Primal-dual splitting algorithm for solving inclusions with mixtures of composite, Lipschitzian, and parallel-sum type monotone operators*

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Abstract

We propose a primal-dual splitting algorithm for solving monotone inclusions involving a mixture of sums, linear compositions, and parallel sums of set-valued and Lipschitzian operators. An important feature of the algorithm is that the Lipschitzian operators present in the formulation can be processed individually via explicit steps, while the set-valued operators are processed individually via their resolvents. In addition, the algorithm is highly parallel in that most of its steps can be executed simultaneously. This work brings together and notably extends various types of structured monotone inclusion problems and their solution methods. The application to convex minimization problems is given special attention.

Keywords maximal monotone operator, monotone inclusion, nonsmooth convex optimization, parallel sum, set-valued duality, splitting algorithm

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1 Introduction

Duality theory occupies a central place in classical optimization [19, 24, 33, 40, 41]. Since the mid 1960s it has expanded in various directions, e.g., variational inequalities [2, 17, 21, 23, 26, 34], minimax and saddle point problems [27, 29, 32, 39], and, from a more global perspective, monotone inclusions [5, 9, 10, 16, 31, 37, 38]. In the present paper, we propose an algorithm for solving the following structured duality framework for monotone inclusions that encompasses the above cited works. In this formulation, we denote by $B \,\square\, D$ the parallel sum of two set-valued operators B and D (see (2.5)). This operation plays a central role in convex analysis and monotone operator theory. In particular, $B \,\square\, D$ can be seen as a regularization of B by D, and $\square$ is naturally connected to addition through duality since $(B+D)^{-1} = B^{-1} \,\square\, D^{-1}$. It is also strongly related to the infimal convolution of functions through subdifferentials. We refer the reader to [8, 28, 35, 36, 43] and the references therein for background on the parallel sum.

Problem 1.1 Let $\mathcal{H}$ be a real Hilbert space, let $z \in \mathcal{H}$, let m be a strictly positive integer, let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone, and let $C\colon \mathcal{H} \to \mathcal{H}$ be monotone and μ-Lipschitzian for some $\mu \in \left]0,+\infty\right[$. For every $i \in \{1,\ldots,m\}$, let $\mathcal{G}_i$ be a real Hilbert space, let $r_i \in \mathcal{G}_i$, let $B_i\colon \mathcal{G}_i \to 2^{\mathcal{G}_i}$ be maximally monotone, let $D_i\colon \mathcal{G}_i \to 2^{\mathcal{G}_i}$ be monotone and such that D_i^{-1} is ν_i-Lipschitzian, for some $\nu_i \in \left]0,+\infty\right[$, and suppose that $L_i\colon \mathcal{H} \to \mathcal{G}_i$ is a nonzero bounded linear operator. The problem is to solve the primal inclusion

$$\text{find} \quad \overline{x} \in \mathcal{H} \;\text{ such that }\; z \in A\overline{x} + \sum_{i=1}^m L_i^*\big((B_i \,\square\, D_i)(L_i\overline{x} - r_i)\big) + C\overline{x}, \tag{1.1}$$

together with the dual inclusion

$$\text{find } \overline{v_1} \in \mathcal{G}_1,\ \ldots,\ \overline{v_m} \in \mathcal{G}_m \;\text{ such that }\; (\exists\, x \in \mathcal{H}) \begin{cases} z - \sum_{i=1}^m L_i^*\overline{v_i} \in Ax + Cx \\ (\forall i \in \{1,\ldots,m\})\ \overline{v_i} \in (B_i \,\square\, D_i)(L_i x - r_i). \end{cases} \tag{1.2}$$

Problem 1.1 captures and extends various existing problem formulations. Here are some examples that illustrate its versatility and the breadth of its scope.

Example 1.2 In Problem 1.1 set

$$m = 1, \quad C\colon x \mapsto 0, \quad \text{and} \quad D_1\colon y \mapsto \begin{cases} \mathcal{G}_1, & \text{if } y = 0; \\ \varnothing, & \text{if } y \neq 0. \end{cases} \tag{1.3}$$

Then we recover a duality framework investigated in [10, 16, 37, 38], namely (we drop the subscript '1' for brevity),

$$\text{find} \;\; (\overline{x},\overline{v}) \in \mathcal{H} \oplus \mathcal{G} \;\text{ such that }\; \begin{cases} z \in A\overline{x} + L^*\big(B(L\overline{x} - r)\big) \\ -r \in -L\big(A^{-1}(z - L^*\overline{v})\big) + B^{-1}\overline{v}. \end{cases} \tag{1.4}$$

Example 1.3 In Example 1.2, let $\mathcal{G} = \mathcal{H}$, $r = z = 0$, and $L = \mathrm{Id}$. Then we obtain the duality setting of [5, 31], i.e.,

$$\text{find} \;\; (\overline{x},\overline{u}) \in \mathcal{H} \oplus \mathcal{H} \;\text{ such that }\; \begin{cases} 0 \in A\overline{x} + B\overline{x} \\ 0 \in -A^{-1}(-\overline{u}) + B^{-1}\overline{u}. \end{cases} \tag{1.5}$$

The special case of variational inequalities was first treated in [34].

Example 1.4 In Example 1.2, let A and B be the subdifferentials of lower semicontinuous convex functions $f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ and $g\colon \mathcal{G} \to \left]-\infty,+\infty\right]$, respectively. Then, under suitable constraint qualification, we obtain the classical Fenchel-Rockafellar duality framework [40], i.e.,

$$\begin{cases} \underset{x\in\mathcal{H}}{\text{minimize}}\;\; f(x) + g(Lx - r) - \langle x \mid z\rangle \\ \underset{v\in\mathcal{G}}{\text{minimize}}\;\; f^*(z - L^*v) + g^*(v) + \langle v \mid r\rangle. \end{cases} \tag{1.6}$$

Example 1.5 In Problem 1.1, set $C\colon x \mapsto 0$, $z = 0$, and $(\forall i \in \{1,\ldots,m\})$ $\mathcal{G}_i = \mathcal{H}$, $r_i = 0$, $L_i = \operatorname{Id}$, and $D_i = \rho_i^{-1}\operatorname{Id}$, where $\rho_i \in \left]0,+\infty\right[$. Then it follows from [8, Proposition 23.6(ii)] that, for every $i \in \{1,\ldots,m\}$, $B_i \,\square\, D_i = (\operatorname{Id} - J_{\rho_i B_i})/\rho_i = {}^{\rho_i}B_i$ is the Yosida approximation of index ρ_i of B_i. Thus, (1.1) reduces to

$$\text{find}\quad \overline{x} \in \mathcal{H} \;\;\text{such that}\quad 0 \in A\overline{x} + \sum_{i=1}^{m} {}^{\rho_i}B_i\overline{x}. \tag{1.7}$$

This primal problem is investigated in [13, Section 6.3]. In the case when $m = 1$, we obtain the primal-dual problem (we drop the subscript '1' for brevity)

$$\text{find}\;\; (\overline{x},\overline{u}) \in \mathcal{H}\oplus\mathcal{H} \;\;\text{such that}\; \begin{cases} 0 \in A\overline{x} + {}^{\rho}B\overline{x} \\ 0 \in -A^{-1}(-\overline{u}) + B^{-1}\overline{u} + \rho\overline{u} \end{cases} \tag{1.8}$$

investigated in [9].

Example 1.6 In Problem 1.1, set $m = 1$, $\mathcal{G}_1 = \mathcal{G}$, $L_1 = L$, $z = 0$, and $r_1 = 0$, and let A and B_1 be the subdifferentials of lower semicontinuous convex functions $f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ and $g\colon \mathcal{G} \to \left]-\infty,+\infty\right]$, respectively. In addition, let C be the gradient of a differentiable convex function $h\colon \mathcal{H} \to \mathbb{R}$, and let D be the subdifferential of a lower semicontinuous strongly convex function $\ell\colon \mathcal{G} \to \left]-\infty,+\infty\right]$. Then, under suitable constraint qualification, (1.1) assumes the form of the minimization problem

$$\underset{x\in\mathcal{H}}{\text{minimize}}\;\; f(x) + (g \,\square\, \ell)(Lx) + h(x), \tag{1.9}$$

which can be rewritten as

$$\underset{x\in\mathcal{H},\, y\in\mathcal{G}}{\text{minimize}}\;\; f(x) + h(x) + g(y) + \ell(Lx - y). \tag{1.10}$$

In the special case when $h = 0$, $\mathcal{G} = \mathcal{H}$, $L = \operatorname{Id}$, and ℓ is a quadratic coupling function, such formulations have been investigated in [1, 4, 6, 12, 15].

Example 1.7 In Problem 1.1, set $m = 1$, $\mathcal{G}_1 = \mathcal{H}$, $L_1 = \operatorname{Id}$, $B_1 = D_1^{-1}\colon x \mapsto \{0\}$, and $z = r_1 = 0$. Then (1.1) yields the inclusion $0 \in A\overline{x} + C\overline{x}$ studied in [45], where an algorithm using explicit steps for C was proposed.

