{\text{dom}}\,\varphi^*}\left(\frac{u}{\gamma}\right)\right)\right)s^{\blacktriangle}}v\Big) = 0 \;\; \textit{and} \;\; \varphi^*\Big(\text{proj}_{\overline{\text{dom}}\,\varphi^*}\Big(\dfrac{u}{\gamma}\Big)\Big) < 0\Big\}\\
\Xi_4 = (\mathcal{H} \times \mathcal{G}) \smallsetminus (\Xi_1 \cup \Xi_2 \cup \Xi_3).
\end{cases} \tag{4.44}$$

Then exactly one of the following holds:

(i) $(x,y) \in \Xi_1$, $\eta = 0$, *and* $\text{prox}_{\gamma(\varphi \ltimes s)}(x,y) = \Big(x - \gamma\,\text{proj}_{\overline{\text{dom}}\,\varphi^*}(x/\gamma), \text{proj}_{\overline{\text{conv}}\,S}y\Big)$.

(ii) $(x,y) \in \Xi_2$, $\eta = 0$, *and* $\text{prox}_{\gamma(\varphi \ltimes s)}(x,y) = \Big(x - \gamma\,\text{prox}_{\frac{1}{\gamma}s^{\blacktriangle}(\text{proj}_{\overline{\text{conv}}\,S}y)\varphi^*}(x/\gamma), \text{proj}_{\overline{\text{conv}}\,S}y\Big)$.

(iii) $(x,y) \in \Xi_3$, $\eta = -\varphi^*\big(\text{proj}_{\overline{\text{dom}}\,\varphi^*}(x/\gamma)\big) > 0$, *and*

$$\text{prox}_{\gamma(\varphi \ltimes s)}(x,y) = \Big(x - \gamma\,\text{proj}_{\overline{\text{dom}}\,\varphi^*}\Big(\frac{x}{\gamma}\Big), \text{prox}_{\left(-\gamma\varphi^*\left(\text{proj}_{\overline{\text{dom}}\,\varphi^*}\left(\frac{x}{\gamma}\right)\right)\right)s^{\blacktriangle}}y\Big). \tag{4.45}$$

(iv) $(x,y) \in \Xi_4$, $\eta > 0$ *solves*

$$\varphi^*\Big(\text{prox}_{\frac{1}{\gamma}s^{\blacktriangle}(\text{prox}_{\gamma\eta s^{\blacktriangle}}y)\varphi^*}\Big(\frac{x}{\gamma}\Big)\Big) + \eta = 0 \tag{4.46}$$

and

$$\text{prox}_{\gamma(\varphi \ltimes s)}(x,y) = \Big(x - \gamma\,\text{prox}_{\frac{1}{\gamma}s^{\blacktriangle}(\text{prox}_{\gamma\eta s^{\blacktriangle}}y)\varphi^*}\Big(\frac{x}{\gamma}\Big), \text{prox}_{\gamma\eta s^{\blacktriangle}}y\Big). \tag{4.47}$$

Proof. It follows from Lemma 2.5(v) that

$$\overline{\operatorname{dom}}\, s^{\blacktriangle} = \overline{\operatorname{conv}}\, S. \tag{4.48}$$

(i): Since Lemma 3.1(i) and (4.48) yield

$$\varphi^*\left(\operatorname{prox}_{\frac{1}{\gamma}s^{\blacktriangle}(\operatorname{proj}_{\overline{\operatorname{conv}}\, S}y)\odot\varphi^*}\left(\frac{x}{\gamma}\right)\right) = \varphi^*\left(\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}\left(\frac{x}{\gamma}\right)\right) = 0, \tag{4.49}$$

we deduce from (4.6) and Lemma 3.1(i) that $\eta = 0$. The claim therefore follows from (4.7).

(ii): Lemma 3.1(i) yields

$$\varphi^*\left(\operatorname{prox}_{\frac{1}{\gamma}s^{\blacktriangle}(\operatorname{proj}_{\overline{\operatorname{conv}}\, S}y)\odot\varphi^*}\left(\frac{x}{\gamma}\right)\right) = \varphi^*\left(\operatorname{prox}_{\frac{1}{\gamma}s^{\blacktriangle}(\operatorname{proj}_{\overline{\operatorname{conv}}\, S}y)\varphi^*}\left(\frac{x}{\gamma}\right)\right) = 0, \tag{4.50}$$

and we deduce from (4.6) and Lemma 3.1(i) that $\eta = 0$. Therefore, the claim follows from (4.7).

