The Mathematical Foundations of Projection Techniques and Their Application to Set Theoretic Estimation

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1. Introduction:

A large body of digital signal processing problems can be abstracted into a single one which is to select from a prescribed class S an object y which is the closest, in some sense, to a fixed element x. In the mathematical literature, such a problem is often referred to as a best approximation or a nearest projection problem. In signal processing, projections are frequently used. For instance the notion of projections onto subspaces is central to any least-squares prediction procedure: in adaptive signal processing, the update of the filters are based on projections onto subspaces spanned by data points. Projections can also be used in digital filtering: suppose that x represents an autoregressive filter that has been computed and that turns out to be unstable. The problem is to stabilize this filter, i.e. find a best approximation y to x in the class S of stable filters.

The projection problem can be stated as follows: given a point x and a set S in a metric space (Ξ, d), determine a point y of S of minimum distance from x. There are three basic questions which arise in connection with this problem:
- Does such a y exist?
- Is such a y unique?
- What characterizes such a y?

Although projection techniques have come to play an increasing rôle in digital signal processing, their mathematical properties remain widely unknown and the purpose of this report is to give a general account of this subject. In particular the three questions mentioned above will be answered in detail and new results will be established. The application of projection methods to set theoretic estimation will also be addressed. Most of the materials presented herein will be purely mathematical and will be introduced in a unified and systematic manner, which should promote a better understanding. Although no mention will be made of any particular signal processing problem, the reader should constantly bear in mind the geometrical interpretation of the results and should make the natural connection with potential applications.

The report is organized as follows: Section 2 provides results which are needed in the remainder; Section 3 addresses the basic properties of projections in metric spaces; Section 4 is devoted to the study of projections in the Cartesian space $\mathbb{R}^k$; Section 5 focuses on the application to set theoretic estimation. The reader who is not familiar with some of the notations or notions used in this report is referred to the appendix, where the necessary background is provided.

2. Preliminary results:

2.1. Remarks on closedness, compactness, and convexity:

Since the notions of closedness, compactness, and convexity are central to projection methods we give some useful criteria under which one of these properties can easily be identified (one may also refer to the appendix for defining properties).

Proposition 1: A nonempty subset S of a metric space Ξ is closed if any of the following conditions holds:

(i) S is a finite reunion or an arbitrary intersection of closed sets.

(ii) There exists a real-valued continuous functions f defined on Ξ and a closed subset C of $\mathbb{R}$ such that: $S = \{x \in \Xi \mid f(x) \in C\}$. In particular, C is of the form $\{\beta\}$, $]-\infty,\beta]$, $[\beta,+\infty[$, or $[\alpha,\beta]$.

Proof: (i) see Appendix; (ii) is proven by writing S as $f^{-1}(C)$ where C is closed in $\mathbb{R}$ and f continuous.

We shall often use the following theorem in connection with compactness:

Theorem 1 [Schw80]: In a metric space Ξ every compact set is closed and bounded. Conversely, every closed and bounded set of Ξ is compact if and only if the closed balls of Ξ are compact (if Ξ is a normed vector space, this holds if and only if its dimension is finite).

Proposition 2 [Schw80]: A nonempty subset S of a metric space Ξ is compact if any of the following conditions holds:

(i) All the closed balls in Ξ are compact and S is closed and bounded.

(ii) S is a closed subset of a compact set.

(iii) S is a finite reunion or an arbitrary intersection of compact sets.

(iv) There exists a continuous function f of a real variable mapping into Ξ and a compact subset K of $\mathbb{R}$ such that: $S = f(K)$.

Proposition 3 [Bour81]: A nonempty subset S of a vector space Ξ is convex if any of the following conditions holds:

(i) S is an arbitrary intersection of convex sets.

(ii) S is closed and: $(\forall (x,y) \in S^2) \quad \frac{1}{2}(x+y) \in S$.

(iii) There exists a real-valued convex function f defined on Ξ and a real number β such that S can be written either as $f^{-1}(]-\infty,\beta])$ or $f^{-1}(]-\infty,\beta[)$.

(iv) Let Ξ' be another vector space and let f be a linear mapping from Ξ into Ξ'. S is the image of a convex set of Ξ under f or the inverse image of a convex set of Ξ' under f. A special case is $\Xi' = \mathbb{R}$, where it is known that the intervals are the only convex sets.

2.2. Bounded and approximative compactness:

A set in a metric space is said to be boundedly compact if its intersection with an arbitrary closed ball is empty or compact; a set in a normed vector space (n.v.s.) is said to be boundedly weakly compact if its intersection with an arbitrary closed ball is empty or weakly compact [Bros69],[Vlas73].

Approximative compactness was introduced in [Efim61] for real Banach spaces and was then trivially extended to arbitrary metric spaces in [Sing64]: let S be a nonempty subset of a metric space (Ξ,d) and let x be a point in Ξ. A sequence $\{y_n\}_{n \geq 0}$ of points in S is said to be minimizing for x in S if: $\lim_{n \to +\infty} d(x,y_n) = \phi_S(x)$ [Yosi80]. A set S in a metric space Ξ is called approximately compact if for every point x in Ξ any minimizing sequence of elements in S contains a subsequence converging to a point in S [Efim61].
A set S in a n.v.s. Ξ is called approximately weakly compact if for every point x in Ξ any minimizing sequence of elements in S contains a subsequence converging weakly to a

point in S [Brec68],[Vlas73]. It is readily noted that in a metric space:
compact $\Longrightarrow$ boundedly compact $\Longrightarrow$ approximately compact $\Longrightarrow$ closed,
and that in a n.v.s.:
weakly compact $\Longrightarrow$ boundedly weakly compact $\Longrightarrow$ approximately weakly compact $\Longrightarrow$ closed.
Moreover, it follows from the definitions that in a n.v.s. a type of compactness implies its weak counterpart; in finite dimensional spaces, the distinction disappears. Conditions under which an approximately weakly compact set is approximately compact in infinite dimension can be found in [Brec68].