Example 1.8 In Problem 1.1, set $A\colon x \mapsto \{0\}$ and $C = \operatorname{Id}$. Furthermore, for every $i \in \{1,\ldots,m\}$, let B_i be the subdifferential of a lower semicontinuous convex function $g_i\colon \mathcal{G}_i \to \left]-\infty,+\infty\right]$ and let $D_i^{-1}\colon y \mapsto \{0\}$. Then, under suitable constraint qualification, we obtain the primal-dual pair considered in [14], namely

$$\underset{x\in\mathcal{H}}{\text{minimize}} \;\; \sum_{i=1}^m g_i(L_i x - r_i) + \frac{1}{2}\|x-z\|^2 \tag{1.11}$$

and

$$\underset{v_1\in\mathcal{G}_1,\ldots,v_m\in\mathcal{G}_m}{\text{minimize}} \;\; \frac{1}{2}\left\|z-\sum_{i=1}^m L_i^* v_i\right\|^2 + \sum_{i=1}^m \big(g_i^*(v_i)+\langle v_i \mid r_i\rangle\big). \tag{1.12}$$

Example 1.9 The special case of Problem 1.1 in which

$$A\colon x\mapsto\{0\},\; C\colon x\mapsto 0, \quad\text{and}\quad (\forall i\in\{1,\ldots,m\})\quad D_i\colon y\mapsto \begin{cases}\mathcal{G}_i, & \text{if } \; y=0;\\ \varnothing, & \text{if }\; y\neq 0\end{cases} \tag{1.13}$$

yields the primal-dual pair

$$\text{find}\quad \overline{x}\in\mathcal{H}\;\text{ such that }\quad z\in\sum_{i=1}^m L_i^*\big(B_i(L_i\overline{x}-r_i)\big) \tag{1.14}$$

and

$$\text{find }\; \overline{v_1}\in\mathcal{G}_1,\ldots,\overline{v_m}\in\mathcal{G}_m \quad\text{such that}\; \begin{cases}\sum_{i=1}^m L_i^*\overline{v_i}=z\\ (\exists\, x\in\mathcal{H})(\forall i\in\{1,\ldots,m\})\;\; \overline{v_i}\in B_i(L_i x-r_i).\end{cases} \tag{1.15}$$

This framework is considered in [10, Theorem 3.8].

Conceptually, the primal problem (1.1) could be recast in the form of (1.14), namely

$$\text{find}\quad \overline{x}\in\mathcal{H}\;\text{ such that }\quad z\in\sum_{i=0}^m L_i^*\big(E_i(L_i\overline{x}-r_i)\big), \tag{1.16}$$

where

$$\mathcal{G}_0=\mathcal{H},\;\; E_0=A+C,\;\; L_0=\operatorname{Id},\;\; r_0=0,\quad\text{and}\quad (\forall i\in\{1,\ldots,m\})\quad E_i=B_i \,\square\, D_i. \tag{1.17}$$

In turn, one could contemplate the possibility of using the primal-dual algorithm proposed in [10, Theorem 3.8] to solve Problem 1.1. However, this algorithm requires the computation of the resolvents of the operators $A+C$ and $(B_i^{-1}+D_i^{-1})_{1\leq i\leq m}$, which are usually intractable. Thus, for numerical purposes, Problem 1.1 cannot be reduced to Example 1.9. Let us stress that, even in the instance of the simple inclusion $0\in A\overline{x}+C\overline{x}$, it is precisely the objective of the forward-backward splitting algorithm and its variants [8, 15, 30, 44, 45] to circumvent the computation of the resolvent of $A+C$, as would impose a naive application of the proximal point algorithm [42].

The goal of this paper is to propose a fully split algorithm for solving Problem 1.1 that employs the operators A, $(L_i)_{1\leq i\leq m}$, $(B_i)_{1\leq i\leq m}$, $(D_i)_{1\leq i\leq m}$, and C separately. An important feature of the algorithm is to activate the single-valued operators $(L_i)_{1\leq i\leq m}$, $(D_i^{-1})_{1\leq i\leq m}$, and C through explicit steps. In addition, it exhibits a highly parallel structure which allows for the simultaneous activation of the operators involved. This new splitting method goes significantly beyond the state-of-the-art, which is limited to specific subclasses of Problem 1.1.

In Section 2, we briefly set our notation. The new splitting method is proposed in Section 3, where we also prove its convergence. The special case of minimization problems is discussed in Section 4.

2 Notation and background

Our notation is standard. We refer the reader to [8, 46] for background on convex analysis and monotone operator theory. Hereafter, $\mathcal{K}$ is a real Hilbert space.

We denote the scalar product of a Hilbert space by $\langle\cdot \mid \cdot\rangle$ and the associated norm by $\|\cdot\|$. The symbols $\rightharpoonup$ and $\to$ denote respectively weak and strong convergence. Moreover, $\mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_m$ is the Hilbert direct sum of the Hilbert spaces $(\mathcal{G}_i)_{1\leq i\leq m}$ in Problem 1.1, i.e., their product space equipped with the norm $(y_i)_{1\leq i\leq m} \mapsto \sqrt{\sum_{i=1}^m \|y_i\|^2}$. For every $i \in \{1,\ldots,m\}$, let T_i be a mapping from $\mathcal{G}_i$ to some set $\mathcal{R}$. Then

$$\bigoplus_{i=1}^m T_i\colon \bigoplus_{i=1}^m \mathcal{G}_i \to \mathcal{R}\colon (y_i)_{1\leq i\leq m} \mapsto \sum_{i=1}^m T_i y_i. \tag{2.1}$$

Let $M\colon \mathcal{K} \to 2^{\mathcal{K}}$ be a set-valued operator. We denote by $\operatorname{ran} M = \big\{u \in \mathcal{K} \;\big|\; (\exists\, x \in \mathcal{K})\; u \in Mx\big\}$ the range of M, by $\operatorname{dom} M = \big\{x \in \mathcal{K} \;\big|\; Mx \neq \varnothing\big\}$ its domain, by $\operatorname{zer} M = \big\{x \in \mathcal{K} \;\big|\; 0 \in Mx\big\}$ its set of zeros, by $\operatorname{gra} M = \big\{(x,u) \in \mathcal{K}\times\mathcal{K} \;\big|\; u \in Mx\big\}$ its graph, and by M^{-1} its inverse, i.e., the set-valued operator with graph $\big\{(u,x) \in \mathcal{K}\times\mathcal{K} \;\big|\; u \in Mx\big\}$. The resolvent of M is

$$J_M = (\operatorname{Id} + M)^{-1}, \tag{2.2}$$

where Id denotes the identity operator on $\mathcal{K}$. Moreover, M is monotone if

$$(\forall (x,u) \in \operatorname{gra} M)(\forall (y,v) \in \operatorname{gra} M) \quad \langle x - y \mid u - v\rangle \geq 0, \tag{2.3}$$

and maximally so if there exists no monotone operator $\widetilde{M}\colon \mathcal{K} \to 2^{\mathcal{K}}$ such that $\operatorname{gra} M \subset \operatorname{gra} \widetilde{M} \neq \operatorname{gra} M$. We say that M is uniformly monotone at $x \in \operatorname{dom} M$ if there exists an increasing function $\phi\colon\ [0,+\infty[\, \to [0,+\infty]$ that vanishes only at 0 such that

$$(\forall u \in Mx)(\forall (y,v) \in \operatorname{gra} M) \quad \langle x - y \mid u - v\rangle \geq \phi(\|x - y\|). \tag{2.4}$$

The parallel sum of two set-valued operators M_1 and M_2 from $\mathcal{K}$ to $2^{\mathcal{K}}$ is

$$M_1 \,\Box\, M_2 = (M_1^{-1} + M_2^{-1})^{-1}. \tag{2.5}$$

We denote by $\Gamma_0(\mathcal{K})$ the class of lower semicontinuous convex functions $\varphi\colon \mathcal{K} \to\,]-\infty,+\infty]$ such that $\operatorname{dom}\varphi = \big\{x \in \mathcal{K} \;\big|\; \varphi(x) < +\infty\big\} \neq \varnothing$. Now let $\varphi \in \Gamma_0(\mathcal{K})$. The conjugate of φ is the function $\varphi^* \in \Gamma_0(\mathcal{K})$ defined by $\varphi^*\colon u \mapsto \sup_{x\in\mathcal{K}}(\langle x \mid u\rangle - \varphi(x))$, and the subdifferential of φ is the maximally monotone operator

$$\partial\varphi\colon \mathcal{K} \to 2^{\mathcal{K}}\colon x \mapsto \big\{u \in \mathcal{K} \;\big|\; (\forall y \in \mathcal{K})\;\; \langle y - x \mid u\rangle + \varphi(x) \leq \varphi(y)\big\} \tag{2.6}$$

with inverse given by

$$(\partial\varphi)^{-1} = \partial\varphi^*. \tag{2.7}$$

Moreover, for every $x \in \mathcal{K}$, $\varphi + \|x - \cdot\|^2/2$ possesses a unique minimizer, which is denoted by $\operatorname{prox}_\varphi x$. We have

$$\operatorname{prox}_\varphi = J_{\partial\varphi}. \tag{2.8}$$

We say that φ is ν-strongly convex for some $\nu \in]0,+\infty[$ if $\varphi - \nu\|\cdot\|^2/2$ is convex, and that φ is uniformly convex at $x \in \operatorname{dom}\varphi$ if there exists an increasing function $\phi\colon [0,+\infty[\to [0,+\infty]$ that vanishes only at 0 such that

$$(\forall y \in \operatorname{dom}\varphi)(\forall \alpha \in]0,1[) \quad \varphi(\alpha x + (1-\alpha)y) + \alpha(1-\alpha)\phi(\|x-y\|) \leq \alpha\varphi(x) + (1-\alpha)\varphi(y). \tag{2.9}$$

The infimal convolution of two functions φ_1 and φ_2 from $\mathcal{K}$ to $]-\infty,+\infty]$ is

$$\varphi_1 \,\square\, \varphi_2 \colon \mathcal{K} \to [-\infty,+\infty] : x \mapsto \inf_{y\in\mathcal{K}} \big(\varphi_1(y) + \varphi_2(x-y)\big). \tag{2.10}$$

Finally, let S be a convex subset of $\mathcal{K}$. The strong relative interior of S, i.e., the set of points $x \in S$ such that the cone generated by $-x+S$ is a closed vector subspace of $\mathcal{K}$, is denoted by $\operatorname{sri} S$, and the relative interior of S, i.e., the set of points $x \in S$ such that the cone generated by $-x+S$ is a vector subspace of $\mathcal{K}$, is denoted by $\operatorname{ri} S$.

3 Main result

Our main result is the following theorem, which presents our new splitting algorithm and describes its asymptotic behavior.