(iii): Since $\varphi^*(\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}(\frac{x}{\gamma})) \in \left]-\infty, 0\right[$, Lemma 3.1(i) yields

$$\begin{aligned}\varphi^*&\left(\operatorname{prox}_{\frac{1}{\gamma}s^{\blacktriangle}(\operatorname{prox}_{\gamma(-\varphi^*(\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}(\frac{x}{\gamma})))\odot s^{\blacktriangle}}y)\odot\varphi^*}\left(\frac{x}{\gamma}\right)\right)\\ &= \varphi^*\left(\operatorname{prox}_{\frac{1}{\gamma}s^{\blacktriangle}(\operatorname{prox}_{\gamma(-\varphi^*(\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}(\frac{x}{\gamma})))s^{\blacktriangle}}y)\odot\varphi^*}\left(\frac{x}{\gamma}\right)\right)\\ &= \varphi^*\left(\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}\left(\frac{x}{\gamma}\right)\right)\end{aligned} \tag{4.51}$$

and we deduce from (4.6) that $\eta = -\varphi^*(\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}(x/\gamma)) > 0$. Therefore, the claim follows from (4.7).

(iv): Suppose that $\eta = 0$, hence $s^{\blacktriangle}(\operatorname{proj}_{\overline{\operatorname{conv}}\, S}y) < +\infty$. Then it follows from (4.6) that

$$0 = \varphi^*\left(\operatorname{prox}_{\frac{1}{\gamma}s^{\blacktriangle}(\operatorname{proj}_{\overline{\operatorname{conv}}\, S}y)\odot\varphi^*}\left(\frac{x}{\gamma}\right)\right). \tag{4.52}$$

Therefore, if $s^{\blacktriangle}(\operatorname{proj}_{\overline{\operatorname{conv}}\, S}y) = 0$, then (4.52) yields $0 = \varphi^*(\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}(x/\gamma))$, which implies that $(x, y) \in \Xi_1$. On the other hand, if $s^{\blacktriangle}(\operatorname{proj}_{\overline{\operatorname{conv}}\, S}y) > 0$, then (4.52) yields

$$0 = \varphi^*\left(\operatorname{prox}_{\frac{1}{\gamma}s^{\blacktriangle}(\operatorname{proj}_{\overline{\operatorname{conv}}\, S}y)\varphi^*}\left(\frac{x}{\gamma}\right)\right) \tag{4.53}$$

and thus $(x, y) \in \Xi_2$. At the same time, since $(x, y) \in \Xi_4$, we have $(x, y) \notin \Xi_1 \cup \Xi_2$. This contradiction shows that $\eta > 0$. In turn, (4.6) reduces to

$$-\eta = \varphi^*\left(\operatorname{prox}_{\frac{1}{\gamma}s^{\blacktriangle}(\operatorname{prox}_{\gamma\eta s^{\blacktriangle}}y)\odot\varphi^*}\left(\frac{x}{\gamma}\right)\right). \tag{4.54}$$

Hence, if $s^{\blacktriangle}(\operatorname{prox}_{\gamma\eta s^{\blacktriangle}}y) = 0$, $0 > -\eta = \varphi^*(\operatorname{proj}_{\overline{\operatorname{dom}}\,\varphi^*}(x/\gamma))$, which yields $(x, y) \in \Xi_3$. However, since $(x, y) \in \Xi_4$, we have $s^{\blacktriangle}(\operatorname{prox}_{\gamma\eta s^{\blacktriangle}}y) > 0$. Consequently, the claim follows from Lemma 3.1(i).

To conclude the proof, we observe that (4.44) yields $\Xi_1 \cap \Xi_2 = \varnothing$ and $\Xi_1 \cap \Xi_3 = \varnothing$, and we infer from (ii) and (iii) that $\Xi_2 \cap \Xi_3 = \varnothing$. Altogether, $(\Xi_i)_{1 \leqslant i \leqslant 4}$ is a partition of $\mathcal{H} \times \mathcal{G}$. ☐

Remark 4.6 In cases (i)–(iii) of Proposition 4.4, the computation of $\operatorname{prox}_{\gamma(\varphi \ltimes s)}(x, y)$ requires only the ability to compute the projection operators onto $\overline{\operatorname{dom}}\,\varphi^*$ and $\overline{S}$, as well as the proximity operators of φ^* and $(-s)^{\blacktriangledown}$. Examples of explicit formulas for these operators can be found in [8, 23]. The case (iv) requires additionally the solution $\eta \in\,]0, +\infty[$ to (4.33). To determine η, let us define ϕ_1 and ϕ_2 as in (4.22) and note that it is the root of $T = \phi_1 \circ \phi_2 + \mathrm{Id}$. Since Lemma 3.3 implies that T is strictly increasing and continuous on $]0, +\infty[$, η can be found via standard one-dimensional root-finding routines [50, Chapter 9]. A similar observation can be made for Proposition 4.5.