Theorem 3.1 *In Problem 1.1, suppose that*

$$z \in \operatorname{ran}\left(A + \sum_{i=1}^m L_i^*(B_i \,\square\, D_i)(L_i\cdot - r_i) + C\right). \tag{3.1}$$

Let $(a_{1,n})_{n\in\mathbb{N}}$, $(b_{1,n})_{n\in\mathbb{N}}$, and $(c_{1,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{H}$ and, for every $i \in \{1,\ldots,m\}$, let $(a_{2,i,n})_{n\in\mathbb{N}}$, $(b_{2,i,n})_{n\in\mathbb{N}}$, and $(c_{2,i,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{G}_i$. Furthermore, set

$$\beta = \max\{\mu,\nu_1,\ldots,\nu_m\} + \sqrt{\sum_{i=1}^m \|L_i\|^2}, \tag{3.2}$$

let $x_0 \in \mathcal{H}$, let $(v_{1,0},\ldots,v_{m,0}) \in \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_m$, let $\varepsilon \in]0,1/(\beta+1)[$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon,(1-\varepsilon)/\beta]$, and set

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_{1,n} = x_n - \gamma_n\big(Cx_n + \sum_{i=1}^m L_i^* v_{i,n} + a_{1,n}\big) \\ p_{1,n} = J_{\gamma_n A}(y_{1,n} + \gamma_n z) + b_{1,n} \\ \text{For } i = 1,\ldots,m \\ \left\lfloor \begin{array}{l} y_{2,i,n} = v_{i,n} + \gamma_n\big(L_i x_n - D_i^{-1} v_{i,n} + a_{2,i,n}\big) \\ p_{2,i,n} = J_{\gamma_n B_i^{-1}}(y_{2,i,n} - \gamma_n r_i) + b_{2,i,n} \\ q_{2,i,n} = p_{2,i,n} + \gamma_n\big(L_i p_{1,n} - D_i^{-1} p_{2,i,n} + c_{2,i,n}\big) \\ v_{i,n+1} = v_{i,n} - y_{2,i,n} + q_{2,i,n}. \end{array} \right. \\ q_{1,n} = p_{1,n} - \gamma_n\big(Cp_{1,n} + \sum_{i=1}^m L_i^* p_{2,i,n} + c_{1,n}\big) \\ x_{n+1} = x_n - y_{1,n} + q_{1,n}. \end{array} \right. \tag{3.3}$$

Then the following hold.

(i) $\sum_{n\in\mathbb{N}}\|x_n-p_{1,n}\|^2<+\infty$ *and* $(\forall i\in\{1,\ldots,m\})$ $\sum_{n\in\mathbb{N}}\|v_{i,n}-p_{2,i,n}\|^2<+\infty$.

(ii) *There exist a solution* $\overline{x}$ *to* (1.1) *and a solution* $(\overline{v_1},\ldots,\overline{v_m})$ *to* (1.2) *such that the following hold.*

(a) $z-\sum_{j=1}^m L_j^*\overline{v_j}\in A\overline{x}+C\overline{x}$ *and* $(\forall i\in\{1,\ldots,m\})$ $L_i\overline{x}-r_i\in B_i^{-1}\overline{v_i}+D_i^{-1}\overline{v_i}$.

(b) $(\forall i\in\{1,\ldots,m\})$ $-r_i\in -L_i\big((A^{-1}\,\square\,C^{-1})\big(z-\sum_{j=1}^m L_j^*\overline{v_j}\big)\big)+B_i^{-1}\overline{v_i}+D_i^{-1}\overline{v_i}$.

(c) $x_n\rightharpoonup\overline{x}$ *and* $p_{1,n}\rightharpoonup\overline{x}$.

(d) $(\forall i\in\{1,\ldots,m\})$ $v_{i,n}\rightharpoonup\overline{v_i}$ *and* $p_{2,i,n}\rightharpoonup\overline{v_i}$.

(e) *Suppose that* A *or* C *is uniformly monotone at* $\overline{x}$. *Then* $x_n\to\overline{x}$ *and* $p_{1,n}\to\overline{x}$.

(f) *Suppose that, for some* $i\in\{1,\ldots,m\}$, B_i^{-1} *or* D_i^{-1} *is uniformly monotone at* $\overline{v_i}$. *Then* $v_{i,n}\to\overline{v_i}$ *and* $p_{2,i,n}\to\overline{v_i}$.

Proof. Let us first rewrite (3.3) as

$$(\forall n\in\mathbb{N})\quad \left\lfloor\begin{array}{l} y_{1,n}=x_n-\gamma_n\big(Cx_n+\sum_{i=1}^m L_i^*v_{i,n}+a_{1,n}\big)\\ \text{For } i=1,\ldots,m\\ \left\lfloor\; y_{2,i,n}=v_{i,n}+\gamma_n\big(L_ix_n-D_i^{-1}v_{i,n}+a_{2,i,n}\big)\right.\\ p_{1,n}=J_{\gamma_nA}(y_{1,n}+\gamma_nz)+b_{1,n}\\ \text{For } i=1,\ldots,m\\ \left\lfloor\; p_{2,i,n}=J_{\gamma_nB_i^{-1}}(y_{2,i,n}-\gamma_nr_i)+b_{2,i,n}\right.\\ q_{1,n}=p_{1,n}-\gamma_n\big(Cp_{1,n}+\sum_{i=1}^m L_i^*p_{2,i,n}+c_{1,n}\big)\\ \text{For } i=1,\ldots,m\\ \left\lfloor\; q_{2,i,n}=p_{2,i,n}+\gamma_n\big(L_ip_{1,n}-D_i^{-1}p_{2,i,n}+c_{2,i,n}\big)\right.\\ x_{n+1}=x_n-y_{1,n}+q_{1,n}\\ \text{For } i=1,\ldots,m\\ \left\lfloor\; v_{i,n+1}=v_{i,n}-y_{2,i,n}+q_{2,i,n}.\right. \end{array}\right. \tag{3.4}$$

Next, let us introduce the Hilbert space

$$\boldsymbol{\mathcal{K}}=\mathcal{H}\oplus\mathcal{G}_1\oplus\cdots\oplus\mathcal{G}_m, \tag{3.5}$$

and the operators

$$\begin{array}{rcl}\boldsymbol{M}\colon\boldsymbol{\mathcal{K}} &\to& 2^{\boldsymbol{\mathcal{K}}}\\ (x,v_1,\ldots,v_m)&\mapsto&(-z+Ax)\times(r_1+B_1^{-1}v_1)\times\cdots\times(r_m+B_m^{-1}v_m)\end{array} \tag{3.6}$$

and

$$\begin{array}{rcl}\boldsymbol{Q}\colon\boldsymbol{\mathcal{K}} &\to& \boldsymbol{\mathcal{K}}\\ (x,v_1,\ldots,v_m)&\mapsto&\big(Cx+L_1^*v_1+\cdots+L_m^*v_m,-L_1x+D_1^{-1}v_1,\ldots,-L_mx+D_m^{-1}v_m\big).\end{array} \tag{3.7}$$

Since the operator A and $(B_i)_{1\le i\le m}$ are maximally monotone, so is $\boldsymbol{M}$ by [8, Propositions 20.22 and 20.23]. In addition, [8, Propositions 23.15(ii) and 23.16] yield

$$(\forall\gamma\in\,]0,+\infty[)(\forall(x,v_1,\ldots,v_m)\in\boldsymbol{\mathcal{K}})\quad J_{\gamma\boldsymbol{M}}(x,v_1,\ldots,v_m)=\big(J_{\gamma A}(x+\gamma z),J_{\gamma B_1^{-1}}(v_1-\gamma r_1),\ldots,J_{\gamma B_m^{-1}}(v_m-\gamma r_m)\big). \tag{3.8}$$

Let us now examine the properties of $\boldsymbol{Q}$. To this end, let $(x, v_1, \ldots, v_m)$ and $(y, w_1, \ldots, w_m)$ be two points in $\boldsymbol{\mathcal{K}}$. Using the monotonicity of the operators C and $(D_i^{-1})_{1\leq i\leq m}$, we derive from (3.7) that

$$
\begin{aligned}
&\langle (x, v_1, \ldots, v_m) - (y, w_1, \ldots, w_m) \mid \boldsymbol{Q}(x, v_1, \ldots, v_m) - \boldsymbol{Q}(y, w_1, \ldots, w_m)\rangle \\
&\quad = \Big\langle (x-y, v_1 - w_1, \ldots, v_m - w_m) \,\Big|\, \big(Cx - Cy + L_1^*(v_1 - w_1) + \cdots + L_m^*(v_m - w_m), \\
&\qquad\qquad - L_1(x-y) + D_1^{-1}v_1 - D_1^{-1}w_1, \ldots, -L_m(x-y) + D_m^{-1}v_m - D_m^{-1}w_m\big)\Big\rangle \\
&\quad = \langle x - y \mid Cx - Cy\rangle + \sum_{i=1}^m \big\langle v_i - w_i \mid D_i^{-1}v_i - D_i^{-1}w_i\big\rangle \\
&\qquad + \sum_{i=1}^m \big(\langle x - y \mid L_i^*(v_i - w_i)\rangle - \langle v_i - w_i \mid L_i(x-y)\rangle\big) \\
&\quad = \langle x - y \mid Cx - Cy\rangle + \sum_{i=1}^m \big\langle v_i - w_i \mid D_i^{-1}v_i - D_i^{-1}w_i\big\rangle \\
&\quad \geq 0. \qquad (3.9)
\end{aligned}
$$

Hence, $\boldsymbol{Q}$ is monotone. Using the triangle inequality, the Lipschitzianity assumptions, the Cauchy-Schwarz inequality, and (3.2), we obtain

$$
\begin{aligned}
&\|\boldsymbol{Q}(x, v_1, \ldots, v_m) - \boldsymbol{Q}(y, w_1, \ldots, w_m)\| \\
&\quad = \bigg\| \Big(Cx - Cy, D_1^{-1}v_1 - D_1^{-1}w_1, \ldots, D_m^{-1}v_m - D_m^{-1}w_m\Big) \\
&\qquad + \bigg(\sum_{i=1}^m L_i^*(v_i - w_i), -L_1(x-y), \ldots, -L_m(x-y)\bigg)\bigg\| \\
&\quad \leq \Big\| \Big(Cx - Cy, D_1^{-1}v_1 - D_1^{-1}w_1, \ldots, D_m^{-1}v_m - D_m^{-1}w_m\Big)\Big\| \\
&\qquad + \bigg\|\bigg(\sum_{i=1}^m L_i^*(v_i - w_i), -L_1(x-y), \ldots, -L_m(x-y)\bigg)\bigg\| \\
&\quad = \sqrt{\|Cx - Cy\|^2 + \sum_{i=1}^m \big\|D_i^{-1}v_i - D_i^{-1}w_i\big\|^2} + \sqrt{\bigg\|\sum_{i=1}^m L_i^*(v_i - w_i)\bigg\|^2 + \sum_{i=1}^m \|L_i(x-y)\|^2} \\
&\quad \leq \sqrt{\mu^2\|x-y\|^2 + \sum_{i=1}^m \nu_i^2\|v_i - w_i\|^2} + \sqrt{\bigg(\sum_{i=1}^m \|L_i\|\,\|v_i - w_i\|\bigg)^2 + \sum_{i=1}^m \|L_i\|^2\,\|x-y\|^2} \\
&\quad \leq \max\{\mu, \nu_1, \ldots, \nu_m\}\sqrt{\|x-y\|^2 + \sum_{i=1}^m \|v_i - w_i\|^2} \\
&\qquad + \sqrt{\bigg(\sum_{i=1}^m \|L_i\|^2\bigg)\bigg(\sum_{i=1}^m \|v_i - w_i\|^2\bigg) + \bigg(\sum_{i=1}^m \|L_i\|^2\bigg)\|x-y\|^2} \\
&\quad = \beta\|(x, v_1, \ldots, v_m) - (y, w_1, \ldots, w_m)\|. \qquad (3.10)
\end{aligned}
$$

To sum up, we have shown that

$$
\boldsymbol{M} \text{ is maximally monotone and } \boldsymbol{Q} \text{ is monotone and } \beta\text{-Lipschitzian.} \tag{3.11}
$$

Next, let us observe that

$$
\begin{aligned}
(3.1) \Leftrightarrow (\exists\, x \in \mathcal{H}) \quad & z \in Ax + \sum_{i=1}^{m} L_i^*\big((B_i \,\square\, D_i)(L_i x - r_i)\big) + Cx \\
\Leftrightarrow (\exists\, (x, v_1, \ldots, v_m) \in \boldsymbol{\mathcal{K}}) \quad & \begin{cases} z \in Ax + \sum_{i=1}^{m} L_i^* v_i + Cx \\ v_1 \in (B_1 \,\square\, D_1)(L_1 x - r_1) \\ \quad\vdots \\ v_m \in (B_m \,\square\, D_m)(L_m x - r_m) \end{cases} \\
\Leftrightarrow (\exists\, (x, v_1, \ldots, v_m) \in \boldsymbol{\mathcal{K}}) \quad & \begin{cases} 0 \in -z + Ax + \sum_{i=1}^{m} L_i^* v_i + Cx \\ 0 \in r_1 + B_1^{-1} v_1 + D_1^{-1} v_1 - L_1 x \\ \quad\vdots \\ 0 \in r_m + B_m^{-1} v_m + D_m^{-1} v_m - L_m x \end{cases} \\
\Leftrightarrow (\exists\, (x, v_1, \ldots, v_m) \in \boldsymbol{\mathcal{K}}) \quad & (0, \ldots, 0) \in (-z + Ax) \times (r_1 + B_1^{-1} v_1) \times \cdots \times (r_m + B_m^{-1} v_m) \\
& \quad + (L_1^* v_1 + \cdots + L_m^* v_m + Cx, D_1^{-1} v_1 - L_1 x, \ldots, D_m^{-1} v_m - L_m x) \\
\Leftrightarrow (\exists\, (x, v_1, \ldots, v_m) \in \boldsymbol{\mathcal{K}}) \quad & (0, \ldots, 0) \in (\boldsymbol{M} + \boldsymbol{Q})(x, v_1, \ldots, v_m).
\end{aligned} \tag{3.12}
$$

In other words,

$$
\operatorname{zer}(\boldsymbol{M} + \boldsymbol{Q}) \neq \varnothing. \tag{3.13}
$$

Now, let us set

$$
(\forall n \in \mathbb{N}) \quad \begin{cases} \boldsymbol{x}_n = (x_n, v_{1,n}, \ldots, v_{1,n}) \\ \boldsymbol{y}_n = (y_{1,n}, y_{2,1,n}, \ldots, y_{2,m,n}) \\ \boldsymbol{p}_n = (p_{1,n}, p_{2,1,n}, \ldots, p_{2,m,n}) \\ \boldsymbol{q}_n = (q_{1,n}, q_{2,1,n}, \ldots, q_{2,m,n}) \end{cases} \quad \text{and} \quad \begin{cases} \boldsymbol{a}_n = (a_{1,n}, a_{2,1,n}, \ldots, a_{2,m,n}) \\ \boldsymbol{b}_n = (b_{1,n}, b_{2,1,n}, \ldots, b_{2,m,n}) \\ \boldsymbol{c}_n = (c_{1,n}, c_{2,1,n}, \ldots, c_{2,m,n}). \end{cases} \tag{3.14}
$$

We first observe that our assumptions imply that

$$
\sum_{n \in \mathbb{N}} \|\boldsymbol{a}_n\| < +\infty, \quad \sum_{n \in \mathbb{N}} \|\boldsymbol{b}_n\| < +\infty, \quad \text{and} \quad \sum_{n \in \mathbb{N}} \|\boldsymbol{c}_n\| < +\infty. \tag{3.15}
$$

Furthermore, it follows from (3.7), (3.8), and (3.14), that (3.4) assumes in $\boldsymbol{\mathcal{K}}$ the form of the error-tolerant forward-backward-forward algorithm

$$
(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} \boldsymbol{y}_n = \boldsymbol{x}_n - \gamma_n(\boldsymbol{Q}\boldsymbol{x}_n + \boldsymbol{a}_n) \\ \boldsymbol{p}_n = J_{\gamma_n \boldsymbol{M}}\, \boldsymbol{y}_n + \boldsymbol{b}_n \\ \boldsymbol{q}_n = \boldsymbol{p}_n - \gamma_n(\boldsymbol{Q}\boldsymbol{p}_n + \boldsymbol{c}_n) \\ \boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \boldsymbol{y}_n + \boldsymbol{q}_n. \end{array} \right. \tag{3.16}
$$

(i): It follows from (3.11), (3.13), (3.15), (3.16), and [10, Theorem 2.5(i)] that $\sum_{n \in \mathbb{N}} \|\boldsymbol{x}_n - \boldsymbol{p}_n\|^2 < +\infty$.

(ii): It follows from [10, Theorem 2.5(ii)] that there exists $\overline{\boldsymbol{x}} \in \operatorname{zer}(\boldsymbol{M} + \boldsymbol{Q})$ such that

$$
\boldsymbol{x}_n \rightharpoonup \overline{\boldsymbol{x}} \quad \text{and} \quad \boldsymbol{p}_n \rightharpoonup \overline{\boldsymbol{x}}. \tag{3.17}
$$

Let us set

$$\boldsymbol{\overline{x}} = (\overline{x}, \overline{v_1}, \ldots, \overline{v_m}). \tag{3.18}$$

In view of (3.6) and (3.7),

$$\begin{aligned}
\boldsymbol{\overline{x}} \in \operatorname{zer}(\boldsymbol{M} + \boldsymbol{Q}) &\Leftrightarrow \begin{cases} 0 \in -z + A\overline{x} + \sum_{i=1}^m L_i^* \overline{v_i} + C\overline{x} \\ 0 \in r_1 + B_1^{-1}\overline{v_1} + D_1^{-1}\overline{v_1} - L_1\overline{x} \\ \quad\vdots \\ 0 \in r_m + B_m^{-1}\overline{v_m} + D_m^{-1}\overline{v_m} - L_m\overline{x} \end{cases} \\
&\Leftrightarrow \begin{cases} z - \sum_{j=1}^m L_j^* \overline{v_j} \in A\overline{x} + C\overline{x} \\ L_1\overline{x} - r_1 \in (B_1^{-1} + D_1^{-1})\overline{v_1} \\ \qquad\qquad\vdots \\ L_m\overline{x} - r_m \in (B_m^{-1} + D_m^{-1})\overline{v_m} \end{cases} && (3.19) \\
&\Leftrightarrow \begin{cases} z - \sum_{j=1}^m L_j^* \overline{v_j} \in A\overline{x} + C\overline{x} \\ \overline{v_1} \in (B_1 \,\square\, D_1)(L_1\overline{x} - r_1) \\ \qquad\vdots \\ \overline{v_m} \in (B_m \,\square\, D_m)(L_m\overline{x} - r_m) \end{cases} && (3.20) \\
&\Rightarrow \begin{cases} z - \sum_{j=1}^m L_j^* \overline{v_j} \in A\overline{x} + C\overline{x} \\ L_1^*\overline{v_1} \in L_1^*\big((B_1 \,\square\, D_1)(L_1\overline{x} - r_1)\big) \\ \qquad\quad\vdots \\ L_m^*\overline{v_m} \in L_m^*\big((B_m \,\square\, D_m)(L_m\overline{x} - r_m)\big) \end{cases} \\
&\Rightarrow z \in A\overline{x} + \sum_{i=1}^m L_i^*\big((B_i \,\square\, D_i)(L_i\overline{x} - r_i)\big) + C\overline{x} \\
&\Leftrightarrow \overline{x} \text{ solves (1.1).} && (3.21)
\end{aligned}$$

On the other hand, (3.20) means that

$$(\overline{v_1}, \ldots, \overline{v_m}) \ \text{ solves (1.2).} \tag{3.22}$$

(ii)(a): This follows from (3.19).

(ii)(b): We derive from (3.19) that

$$\overline{x} \in (A + C)^{-1}\left(z - \sum_{j=1}^m L_j^*\overline{v_j}\right) \quad \text{and} \quad (\forall i \in \{1, \ldots, m\}) \quad L_i\overline{x} - r_i \in (B_i^{-1} + D_i^{-1})\overline{v_i}. \tag{3.23}$$

Hence,

$$(\forall i \in \{1, \ldots, m\}) \quad \begin{cases} -L_i\overline{x} \in -L_i\big((A^{-1} \,\square\, C^{-1})\big(z - \sum_{j=1}^m L_j^*\overline{v_j}\big)\big) \\ L_i\overline{x} - r_i \in (B_i^{-1} + D_i^{-1})\overline{v_i}. \end{cases} \tag{3.24}$$

Thus,

$$(\forall i \in \{1, \ldots, m\}) \quad -r_i \in -L_i\left((A^{-1} \,\square\, C^{-1})\left(z - \sum_{j=1}^m L_j^*\overline{v_j}\right)\right) + B_i^{-1}\overline{v_i} + D_i^{-1}\overline{v_i}. \tag{3.25}$$

(ii)(c): This follows from (3.17), (3.18), and (3.21).