5 Examples

We illustrate the proposed computation of the proximity operator of perspective functions in the context of applications arising in mean field type control [2, 3], optimal transport [32, 49], statistics [27, 48], and thermostatistics [11, 40]. These applications share a radial base function model, which motivates the following two examples.

Example 5.1 Let $\phi \in \Gamma_0(\mathbb{R})$ be an even coercive function such that $\phi^*(\mathbb{R}) \subset [0, +\infty]$ and $(\phi^*)^{-1}(]0, +\infty[) \neq \varnothing$, and set $\varphi = \phi \circ \|\cdot\|$. Then, $\phi^* \in \Gamma_0(\mathbb{R})$ is even, $0 \in \operatorname{int}\operatorname{dom}\phi^*$ by [8, Proposition 14.16], $\varphi \in \Gamma_0(\mathcal{H})$, and [8, Example 13.8] implies that $\varphi^* = \phi^* \circ \|\cdot\|$. In turn, $\varphi^*(\mathcal{H}) \subset [0, +\infty]$ and $(\varphi^*)^{-1}(]0, +\infty[) \neq \varnothing$. Furthermore, let $-s \in \Gamma_0(\mathcal{G})$, suppose that $s^{-1}(]0, +\infty[) \neq \varnothing$, let $(x, y) \in \mathcal{H} \times \mathcal{G}$, and note that Proposition 3.4(i) asserts that

$$(\varphi \ltimes s)(x, y) = \begin{cases} s(y)\phi\left(\dfrac{\|x\|}{s(y)}\right), & \text{if } 0 < s(y) < +\infty; \\ (\operatorname{rec}\phi)(\|x\|), & \text{if } s(y) = 0; \\ +\infty, & \text{otherwise.} \end{cases} \tag{5.1}$$

Now let $\gamma \in\,]0, +\infty[$. It follows from Lemma 3.2(ii) and Proposition 4.4 that the sets

$$\begin{cases} \Omega_1 = \left\{(u, v) \in \mathcal{H} \times \mathcal{G} \mid \phi^*\left(\operatorname{proj}_{\overline{\operatorname{dom}}\,\phi^*}\left(\dfrac{\|u\|}{\gamma}\right)\right) = 0 \text{ and } s(\operatorname{proj}_{\overline{S}} v) = 0\right\} \\ \Omega_2 = \left\{(u, v) \in \mathcal{H} \times \mathcal{G} \mid \phi^*\left(\operatorname{proj}_{\overline{\operatorname{dom}}\,\phi^*}\left(\dfrac{\|u\|}{\gamma}\right)\right) \in\,]0, +\infty[\text{ and } s\left(\operatorname{prox}_{\gamma\phi^*\left(\operatorname{proj}_{\overline{\operatorname{dom}}\,\phi^*}\left(\frac{\|u\|}{\gamma}\right)\right)(-s)^{\blacktriangledown}} v\right) = 0\right\} \\ \Omega_3 = \left\{(u, v) \in \mathcal{H} \times \mathcal{G} \mid \phi^*\left(\operatorname{prox}_{\frac{s(\operatorname{proj}_{\overline{S}} v)}{\gamma}\phi^*}\left(\dfrac{\|u\|}{\gamma}\right)\right) = 0 \text{ and } s(\operatorname{proj}_{\overline{S}} v) \in\,]0, +\infty[\right\} \\ \Omega_4 = (\mathcal{H} \times \mathcal{G}) \smallsetminus (\Omega_1 \cup \Omega_2 \cup \Omega_3) \end{cases} \tag{5.2}$$

form a partition of $\mathcal{H} \times \mathcal{G}$, which brings up four cases for consideration:

- $(x, y) \in \Omega_1$: We derive from Lemma 3.1(i), Lemma 3.2(i), and Proposition 4.4(i) that

$$\operatorname{prox}_{\gamma(\varphi \ltimes s)}(x, y) = \begin{cases} \left(\left(1 - \dfrac{\gamma}{\|x\|}\operatorname{proj}_{\overline{\operatorname{dom}}\,\phi^*}\left(\dfrac{\|x\|}{\gamma}\right)\right)x, \operatorname{proj}_{\overline{S}}\, y\right), & \text{if } x \neq 0; \\ \left(0, \operatorname{proj}_{\overline{S}}\, y\right), & \text{if } x = 0. \end{cases} \tag{5.3}$$

- $(x,y)\in\Omega_2$: We derive from Lemma 3.1(i), Lemma 3.2(i), and Proposition 4.4(ii) that

$$\operatorname{prox}_{\gamma(\varphi\ltimes s)}(x,y)=\begin{cases}\left(\left(1-\dfrac{\gamma}{\|x\|}\operatorname{proj}_{\overline{\operatorname{dom}}\,\phi^*}\left(\dfrac{\|x\|}{\gamma}\right)\right)x,\operatorname{prox}_{\gamma\phi^*\left(\operatorname{proj}_{\overline{\operatorname{dom}}\,\phi^*}\left(\frac{\|x\|}{\gamma}\right)\right)(-s)^{\blacktriangledown}}y\right), & \text{if } x\neq 0;\\ \left(0,\operatorname{prox}_{\gamma\phi^*(0)(-s)^{\blacktriangledown}}y\right), & \text{if } x=0.\end{cases}\tag{5.4}$$

- $(x,y)\in\Omega_3$: We derive from Lemma 3.1(i), Lemma 3.2(i), and Proposition 4.4(iii) that

$$\operatorname{prox}_{\gamma(\varphi\ltimes s)}(x,y)=\begin{cases}\left(\left(1-\dfrac{\gamma}{\|x\|}\operatorname{prox}_{\frac{s(\operatorname{proj}_{\overline{S}}y)}{\gamma}\phi^*}\left(\dfrac{\|x\|}{\gamma}\right)\right)x,\operatorname{proj}_{\overline{S}}y\right), & \text{if } x\neq 0;\\ \left(0,\operatorname{proj}_{\overline{S}}y\right), & \text{if } x=0.\end{cases}\tag{5.5}$$

- $(x,y)\in\Omega_4$: In view of Lemma 3.2(i) and Lemma 3.1(i), Theorem 4.3(i) and Proposition 4.4(iv) guarantee the existence of a unique solution $\eta\in\,]0,+\infty[$ to

$$\eta=s\left(\operatorname{prox}_{\gamma\phi^*\left(\operatorname{prox}_{\frac{\eta}{\gamma}\phi^*}\left(\frac{\|x\|}{\gamma}\right)\right)(-s)^{\blacktriangledown}}y\right)\tag{5.6}$$

and [8, Proposition 24.32] yields

$$\operatorname{prox}_{\gamma(\varphi\ltimes s)}(x,y)=\begin{cases}\left(\left(1-\dfrac{\gamma}{\|x\|}\operatorname{prox}_{\frac{\eta}{\gamma}\phi^*}\left(\dfrac{\|x\|}{\gamma}\right)\right)x,\operatorname{prox}_{\gamma\phi^*\left(\operatorname{prox}_{\frac{\eta}{\gamma}\phi^*}\left(\frac{\|x\|}{\gamma}\right)\right)(-s)^{\blacktriangledown}}y\right), & \text{if } x\neq 0;\\ \left(0,\operatorname{prox}_{\gamma\phi^*(0)(-s)^{\blacktriangledown}}y\right), & \text{if } x=0.\end{cases}\tag{5.7}$$

Our next example addresses the counterpart of the previous one in which the sign of ϕ^* is flipped.

Example 5.2 Let $\phi\in\Gamma_0(\mathbb{R})$ be an even coercive function such that $\phi^*(\mathbb{R})\subset\,]-\infty,0]\cup\{+\infty\}$ and $(\phi^*)^{-1}(]-\infty,0[)\neq\varnothing$, and set $\varphi=\phi\circ\|\cdot\|$. As in Example 5.1, $\phi^*\in\Gamma_0(\mathbb{R})$ is even, $0\in\operatorname{int}\operatorname{dom}\phi^*$, $\varphi\in\Gamma_0(\mathcal{H})$, and $\varphi^*=\phi^*\circ\|\cdot\|$. In turn, $\varphi^*(\mathcal{H})\subset\,]-\infty,0]\cup\{+\infty\}$ and $(\varphi^*)^{-1}(]-\infty,0[)\neq\varnothing$. In addition, let $s\in\Gamma_0(\mathcal{G})$, suppose that $s^{-1}(]0,+\infty[)\neq\varnothing$, and let $(x,y)\in\mathcal{H}\times\mathcal{G}$. Then, by Proposition 3.4(iii),