(ii)(d): This follows from (3.17), (3.18), and (3.22).

(ii)(e): Let us set

$$(\forall n\in\mathbb{N})\quad \begin{cases}\widetilde{y}_{1,n}=x_n-\gamma_n\big(Cx_n+\sum_{j=1}^m L_j^*v_{j,n}\big)\\ \widetilde{p}_{1,n}=J_{\gamma_nA}(\widetilde{y}_{1,n}+\gamma_nz)\end{cases} \tag{3.26}$$

and

$$(\forall i\in\{1,\ldots,m\})(\forall n\in\mathbb{N})\quad \begin{cases}\widetilde{y}_{2,i,n}=v_{i,n}+\gamma_n(L_ix_n-D_i^{-1}v_{i,n})\\ \widetilde{p}_{2,i,n}=J_{\gamma_nB_i^{-1}}(\widetilde{y}_{2,i,n}-\gamma_nr_i).\end{cases} \tag{3.27}$$

Then, in view of (3.3),

$$(\forall n\in\mathbb{N})\quad \|y_{1,n}-\widetilde{y}_{1,n}\|\leq\gamma_n\|a_{1,n}\|\leq\beta^{-1}\|a_{1,n}\| \tag{3.28}$$

and, using the nonexpansiveness of the resolvents [8, Proposition 23.7], we obtain

$$\begin{aligned}(\forall n\in\mathbb{N})\quad \|p_{1,n}-\widetilde{p}_{1,n}\|&\leq\|J_{\gamma_nA}(y_{1,n}+\gamma_nz)+b_{1,n}-J_{\gamma_nA}(\widetilde{y}_{1,n}+\gamma_nz)\|\\ &\leq\|y_{1,n}-\widetilde{y}_{1,n}\|+\|b_{1,n}\|\\ &\leq\beta^{-1}\|a_{1,n}\|+\|b_{1,n}\|.\end{aligned} \tag{3.29}$$

Since the sequences $(a_{1,n})_{n\in\mathbb{N}}$ and $(b_{1,n})_{n\in\mathbb{N}}$ are absolutely summable, it follows that

$$y_{1,n}-\widetilde{y}_{1,n}\to 0\quad\text{and}\quad p_{1,n}-\widetilde{p}_{1,n}\to 0. \tag{3.30}$$

Using the same arguments, we derive from (3.3) and (3.27) that

$$(\forall i\in\{1,\ldots,m\})\quad y_{2,i,n}-\widetilde{y}_{2,i,n}\to 0\quad\text{and}\quad p_{2,i,n}-\widetilde{p}_{2,i,n}\to 0. \tag{3.31}$$

On the other hand, we deduce from (ii)(a) that there exists $u\in\mathcal{H}$ such that

$$u\in A\overline{x}\quad\text{and}\quad z=u+\sum_{j=1}^m L_j^*\overline{v_j}+C\overline{x}, \tag{3.32}$$

and that

$$(\forall i\in\{1,\ldots,m\})\quad L_i\overline{x}-r_i-D_i^{-1}\overline{v_i}\in B_i^{-1}\overline{v_i}. \tag{3.33}$$

In addition, (3.26) yields

$$(\forall n\in\mathbb{N})\quad \gamma_n^{-1}(x_n-\widetilde{p}_{1,n})-Cx_n-\sum_{j=1}^m L_j^*v_{j,n}+z\in A\widetilde{p}_{1,n} \tag{3.34}$$

while (3.27) yields

$$(\forall i\in\{1,\ldots,m\})(\forall n\in\mathbb{N})\quad \gamma_n^{-1}(v_{i,n}-\widetilde{p}_{2,i,n})+L_ix_n-D_i^{-1}v_{i,n}-r_i\in B_i^{-1}\widetilde{p}_{2,i,n}. \tag{3.35}$$

Now let us set

$$(\forall n\in\mathbb{N})\quad \begin{cases}\alpha_{1,n}=\|x_n-\widetilde{p}_{1,n}\|\big(\varepsilon^{-1}\|\widetilde{p}_{1,n}-\overline{x}\|+\mu\|x_n-\overline{x}\|+\sum_{i=1}^m\|L_i\|\,\|v_{i,n}-\overline{v_i}\|\big)\\ \alpha_{2,n}=\sum_{i=1}^m(\varepsilon^{-1}+\nu_i)\|v_{i,n}-\widetilde{p}_{2,i,n}\|\,\|\widetilde{p}_{2,i,n}-\overline{v_i}\|.\end{cases} \tag{3.36}$$

It follows from (i), (ii)(c), (ii)(d), (3.30), and (3.31) that

$$\alpha_{1,n} \to 0 \quad \text{and} \quad \alpha_{2,n} \to 0. \tag{3.37}$$

Using the Cauchy-Schwarz inequality, and the Lipschitzianity and monotonicity of C, we obtain

$$\begin{aligned}
&(\forall n \in \mathbb{N}) \quad \alpha_{1,n} + \left\langle x_n - \overline{x} \,\middle|\, \sum_{i=1}^m L_i^*(\overline{v_i} - v_{i,n}) \right\rangle \\
&\geq \|x_n - \widetilde{p}_{1,n}\| \big(\varepsilon^{-1}\|\widetilde{p}_{1,n} - \overline{x}\| + \|Cx_n - C\overline{x}\|\big) + \left\langle \widetilde{p}_{1,n} - x_n \,\middle|\, \sum_{i=1}^m L_i^*(\overline{v_i} - v_{i,n}) \right\rangle \\
&\quad + \left\langle x_n - \overline{x} \,\middle|\, \sum_{i=1}^m L_i^*(\overline{v_i} - v_{i,n}) \right\rangle \\
&= \|x_n - \widetilde{p}_{1,n}\| \big(\varepsilon^{-1}\|\widetilde{p}_{1,n} - \overline{x}\| + \|Cx_n - C\overline{x}\|\big) + \left\langle \widetilde{p}_{1,n} - \overline{x} \,\middle|\, \sum_{i=1}^m L_i^*(\overline{v_i} - v_{i,n}) \right\rangle \\
&\geq \left\langle \widetilde{p}_{1,n} - \overline{x} \,\middle|\, \gamma_n^{-1}(x_n - \widetilde{p}_{1,n}) + \sum_{i=1}^m L_i^*(\overline{v_i} - v_{i,n}) \right\rangle + \langle \widetilde{p}_{1,n} - x_n \mid C\overline{x} - Cx_n \rangle \\
&= \left\langle \widetilde{p}_{1,n} - \overline{x} \,\middle|\, \gamma_n^{-1}(x_n - \widetilde{p}_{1,n}) - \sum_{i=1}^m L_i^* v_{i,n} - Cx_n + \sum_{i=1}^m L_i^*\overline{v_i} + C\overline{x} \right\rangle + \langle \overline{x} - x_n \mid C\overline{x} - Cx_n \rangle \\
&= \left\langle \widetilde{p}_{1,n} - \overline{x} \,\middle|\, \gamma_n^{-1}(x_n - \widetilde{p}_{1,n}) - \sum_{i=1}^m L_i^* v_{i,n} - Cx_n + z - u \right\rangle + \langle \overline{x} - x_n \mid C\overline{x} - Cx_n \rangle \\
&\geq \left\langle \widetilde{p}_{1,n} - \overline{x} \,\middle|\, \left(\gamma_n^{-1}(x_n - \widetilde{p}_{1,n}) - \sum_{i=1}^m L_i^* v_{i,n} - Cx_n + z\right) - u \right\rangle.
\end{aligned} \tag{3.38}$$

Now suppose that A is uniformly monotone at $\overline{x}$. Then, in view of (3.32), (3.34), and (3.38), there exists an increasing function $\phi_A\colon [0,+\infty[\to [0,+\infty]$ that vanishes only at 0 such that

$$(\forall n \in \mathbb{N}) \quad \alpha_{1,n} + \left\langle x_n - \overline{x} \,\middle|\, \sum_{i=1}^m L_i^*(\overline{v_i} - v_{i,n}) \right\rangle \geq \phi_A(\|\widetilde{p}_{1,n} - \overline{x}\|). \tag{3.39}$$

On the other hand, it follows from (3.36), the Lipschitzianity of the operators $(D_i^{-1})_{1\leq i\leq m}$, (3.33), (3.35), and the monotonicity of the operators $(B_i^{-1})_{1\leq i\leq m}$ and $(D_i^{-1})_{1\leq i\leq m}$ that

$$\begin{aligned}
&(\forall n \in \mathbb{N}) \quad \alpha_{2,n} + \left\langle x_n - \overline{x} \,\middle|\, \sum_{i=1}^m L_i^*(\widetilde{p}_{2,i,n} - \overline{v_i}) \right\rangle \\
&\qquad \geq \sum_{i=1}^m \big\langle \gamma_n^{-1}(v_{i,n} - \widetilde{p}_{2,i,n}) - D_i^{-1}v_{i,n} + D_i^{-1}\widetilde{p}_{2,i,n} + L_i(x_n - \overline{x}) \mid \widetilde{p}_{2,i,n} - \overline{v_i} \big\rangle \\
&\qquad = \sum_{i=1}^m \Big(\big\langle \gamma_n^{-1}(v_{i,n} - \widetilde{p}_{2,i,n}) + L_i x_n - D_i^{-1}v_{i,n} - r_i - (L_i\overline{x} - r_i - D_i^{-1}\overline{v_i}) \mid \widetilde{p}_{2,i,n} - \overline{v_i} \big\rangle \\
&\qquad\qquad + \big\langle D_i^{-1}\widetilde{p}_{2,i,n} - D_i^{-1}\overline{v_i} \mid \widetilde{p}_{2,i,n} - \overline{v_i} \big\rangle \Big) \\
&\qquad \geq 0.
\end{aligned} \tag{3.40}$$

Adding (3.39) and (3.40) yields

$$(\forall n\in\mathbb{N})\quad \alpha_{1,n}+\alpha_{2,n}+\Big\langle x_n-\overline{x}\;\Big|\;\sum_{i=1}^m L_i^*(\widetilde{p}_{2,i,n}-v_{i,n})\Big\rangle\geq \phi_A(\|\widetilde{p}_{1,n}-\overline{x}\|). \tag{3.41}$$

It then follows from (3.37), (ii)(c), (i), (3.31), and [8, Lemma 2.41(iii)] that $\phi_A(\|\widetilde{p}_{1,n}-\overline{x}\|)\to 0$ and, in turn, that $\widetilde{p}_{1,n}\to\overline{x}$. Hence, in view of (i) and (3.30), we get $x_n\to\overline{x}$ and $p_{1,n}\to\overline{x}$. Likewise, if C is uniformly monotone at $\overline{x}$, there exists an increasing function $\phi_C\colon\ [0,+\infty[\,\to[0,+\infty]$ that vanishes only at 0 such that

$$(\forall n\in\mathbb{N})\quad \alpha_{1,n}+\alpha_{2,n}+\Big\langle x_n-\overline{x}\;\Big|\;\sum_{i=1}^m L_i^*(\widetilde{p}_{2,i,n}-v_{i,n})\Big\rangle\geq \phi_C(\|x_n-\overline{x}\|), \tag{3.42}$$

and we reach the same conclusion.

(ii)(f): Suppose that B_i^{-1} is uniformly monotone at $\overline{v_i}$ for some $i\in\{1,\ldots,m\}$. Then, proceeding as in (3.40), there exists an increasing function $\phi_{B_i}\colon\ [0,+\infty[\,\to[0,+\infty]$ that vanishes only at 0 such that

$$\begin{aligned}(\forall n\in\mathbb{N})\quad &\alpha_{2,n}+\Big\langle x_n-\overline{x}\;\Big|\;\sum_{j=1}^m L_j^*(\widetilde{p}_{2,j,n}-\overline{v_j})\Big\rangle\\
&\geq\sum_{j=1}^m\Big(\Big\langle \gamma_n^{-1}(v_{j,n}-\widetilde{p}_{2,j,n})+L_jx_n-D_j^{-1}v_{j,n}-r_j-(L_j\overline{x}-r_j-D_j^{-1}\overline{v_j})\;|\;\widetilde{p}_{2,j,n}-\overline{v_j}\Big\rangle\\
&\qquad\qquad+\Big\langle D_j^{-1}\widetilde{p}_{2,j,n}-D_j^{-1}\overline{v_j}\;|\;\widetilde{p}_{2,j,n}-\overline{v_j}\Big\rangle\Big)\\
&\geq\sum_{j=1}^m\Big\langle \gamma_n^{-1}(v_{j,n}-\widetilde{p}_{2,j,n})+L_jx_n-D_j^{-1}v_{j,n}-r_j-(L_j\overline{x}-r_j-D_j^{-1}\overline{v_j})\;|\;\widetilde{p}_{2,j,n}-\overline{v_j}\Big\rangle\\
&\geq\big\langle \gamma_n^{-1}(v_{i,n}-\widetilde{p}_{2,i,n})+L_ix_n-D_i^{-1}v_{i,n}-r_i-(L_i\overline{x}-r_i-D_i^{-1}\overline{v_i})\;|\;\widetilde{p}_{2,i,n}-\overline{v_i}\big\rangle\\
&\geq\phi_{B_i}(\|\widetilde{p}_{2,i,n}-\overline{v_i}\|).\end{aligned} \tag{3.43}$$

On the other hand, according to (3.38),

$$(\forall n\in\mathbb{N})\quad \alpha_{1,n}+\Big\langle x_n-\overline{x}\;\Big|\;\sum_{j=1}^m L_j^*(\overline{v_j}-v_{j,n})\Big\rangle\geq 0. \tag{3.44}$$

Hence,

$$(\forall n\in\mathbb{N})\quad \alpha_{1,n}+\alpha_{2,n}+\Big\langle x_n-\overline{x}\;\Big|\;\sum_{j=1}^m L_j^*(\widetilde{p}_{2,j,n}-v_{j,n})\Big\rangle\geq \phi_{B_i}(\|\widetilde{p}_{2,i,n}-\overline{v_i}\|). \tag{3.45}$$

By proceeding as previously, we infer that $\widetilde{p}_{2,i,n}\to\overline{v_i}$ and hence, via (3.31) and (i), that $p_{2,i,n}\to\overline{v_i}$ and $v_{i,n}\to\overline{v_i}$. If D_i^{-1} is uniformly monotone at $\overline{v_i}$, the same arguments lead to these conclusions. ☐

In the following remarks, we comment on the structure of the proposed algorithm and its relation to existing work.

Remark 3.2 Here are some observations regarding the structure of algorithm (3.3).

(i) The algorithm achieves full splitting in that each of the operators appearing in Problem 1.1 is used separately.

(ii) The algorithm uses explicit steps for the single-valued operators and implicit steps for the set-valued operators. Since explicit steps are typically much easier to implement than implicit steps, the algorithm therefore exploits efficiently the properties of the operators.

(iii) The sequences $(a_{1,n})_{n\in\mathbb{N}}$, $(b_{1,n})_{n\in\mathbb{N}}$, and $(c_{1,n})_{n\in\mathbb{N}}$, and, for every $i\in\{1,\ldots,m\}$, $(a_{2,i,n})_{n\in\mathbb{N}}$, $(b_{2,i,n})_{n\in\mathbb{N}}$, and $(c_{2,i,n})_{n\in\mathbb{N}}$ relax the requirement for exact evaluations of the operators over the course of the iterations.

(iv) Most of the elementary steps in (3.3) can be executed in parallel.

(v) The update of the variable $p_{2,i,n}$ can also be carried out using the resolvent of B_i since [8, Proposition 23.18] $J_{\gamma_n B_i^{-1}}(y_{2,i,n}-\gamma_n r_i)=y_{2,i,n}-\gamma_n r_i-\gamma_n J_{\gamma_n^{-1}B_i}(\gamma_n^{-1}(y_{2,i,n}-\gamma_n r_i))$.

Remark 3.3 Some noteworthy connections between Theorem 3.1 and existing work are the following.

(i) Unlike most splitting methods, the proposed algorithm is designed to solve explicitly a dual problem.

(ii) In the special case when $m=1$ and D_1 is as in (1.3), the primal problem (1.1) reduces to (we drop the subscript '1' for brevity)

$$\text{find}\quad \overline{x}\in\mathcal{H}\ \text{ such that }\quad z\in A\overline{x}+L^*\big(B(L\overline{x}-r)\big)+C\overline{x}, \tag{3.46}$$

the dual problem (1.2) reduces to

$$\text{find}\quad \overline{v}\in\mathcal{G}\ \text{ such that }\quad -r\in -L\big((A+C)^{-1}(z-L^*\overline{v})\big)+B^{-1}\overline{v}, \tag{3.47}$$

and the algorithm is governed by the iteration

$$\left\lfloor\begin{array}{l} y_{1,n}=x_n-\gamma_n\big(Cx_n+L^*v_n+a_{1,n}\big)\\ y_{2,n}=v_n+\gamma_n(Lx_n+a_{2,n})\\ p_{1,n}=J_{\gamma_n A}(y_{1,n}+\gamma_n z)+b_{1,n}\\ p_{2,n}=J_{\gamma_n B^{-1}}(y_{2,n}-\gamma_n r)+b_{2,n}\\ q_{1,n}=p_{1,n}-\gamma_n\big(Cp_{1,n}+L^*p_{2,n}+c_{1,n}\big)\\ q_{2,n}=p_{2,n}+\gamma_n(Lp_{1,n}+c_{2,n})\\ x_{n+1}=x_n-y_{1,n}+q_{1,n}\\ v_{n+1}=v_n-y_{2,n}+q_{2,n}. \end{array}\right. \tag{3.48}$$

On the one hand, if $C\colon x\mapsto 0$, we recover the primal-dual setting of [10] and its algorithm ([10, Eq. (3.1)]). On the other hand, if $L\colon x\mapsto 0$, $B\colon y\mapsto\{0\}$, $z=0$, and $r=0$, (3.46) yields the problem studied in [45], and (3.48) without error terms and dual variables yields a primal algorithm proposed in that paper, namely

$$\left\lfloor \begin{array}{l} y_{1,n} = x_n - \gamma_n C x_n \\ p_{1,n} = J_{\gamma_n A} y_{1,n} \\ q_{1,n} = p_{1,n} - \gamma_n C p_{1,n} \\ x_{n+1} = x_n - y_{1,n} + q_{1,n}. \end{array} \right. \tag{3.49}$$

Let us note that, even when we specialize (3.46) to $\mathcal{G} = \mathcal{H}$ and $L = \mathrm{Id}$, there does not appear to exist an alternative algorithm that splits A, B, and C and uses explicit steps on the Lipschitzian operator C.

(iii) When $C\colon x \mapsto 0$ and, for every $i \in \{1,\ldots,m\}$, $D_i^{-1}\colon y \mapsto \{0\}$, we recover the primal-dual setting of [10, Theorem 3.8]. However, the algorithm we obtain is different from that proposed in that paper, and novel.

(iv) In general, the weak convergence results of Theorem 3.1(ii) cannot be improved to strong convergence without additional hypotheses on the operators such as those described in (ii)(e) and (ii)(f). Indeed, in the special case when (1.1) reduces to the problem of finding a zero of A, the primal component of (3.3) reduces to the proximal point algorithm, namely (set $C\colon x \mapsto 0$ in (3.49))

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = J_{\gamma_n A} x_n, \tag{3.50}$$

which is known to converge weakly but not strongly [7, 25].