$$(\varphi\ltimes s)(x,y)=\begin{cases}s(y)\phi\left(\dfrac{\|x\|}{s(y)}\right), & \text{if } 0<s(y)<+\infty;\\ (\operatorname{rec}\phi)(\|x\|), & \text{if } y\in\overline{\operatorname{conv}}\,S \text{ and } s(y)\leqslant 0;\\ +\infty, & \text{otherwise}.\end{cases}\tag{5.8}$$

Now let $\gamma\in\,]0,+\infty[$. It follows from Lemma 3.2(ii) and Proposition 4.5 that the sets

$$\begin{cases}\Xi_1=\left\{(u,v)\in\mathcal{H}\times\mathcal{G}\mid s^{\blacktriangle}(\operatorname{proj}_{\overline{\operatorname{conv}}\,S}v)=0\ \text{ and }\ \phi^*\left(\operatorname{proj}_{\overline{\operatorname{dom}}\,\phi^*}\left(\dfrac{\|u\|}{\gamma}\right)\right)=0\right\}\\ \Xi_2=\left\{(u,v)\in\mathcal{H}\times\mathcal{G}\mid s^{\blacktriangle}(\operatorname{proj}_{\overline{\operatorname{conv}}\,S}v)\in\,]0,+\infty[\ \text{ and }\ \phi^*\left(\operatorname{prox}_{\frac{1}{\gamma}s^{\blacktriangle}(\operatorname{proj}_{\overline{\operatorname{conv}}\,S}v)\phi^*}\left(\dfrac{\|u\|}{\gamma}\right)\right)=0\right\}\\ \Xi_3=\left\{(u,v)\in\mathcal{H}\times\mathcal{G}\mid s^{\blacktriangle}\left(\operatorname{prox}_{\gamma\left(-\phi^*\left(\operatorname{proj}_{\overline{\operatorname{dom}}\,\phi^*}\left(\frac{\|u\|}{\gamma}\right)\right)\right)s^{\blacktriangle}}v\right)=0\ \text{ and }\ \phi^*\left(\operatorname{proj}_{\overline{\operatorname{dom}}\,\phi^*}\left(\dfrac{\|u\|}{\gamma}\right)\right)<0\right\}\\ \Xi_4=(\mathcal{H}\times\mathcal{G})\smallsetminus(\Xi_1\cup\Xi_2\cup\Xi_3)\end{cases}$$

(5.9)

form a partition of $\mathcal{H}\times\mathcal{G}$. This leads us to consider the following cases:

- $(x,y)\in\Xi_1$: We derive from Lemma 3.1(i), Lemma 3.2(i), and Proposition 4.5(i) that

$$\operatorname{prox}_{\gamma(\varphi\ltimes s)}(x,y)=\begin{cases}\left(\left(1-\dfrac{\gamma}{\|x\|}\operatorname{proj}_{\overline{\operatorname{dom}\phi^*}}\left(\dfrac{\|x\|}{\gamma}\right)\right)x,\operatorname{proj}_{\overline{\operatorname{conv}}\,S}y\right), & \text{if } x\neq 0;\\ \left(0,\operatorname{proj}_{\overline{\operatorname{conv}}\,S}y\right), & \text{if } x=0.\end{cases}\tag{5.10}$$

- $(x,y)\in\Xi_2$: We derive from Lemma 3.1(i), Lemma 3.2(i), and Proposition 4.5(ii) that

$$\operatorname{prox}_{\gamma(\varphi\ltimes s)}(x,y)=\begin{cases}\left(\left(1-\dfrac{\gamma}{\|x\|}\operatorname{prox}_{\frac{1}{\gamma}s^{\blacktriangle}(\operatorname{proj}_{\overline{\operatorname{conv}}\,S}y)\phi^*}\left(\dfrac{\|x\|}{\gamma}\right)\right)x,\operatorname{proj}_{\overline{\operatorname{conv}}\,S}y\right), & \text{if } x\neq 0;\\ \left(0,\operatorname{proj}_{\overline{\operatorname{conv}}\,S}y\right), & \text{if } x=0.\end{cases}\tag{5.11}$$