4 Minimization problems

The proposed monotone operator splitting algorithm can be applied to a broader class of problems than that within the reach of existing splitting methods. It has therefore potential applications in the areas in which these methods have been used, e.g., partial differential equations [21, 30], mechanics [22, 31], variational inequalities [8, 18, 44], game theory [11], traffic theory [20], and evolution equations [3]. In this section, we focus on the application of the results of Section 3 to convex minimization problems.

Problem 4.1 Let $\mathcal{H}$ be a real Hilbert space, let $z \in \mathcal{H}$, let m be a strictly positive integer, let $f \in \Gamma_0(\mathcal{H})$, and let $h\colon \mathcal{H} \to \mathbb{R}$ be convex and differentiable with a μ-Lipschitzian gradient for some $\mu \in \left]0,+\infty\right[$. For every $i \in \{1,\ldots,m\}$, let $\mathcal{G}_i$ be a real Hilbert space, let $r_i \in \mathcal{G}_i$, let $g_i \in \Gamma_0(\mathcal{G}_i)$, let $\ell_i \in \Gamma_0(\mathcal{G}_i)$ be $1/\nu_i$-strongly convex, for some $\nu_i \in \left]0,+\infty\right[$, and suppose that $L_i\colon \mathcal{H} \to \mathcal{G}_i$ is a nonzero bounded linear operator. Consider the problem

$$\underset{x\in\mathcal{H}}{\text{minimize}} \;\; f(x) + \sum_{i=1}^{m} (g_i \,\square\, \ell_i)(L_i x - r_i) + h(x) - \langle x \mid z\rangle, \tag{4.1}$$

and the dual problem

$$\underset{v_1\in\mathcal{G}_1,\ldots,v_m\in\mathcal{G}_m}{\text{minimize}} \;\; \big(f^* \,\square\, h^*\big)\bigg(z - \sum_{i=1}^{m} L_i^* v_i\bigg) + \sum_{i=1}^{m} \big(g_i^*(v_i) + \ell_i^*(v_i) + \langle v_i \mid r_i\rangle\big). \tag{4.2}$$

The following result is an offspring of Theorem 3.1.

Theorem 4.2 *In Problem 4.1, suppose that*

$$z \in \operatorname{ran}\Big(\partial f + \sum_{i=1}^{m} L_i^*(\partial g_i \,\square\, \partial \ell_i)(L_i \cdot - r_i) + \nabla h\Big). \tag{4.3}$$

Let $(a_{1,n})_{n\in\mathbb{N}}$, $(b_{1,n})_{n\in\mathbb{N}}$, and $(c_{1,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{H}$ and, for every $i \in \{1,\ldots,m\}$, let $(a_{2,i,n})_{n\in\mathbb{N}}$, $(b_{2,i,n})_{n\in\mathbb{N}}$, and $(c_{2,i,n})_{n\in\mathbb{N}}$ be absolutely summable sequences in $\mathcal{G}_i$. Furthermore, set

$$\beta = \max\{\mu, \nu_1, \ldots, \nu_m\} + \sqrt{\sum_{i=1}^{m} \|L_i\|^2}, \tag{4.4}$$

let $x_0 \in \mathcal{H}$, let $(v_{1,0},\ldots,v_{m,0}) \in \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_m$, let $\varepsilon \in \left]0, 1/(\beta+1)\right[$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (1-\varepsilon)/\beta]$, and set

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_{1,n} = x_n - \gamma_n\big(\nabla h(x_n) + \sum_{i=1}^{m} L_i^* v_{i,n} + a_{1,n}\big) \\ p_{1,n} = \operatorname{prox}_{\gamma_n f}(y_{1,n} + \gamma_n z) + b_{1,n} \\ \text{For } i = 1,\ldots,m \\ \left\lfloor \begin{array}{l} y_{2,i,n} = v_{i,n} + \gamma_n(L_i x_n - \nabla \ell_i^*(v_{i,n}) + a_{2,i,n}) \\ p_{2,i,n} = \operatorname{prox}_{\gamma_n g_i^*}(y_{2,i,n} - \gamma_n r_i) + b_{2,i,n} \\ q_{2,i,n} = p_{2,i,n} + \gamma_n\big(L_i p_{1,n} - \nabla \ell_i^*(p_{2,i,n}) + c_{2,i,n}\big) \\ v_{i,n+1} = v_{i,n} - y_{2,i,n} + q_{2,i,n}. \end{array} \right. \\ q_{1,n} = p_{1,n} - \gamma_n\big(\nabla h(p_{1,n}) + \sum_{i=1}^{m} L_i^* p_{2,i,n} + c_{1,n}\big) \\ x_{n+1} = x_n - y_{1,n} + q_{1,n}. \end{array} \right. \tag{4.5}$$

Then the following hold.

(i) *$\sum_{n\in\mathbb{N}} \|x_n - p_{1,n}\|^2 < +\infty$ and $(\forall i \in \{1,\ldots,m\})$ $\sum_{n\in\mathbb{N}} \|v_{i,n} - p_{2,i,n}\|^2 < +\infty$.*

(ii) *There exist a solution $\overline{x}$ to (4.1) and a solution $(\overline{v_1},\ldots,\overline{v_m})$ to (4.2) such that the following hold.*

   (a) *$z - \sum_{j=1}^{m} L_j^* \overline{v_j} \in \partial f(\overline{x}) + \nabla h(\overline{x})$ and $(\forall i \in \{1,\ldots,m\})$ $L_i \overline{x} - r_i \in \partial g_i^*(\overline{v_i}) + \nabla \ell_i^*(\overline{v_i})$.*

   (b) *$x_n \rightharpoonup \overline{x}$ and $p_{1,n} \rightharpoonup \overline{x}$.*

   (c) *$(\forall i \in \{1,\ldots,m\})$ $v_{i,n} \rightharpoonup \overline{v_i}$ and $p_{2,i,n} \rightharpoonup \overline{v_i}$.*

   (d) *Suppose that f or h is uniformly convex at $\overline{x}$. Then $x_n \to \overline{x}$ and $p_{1,n} \to \overline{x}$.*

   (e) *Suppose that, for some $i \in \{1,\ldots,m\}$, g_i^* or ℓ_i^* is uniformly convex at $\overline{v_i}$. Then $v_{i,n} \to \overline{v_i}$ and $p_{2,i,n} \to \overline{v_i}$.*

Proof. Let us first establish a connection between Problem 4.1 and Problem 1.1. To this end, let us define

$$A = \partial f, \;\; C = \nabla h, \;\; \text{and} \quad (\forall i \in \{1,\ldots,m\}) \quad B_i = \partial g_i \;\text{ and }\; D_i = \partial \ell_i. \tag{4.6}$$

It is clear that (4.3) yields (3.1) and, using (2.7) and (2.8), that (4.5) yields (3.3). Moreover, it follows from [8, Theorem 20.40] that the operators A and $(B_i)_{1\leq i\leq m}$ are maximally monotone, and from [8, Proposition 17.10] that C is monotone. On the other hand, for every $i \in \{1,\ldots,m\}$, it follows from the $1/\nu_i$-strong convexity of ℓ_i and [8, Corollary 13.33 and Theorem 18.15] that ℓ_i^* is Fréchet differentiable on $\mathcal{G}_i$ with a ν_i-Lipschitzian gradient, and from (2.7) that $D_i^{-1} = \nabla \ell_i^*$. Altogether, we can apply Theorem 3.1 to obtain the existence of a point $\overline{x} \in \mathcal{H}$ such that

$$z \in \partial f(\overline{x}) + \sum_{i=1}^m L_i^*\big((\partial g_i \,\Box\, \partial \ell_i)(L_i\overline{x} - r_i)\big) + \nabla h(\overline{x}), \tag{4.7}$$

and of an m-tuple $(\overline{v_1},\ldots,\overline{v_m}) \in \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_m$ such that

$$(\exists\, x \in \mathcal{H}) \quad \begin{cases} z - \sum_{j=1}^m L_j^*\overline{v_j} \in \partial f(x) + \nabla h(x) \\ (\forall i \in \{1,\ldots,m\}) \quad \overline{v_i} \in (\partial g_i \,\Box\, \partial \ell_i)(L_i x - r_i), \end{cases} \tag{4.8}$$

that satisfy (i) and (ii). It remains to show that $\overline{x}$ solve (4.1) and $(\overline{v_1},\ldots,\overline{v_m})$ solves (4.2). We first observe that since, for every $i \in \{1,\ldots,m\}$, $\operatorname{dom} \ell_i^* = \mathcal{G}_i$ [8, Proposition 24.27] yields

$$(\forall i \in \{1,\ldots,m\}) \quad \partial g_i \,\Box\, \partial \ell_i = \partial(g_i \,\Box\, \ell_i). \tag{4.9}$$

On the other hand, it follows from [8, Corollary 16.38(iii) and Proposition 17.26(i)] that

$$\partial\big(f + h - \langle \cdot \mid z\rangle\big) = \partial f + \nabla h - z. \tag{4.10}$$

As a result, we derive from (4.7) that

$$0 \in \partial\big(f + h - \langle \cdot \mid z\rangle\big)(\overline{x}) + \sum_{i=1}^m L_i^*\big(\partial(g_i \,\Box\, \ell_i)(L_i\overline{x} - r_i)\big). \tag{4.11}$$

However, since (4.3) and [8, Proposition 16.5(ii)] imply that

$$\partial\big(f + h - \langle \cdot \mid z\rangle\big) + \sum_{i=1}^m L_i^*\big(\partial(g_i \,\Box\, \ell_i)\big)(L_i \cdot - r_i) \subset \partial\left(f + h - \langle \cdot \mid z\rangle + \sum_{i=1}^m (g_i \,\Box\, \ell_i) \circ (L_i \cdot - r_i)\right), \tag{4.12}$$

it follows from (4.11) that

$$0 \in \partial\left(f + h - \langle \cdot \mid z\rangle + \sum_{i=1}^m (g_i \,\Box\, \ell_i) \circ (L_i \cdot - r_i)\right)(\overline{x}). \tag{4.13}$$

Thus, Fermat's rule [8, Theorem 16.2] asserts that $\overline{x}$ solves (4.1). Finally, to show that $(\overline{v_1},\ldots,\overline{v_m})$ solves (4.2), we first note that it follows from (4.10), (2.7), and [8, Proposition 15.2] that

$$\big(\partial f + \nabla h\big)^{-1} = \big(\partial(f+h)\big)^{-1} = \partial(f+h)^* = \partial\big(f^* \,\Box\, h^*\big). \tag{4.14}$$

Likewise, (4.9) and [8, Proposition 13.21(i)] yield

$$(\forall i \in \{1,\ldots,m\}) \quad \big(\partial g_i \,\Box\, \partial \ell_i\big)^{-1} = \partial\big(g_i \,\Box\, \ell_i\big)^* = \partial\big(g_i^* + \ell_i^*\big). \tag{4.15}$$

Hence, combining (4.8), (4.14), and (4.15), we obtain

$$(\exists\, x \in \mathcal{H}) \quad \begin{cases} x \in \partial(f^* \,\Box\, h^*)\big(z - \sum_{j=1}^m L_j^*\overline{v_j}\big) \\ (\forall i \in \{1,\ldots,m\}) \quad L_i x - r_i \in \partial\big(g_i^* + \ell_i^*\big)(\overline{v_i}) \end{cases} \tag{4.16}$$

and therefore

$$(\exists\, x \in \mathcal{H}) \quad \begin{cases} -(L_i x)_{1\le i\le m} \in -\Big(\bigtimes_{i=1}^m L_i\Big)\Big(\partial(f^* \,\Box\, h^*)\big(z - \sum_{j=1}^m L_j^* \overline{v_j}\big)\Big) \\ (L_i x)_{1\le i\le m} \in \bigtimes_{i=1}^m \partial\big(g_i^* + \ell_i^* + \langle \cdot \mid r_i\rangle\big)(\overline{v_i}). \end{cases} \tag{4.17}$$

Hence, using [8, Propositions 16.5(ii) and 16.8] and the notation (2.1),

$$\begin{aligned}(0,\ldots,0) &\in -\Big(\bigtimes_{i=1}^m L_i\Big)\Bigg(\partial(f^* \,\Box\, h^*)\Bigg(z - \sum_{j=1}^m L_j^* \overline{v_j}\Bigg)\Bigg) + \bigtimes_{i=1}^m \partial\big(g_i^* + \ell_i^* + \langle \cdot \mid r_i\rangle\big)(\overline{v_i}) \\ &= -\Bigg(\bigoplus_{i=1}^m L_i^*\Bigg)^*\Bigg(\partial(f^* \,\Box\, h^*)\Bigg(z - \Bigg(\bigoplus_{i=1}^m L_i^*\Bigg)(\overline{v_1},\ldots,\overline{v_m})\Bigg)\Bigg) \\ &\quad + \partial\Bigg(\bigoplus_{i=1}^m \big(g_i^* + \ell_i^* + \langle \cdot \mid r_i\rangle\big)\Bigg)(\overline{v_1},\ldots,\overline{v_m}) \\ &\subset \partial\Bigg((f^* \,\Box\, h^*)\Bigg(z - \Bigg(\bigoplus_{i=1}^m L_i^*\Bigg)\cdot\Bigg) + \bigoplus_{i=1}^m \big(g_i^* + \ell_i^* + \langle \cdot \mid r_i\rangle\big)\Bigg)(\overline{v_1},\ldots,\overline{v_m}). \end{aligned} \tag{4.18}$$

In other words, by Fermat's rule, $(\overline{v_1},\ldots,\overline{v_m})$ solves (4.2). Finally, the strong convergence claims in (ii)(d) and (ii)(e) follow from Theorem 3.1(ii)(e)&(ii)(f) since the uniform convexity of a function $\varphi \in \Gamma_0(\mathcal{H})$ at a point of the domain of $\partial\varphi$ implies the uniform monotonicity of $\partial\varphi$ at that point [46, Section 3.4]. ☐

In the following proposition we give conditions under which (4.3) is satisfied.

Proposition 4.3 *Suppose that* (4.1) *has at least one solution and set*

$$S = \big\{(L_i x - y_i)_{1\le i\le m} \;\big|\; x \in \operatorname{dom} f \text{ and } (\forall i \in \{1,\ldots,m\})\;\; y_i \in \operatorname{dom} g_i + \operatorname{dom} \ell_i\big\}. \tag{4.19}$$

Then (4.3) *is satisfied if one of the following holds.*

(i) $(r_1,\ldots,r_m) \in \operatorname{sri} S$.

(ii) *For every* $i \in \{1,\ldots,m\}$*,* g_i *or* ℓ_i *is real-valued.*

(iii) $\mathcal{H}$ *and* $(\mathcal{G}_i)_{1\le i\le m}$ *are finite-dimensional, and there exists* $x \in \operatorname{ri}\operatorname{dom} f$ *such that*

$$(\forall i \in \{1,\ldots,m\}) \quad L_i x - r_i \in \operatorname{ri}\operatorname{dom} g_i + \operatorname{ri}\operatorname{dom} \ell_i. \tag{4.20}$$

Proof. It follows from (4.19) and [8, Proposition 12.6(ii)] that

$$\begin{aligned} S &= \big\{(L_i x - y_i)_{1\le i\le m} \;\big|\; x \in \operatorname{dom} f \text{ and } (\forall i \in \{1,\ldots,m\})\;\; y_i \in \operatorname{dom}(g_i \,\Box\, \ell_i)\big\} \\ &= \Bigg\{(L_i x - y_i)_{1\le i\le m} \;\Bigg|\; x \in \operatorname{dom}(f + h - \langle \cdot \mid z\rangle) \text{ and } (y_i)_{1\le i\le m} \in \bigtimes_{i=1}^m \operatorname{dom}(g_i \,\Box\, \ell_i)\Bigg\} \\ &= \Bigg(\bigtimes_{i=1}^m L_i\Bigg)\Big(\operatorname{dom}\big(f + h - \langle \cdot \mid z\rangle\big)\Big) - \operatorname{dom}\bigoplus_{i=1}^m (g_i \,\Box\, \ell_i). \end{aligned} \tag{4.21}$$

(i): In view of (4.21),

$$
\begin{aligned}
&(r_1,\ldots,r_m)\in \operatorname{sri} S\\
&\qquad\Rightarrow\quad (0,\ldots,0)\in \operatorname{sri}\bigg(\bigg(\bigtimes_{i=1}^m L_i\bigg)\Big(\operatorname{dom}\big(f+h-\langle\cdot\mid z\rangle\big)\Big)-\operatorname{dom}\bigoplus_{i=1}^m (g_i\,\Box\,\ell_i)(\cdot-r_i)\bigg). \qquad (4.22)
\end{aligned}
$$

Hence, since $(\bigtimes_{i=1}^m L_i)^* = \bigoplus_{i=1}^m L_i^*$, it follows from (4.9), (4.10), and [8, Theorem 16.37(i)] that

$$
\begin{aligned}
\partial f+\sum_{i=1}^m L_i^*(\partial g_i\,\Box\,\partial\ell_i)(L_i\cdot-r_i)+\nabla h-z &= \partial\big(f+h-\langle\cdot\mid z\rangle\big)+\sum_{i=1}^m L_i^*\big(\partial(g_i\,\Box\,\ell_i)\big)(L_i\cdot-r_i)\\
&=\partial\bigg(f+h-\langle\cdot\mid z\rangle+\sum_{i=1}^m (g_i\,\Box\,\ell_i)\circ(L_i\cdot-r_i)\bigg). \qquad (4.23)
\end{aligned}
$$

Since (4.1) has at least one solution it follows from Fermat's rule that 0 is in the range of the right-hand side of (4.23), which shows that (4.3) holds.

(ii)⇒(i): We have $(\forall i\in\{1,\ldots,m\})\ \operatorname{dom} g_i+\operatorname{dom}\ell_i=\mathcal{G}_i$. Therefore (4.19) yields $S=\bigoplus_{i=1}^m \mathcal{G}_i$.

(iii)⇒(i): We have $\operatorname{sri} S=\operatorname{ri} S$. However, it follows from (4.21) and [8, Corollary 6.15] that

$$
\begin{aligned}
\operatorname{ri} S &= \operatorname{ri}\bigg(\bigg(\bigtimes_{i=1}^m L_i\bigg)\Big(\operatorname{dom}\big(f+h-\langle\cdot\mid z\rangle\big)\Big)-\operatorname{dom}\bigoplus_{i=1}^m (g_i\,\Box\,\ell_i)\bigg)\\
&=\operatorname{ri}\bigg(\bigtimes_{i=1}^m L_i\bigg)\big(\operatorname{dom} f\big)-\operatorname{ri}\operatorname{dom}\bigoplus_{i=1}^m (g_i\,\Box\,\ell_i)\\
&=\bigg(\bigtimes_{i=1}^m L_i\bigg)\big(\operatorname{ri}\operatorname{dom} f\big)-\bigtimes_{i=1}^m \operatorname{ri}\operatorname{dom}(g_i\,\Box\,\ell_i)\\
&=\bigg(\bigtimes_{i=1}^m L_i\bigg)\big(\operatorname{ri}\operatorname{dom} f\big)-\bigtimes_{i=1}^m \big(\operatorname{ri}\operatorname{dom} g_i+\operatorname{ri}\operatorname{dom}\ell_i\big). \qquad (4.24)
\end{aligned}
$$

Hence $(r_1,\ldots,r_m)\in\operatorname{sri} S \Leftrightarrow (\exists\, x\in\operatorname{ri}\operatorname{dom} f)(\forall i\in\{1,\ldots,m\})\ L_ix-r_i\in\operatorname{ri}\operatorname{dom} g_i+\operatorname{ri}\operatorname{dom}\ell_i$. ☐

Remark 4.4 In Problem 4.1, if each function ℓ_i is the indicator function of $\{0\}$, then (4.1) reduces to

$$
\underset{x\in\mathcal{H}}{\text{minimize}}\ \ f(x)+\sum_{i=1}^m g_i(L_ix-r_i)+h(x)-\langle x\mid z\rangle. \qquad (4.25)
$$

Even in this special case, the algorithm resulting from (4.5) is new. This observation remains valid if we further assume that $h\colon x\mapsto 0$.

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