- $(x,y)\in\Xi_3$: We derive from Lemma 3.1(i), Lemma 3.2(i), and Proposition 4.5(iii) that

$$\operatorname{prox}_{\gamma(\varphi\ltimes s)}(x,y)=\begin{cases}\left(\left(1-\dfrac{\gamma}{\|x\|}\operatorname{proj}_{\overline{\operatorname{dom}\phi^*}}\left(\dfrac{\|x\|}{\gamma}\right)\right)x,\operatorname{prox}_{\left(-\gamma\phi^*\left(\operatorname{proj}_{\overline{\operatorname{dom}\phi^*}}\left(\frac{\|x\|}{\gamma}\right)\right)\right)s^{\blacktriangle}}y\right), & \text{if } x\neq 0;\\ \left(0,\operatorname{prox}_{(-\gamma\phi^*(0))s^{\blacktriangle}}y\right), & \text{if } x=0.\end{cases}\tag{5.12}$$

- $(x,y)\in\Xi_4$: By virtue of Lemma 3.2(i) and Lemma 3.1(i), it follows from Theorem 4.3(iii) and Proposition 4.5(iv) that there exists a unique solution $\eta\in\left]0,+\infty\right[$ to

$$\eta=-\phi^*\left(\operatorname{prox}_{\frac{1}{\gamma}s^{\blacktriangle}(\operatorname{prox}_{\gamma\eta s^{\blacktriangle}}y)\phi^*}\left(\frac{\|x\|}{\gamma}\right)\right)\tag{5.13}$$

and, hence, [8, Proposition 24.32] yields

$$\operatorname{prox}_{\gamma(\varphi\ltimes s)}(x,y)=\begin{cases}\left(\left(1-\dfrac{\gamma}{\|x\|}\operatorname{prox}_{\frac{1}{\gamma}s^{\blacktriangle}(\operatorname{prox}_{\gamma\eta s^{\blacktriangle}}y)\phi^*}\left(\dfrac{\|x\|}{\gamma}\right)\right)x,\operatorname{prox}_{\gamma\eta s^{\blacktriangle}}y\right), & \text{if } x\neq 0;\\ \left(0,\operatorname{prox}_{(-\gamma\phi^*(0))s^{\blacktriangle}}y\right), & \text{if } x=0.\end{cases}\tag{5.14}$$

Starting with the work [9], convex optimization problems involving the perspective function with linear scaling (1.2) appear in optimal transport theory and in mean field games [1, 10, 39, 53]. In this context, numerical methods employing its proximity operator are investigated in [15]. Extensions to variational models with qth root scaling functions have been proposed to address optimal control of McKean–Vlasov systems with congestion [2, 3], as well as optimal transport with nonlinear mobilities [32]. In the following example, we compute the proximity operator of perspective functions with such scaling functions and incorporate a scale constraint which can be used, in particular, to model density constraints [17, 30, 31, 45].

Example 5.3 Let $p\in\left]1,+\infty\right[$ and $q\in\left]0,1\right[$. Set $p^*=p/(p-1)$, and $\phi=|\cdot|^p/p$. Let $I\subset\left[0,+\infty\right[$ be a closed interval with nonempty interior such that $0\in I$ and define

$$\psi\colon\mathbb{R}\to\{-\infty\}\cup\left[0,+\infty\right[: y\mapsto\begin{cases}y^q, & \text{if } y\geqslant 0;\\ -\infty, & \text{if } y<0,\end{cases}\quad\text{and set}\quad s=\psi-\iota_I.\tag{5.15}$$

Then $\phi^* = |\cdot|^{p^*}/p^*$, $(-s)^{\blacktriangledown} = -s$, $\operatorname{dom}\phi^* = \mathbb{R}$, and $\overline{S} = I$. This places us in the framework of Example 5.1 with $\mathcal{G} = \mathbb{R}$, (5.1) reduces to

$$(\varphi \ltimes s)(x,y) = \begin{cases} \dfrac{\|x\|^p}{p\,y^{q(p-1)}}, & \text{if } 0 < y \in I;\\ 0, & \text{if } x = 0 \text{ and } y = 0;\\ +\infty, & \text{otherwise,}\end{cases} \tag{5.16}$$

and (5.2) reduces to

$$\begin{cases} \Omega_1 = \{0\} \times \left]-\infty, 0\right] \\ \Omega_2 = \left\{(u,v) \in \mathcal{H} \times \mathbb{R} \;\middle|\; u \neq 0 \text{ and } \operatorname{prox}_{\frac{\gamma}{p^*}\left|\frac{\|u\|}{\gamma}\right|^{p^*}(-s)} v = 0\right\} \\ \Omega_3 = \{0\} \times \left]0, +\infty\right[\\ \Omega_4 = \big((\mathcal{H} \smallsetminus \{0\}) \times \mathbb{R}\big) \smallsetminus \Omega_2. \end{cases} \tag{5.17}$$

Note that, for every $\mu \in \left]0,+\infty\right[$ and every $v \in \mathbb{R}$, Lemma 3.1(iii) yields $\operatorname{prox}_{-\mu s} v \in \operatorname{dom}(-s) = \left[0,+\infty\right[$. Moreover, if $\operatorname{prox}_{-\mu s} v = 0$, Lemma 3.1(iii) implies

$$(\forall y \in \left[0,+\infty\right[) \quad y^q(y^{1-q}v + \mu) \leqslant 0, \tag{5.18}$$

which is not possible. Hence, $\operatorname{prox}_{-\mu s} v \in \left]0,+\infty\right[$ and we deduce $\Omega_2 = \varnothing$ and $\Omega_4 = (\mathcal{H} \smallsetminus \{0\}) \times \mathbb{R}$. Therefore, Example 5.1 and [8, Proposition 24.47] yield

$$\operatorname{prox}_{\gamma(\varphi \ltimes s)}(x,y) = \begin{cases} \big(0, \operatorname{proj}_I y\big), & \text{if } x = 0;\\ \left(\left(1 - \dfrac{\gamma}{\|x\|}\operatorname{prox}_{\frac{\eta}{\gamma}\phi^*}\left(\dfrac{\|x\|}{\gamma}\right)\right)x, \operatorname{proj}_I\left(\operatorname{prox}_{\frac{\gamma}{p^*}\left|\operatorname{prox}_{\frac{\eta}{\gamma}\phi^*}\left(\frac{\|x\|}{\gamma}\right)\right|^{p^*}(-\psi)} y\right)\right), & \text{if } x \neq 0, \end{cases} \tag{5.19}$$

where, if $x \neq 0$, η is the unique solution in $\left]0,+\infty\right[$ to

$$\eta = \left|\operatorname{proj}_I\left(\operatorname{prox}_{\frac{\gamma}{p^*}\left|\operatorname{prox}_{\frac{\eta}{\gamma}\phi^*}\left(\frac{\|x\|}{\gamma}\right)\right|^{p^*}(-\psi)} y\right)\right|^q. \tag{5.20}$$

Note that, in view of (5.20), (5.19) can be written as

$$\operatorname{prox}_{\gamma(\varphi \ltimes s)}(x,y) = \begin{cases} \big(0, \operatorname{proj}_I y\big), & \text{if } x = 0;\\ \left(\left(1 - \dfrac{\gamma}{\|x\|}\operatorname{prox}_{\frac{\eta}{\gamma}\phi^*}\left(\dfrac{\|x\|}{\gamma}\right)\right)x, \eta^{1/q}\right), & \text{if } x \neq 0. \end{cases} \tag{5.21}$$

In the case when $x \neq 0$, let us point out that, given $\xi \in \left]0,+\infty\right[$, [8, Example 24.38] asserts that $\rho(\xi) = \operatorname{prox}_{\frac{\xi}{\gamma}\phi^*}(\|x\|/\gamma)$ is the unique solution to

$$\|x\| = \rho\gamma + \xi\rho^{p^*-1}. \tag{5.22}$$

On the other hand, for every $\mu \in \left]0,+\infty\right[$, in view of Lemma 3.1(iii), $z(\mu) = \operatorname{prox}_{\gamma\mu(-\psi)} y \in \left]0,+\infty\right[$ is the unique solution to

$$y = z - q\gamma\mu z^{q-1}. \tag{5.23}$$

Therefore, finding $\eta \in \,]0,+\infty[$ such that (5.20) holds amounts to solving $\eta = |\mathrm{proj}_I(z(\rho(\eta)^{p^*}/p^*))|^q$, that is,

$$\eta = \left|\min\left\{z\left(\frac{\rho(\eta)^{p^*}}{p^*}\right), \sup I\right\}\right|^q, \tag{5.24}$$

which can be handled by one-dimensional root-finding methods.

As mentioned in Section 1, problems in statistical inference and in robust statistics involve joint location/scale estimation; see [4, 27, 28, 38, 48] for further instances of this model. Most of these models involve the perspective function of the Huber function with a scalar scale. Our analysis allows us to extend it to nonlinear scales. An illustration of Example 5.2 in this context is provided in the following example, where the proximity operator of the resulting function, a central piece in algorithms for solving concomitant estimation problems [26, 27, 28], is computed.

Example 5.4 Let ϕ is the Huber function with parameter $\alpha \in \,]0,+\infty[$, that is,

$$\phi\colon \xi \mapsto \begin{cases} \alpha|\xi|, & \text{if } |\xi| > \alpha;\\ \dfrac{|\xi|^2+\alpha^2}{2}, & \text{if } |\xi| \leqslant \alpha. \end{cases} \tag{5.25}$$

It follows from [8, Example 13.7] that

$$\phi^*\colon \xi^* \mapsto \begin{cases} +\infty, & \text{if } |\xi^*| > \alpha;\\ \dfrac{|\xi^*|^2-\alpha^2}{2}, & \text{if } |\xi^*| \leqslant \alpha. \end{cases} \tag{5.26}$$

Therefore $\operatorname{dom}\phi^* = [-\alpha,\alpha]$ and we deduce from [8, Example 24.9] and Lemma 3.1(vi) that

$$\mathrm{prox}_{\gamma\phi^*}\colon \xi \mapsto \begin{cases} \alpha\,\mathrm{sign}(\xi), & \text{if } |\xi| > (\gamma+1)\alpha;\\ \xi/(\gamma+1), & \text{if } |\xi| \leqslant (\gamma+1)\alpha. \end{cases} \tag{5.27}$$

Furthermore, let $\beta \in \,]0,+\infty[$ and set

$$s\colon \mathbb{R} \to [0,+\infty[\,\colon y \mapsto \sqrt{\beta+|y|^2}. \tag{5.28}$$

Then $s^{\blacktriangle} = s$ and $\overline{\mathrm{conv}}\, S = \mathbb{R}$. Altogether, we are in the framework of Example 5.2 with $\mathcal{G} = \mathbb{R}$, (5.8) reduces to

$$(\varphi \ltimes s)(x,y) = \begin{cases} \alpha\|x\|, & \text{if } \|x\| > \alpha\sqrt{\beta+|y|^2};\\ \dfrac{\|x\|^2+\alpha^2(\beta+y^2)}{2\sqrt{\beta+|y|^2}} & \text{if } \|x\| \leqslant \alpha\sqrt{\beta+|y|^2}, \end{cases} \tag{5.29}$$

and (5.9) reduces to

$$\begin{cases} \Xi_1 = \Xi_3 = \varnothing\\ \Xi_2 = \big\{(u,v) \in \mathcal{H}\times\mathbb{R} \;\big|\; \|u\| \geqslant \alpha\big(\sqrt{\beta+v^2}+\gamma\big)\big\}\\ \Xi_4 = \big\{(u,v) \in \mathcal{H}\times\mathbb{R} \;\big|\; \|u\| < \alpha\big(\sqrt{\beta+v^2}+\gamma\big)\big\}. \end{cases} \tag{5.30}$$

Therefore, we deduce from Example 5.2 that

$$\operatorname{prox}_{\gamma(\varphi\blacktriangleright s)}(x,y)=\begin{cases}\left(\left(1-\dfrac{\alpha\gamma}{\|x\|}\right)x,y\right), & \text{if } \|x\|\geqslant\alpha\big(\sqrt{\beta+y^2}+\gamma\big);\\ \left(\left(\dfrac{\sqrt{\beta+q(\eta,y)^2}}{\gamma+\sqrt{\beta+q(\eta,y)^2}}\right)x,q(\eta,y)\right), & \text{if } \|x\|<\alpha\big(\sqrt{\beta+y^2}+\gamma\big),\end{cases}\tag{5.31}$$

where η is the unique solution in $]0,+\infty[$ to

$$\eta=\frac{\alpha^2|\gamma+\sqrt{\beta+q(\eta,y)^2}|^2-\|x\|^2}{2|\gamma+\sqrt{\beta+q(\eta,y)^2}|^2}\tag{5.32}$$

and $q(\eta,y)$ is the unique solution to the quartic equation

$$q^4-2yq^3+(y^2+\beta-\gamma^2\eta^2)q^2-2\beta yq+\beta y^2=0,\tag{5.33}$$

in $[0,y]$ if $y\geqslant 0$ and in $[y,0]$ if $y<0$. In view of [58, Section 2.2.3], (5.33) can be solved explictly. We then solve (5.32) via one-dimensional root-finding algorithms [50, Chapter 9].

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