2.3. Complements on Banach spaces:

Let $(\Xi,||.||)$ be a Banach space. The unit ball in Ξ is denoted by $B=\{x\in\Xi \mid ||x|| \leq 1\}$. The unit sphere in Ξ is denoted by $U=\{x\in\Xi \mid ||x|| = 1\}$. Ξ is said to be strictly convex if [Goeb84]:

$$(\forall(x,y)\in U^2) \quad ||\frac{x+y}{2}|| = 1 \quad \Longrightarrow \quad x = y \tag{1}$$

Ξ is said to be uniformly convex if [Goeb84]:

$$(\forall\eta\in\mathbb{R}_+^*)(\exists\delta_\eta\in\mathbb{R}_+^*)(\forall(x,y)\in U^2) \quad ||x-y||\geq\eta \quad \Longrightarrow \quad ||\frac{x+y}{2}||\leq 1-\delta_\eta \tag{2}$$

A uniformly convex Banach space Ξ is strictly convex; if the dimension of Ξ is finite the converse is also true. Hilbert spaces and $L^p(\mu)$ spaces, with $1<p<+\infty$, are uniformly convex [Raom87]. Ξ is said to be locally uniformly convex if [Daym73]:

$$(\forall\eta\in\mathbb{R}_+^*)(\forall x\in U)(\exists\delta_{\eta,x}\in\mathbb{R}_+^*)(\forall y\in U) \quad ||x-y||\geq\eta \quad \Longrightarrow \quad ||\frac{x+y}{2}||\leq 1-\delta_{\eta,x} \tag{3}$$

Finally, a uniformly convex Banach space Ξ is said to be smooth if its dual Ξ' is strictly convex [Daym73].

2.4. Upper semi-continuity of set-valued maps:

We review here a few facts about set-valued maps. For a full account, see Kuratowski [Kura66],[Kura68] and Berge [Berg59].

Let Ξ_1, Ξ_2, and Ξ_3 be spaces. A set-valued map from Ξ_1 into ζ^{Ξ_2} is a map T which assigns to each point of Ξ_1 a subset of Ξ_2. By the image of a set S of Ξ_1 we mean the subset of Ξ_2 defined by: $TS = \bigcup_{x\in S} T(x)$. If $T_1:\Xi_1\rightarrow\zeta^{\Xi_2}$ and $T_2:\Xi_2\rightarrow\zeta^{\Xi_3}$ are two set-valued maps, the composition $T=T_2\circ T_1$ is defined by:

$$\begin{aligned} &T:\ \Xi_1 \rightarrow \zeta^{\Xi_3} \\ &\quad x \mapsto T_2T_1(x) = \bigcup_{y\in T_1(x)} T_2(y) \end{aligned} \tag{4}$$

Let (Ξ_1,τ_1) and (Ξ_2,τ_2) be two topological spaces. Following K. Kuratowski [Kura66],[Kura68], a set-valued map T from Ξ_1 into 2^{Ξ_2} is said to be upper semi-

continuous (u.s.c.) if the set $\{x \in \Xi_1 \mid T(x) \subset Q\}$ is open in Ξ_1 for any open set Q in Ξ_2, that is:

$$(\forall Q \in \tau_2) \quad \{x \in \Xi \mid T(x) \subset Q\} \in \tau_1 \tag{5}$$

In [Berg59], C. Berge defines u.s.c. maps in a slightly different manner by imposing that, in addition to (5), the set $T(x)$ be compact in Ξ_2 for all x in Ξ_1. Thereafter, we shall call such a map upper Berge semi-continuous (u.B.s.c.).
If T is u.s.c. and if we further assume that Ξ_2 is a metric space, then T is closed in the sense that the set $\{(x,y) \in \Xi_1 \times \Xi_2 \mid y \in T(x)\}$ is closed in the topological product $\Xi_1 \times \Xi_2$ [Kura66]. Consequently if T is closed the following property holds at any point x in Ξ_1: let $\{x_n\}_{n \geq 0}$ be an Ξ_1-sequence converging to x and $\{y_n\}_{n \geq 0}$ be an Ξ_2-sequence converging to y with: $(\forall n \in \mathbb{N}) \; y_n \in T(x_n)$, then we have: $y \in T(x)$. We have: u.B.s.c. $\Longrightarrow$ u.s.c. $\Longrightarrow$ closed. Moreover, if the metric space Ξ_2 is compact, T is u.s.c. if and only if it is closed [Kura68].

Theorem 2: Let Ξ_1 and Ξ_2 be two topological spaces and let T be a u.s.c. [respectively u.B.s.c.] set-valued map from Ξ_1 into 2^{Ξ_2}. Then the restriction T_S of T to a nonempty subset S of Ξ_1 is a u.s.c. [respectively u.B.s.c.] map from S into 2^{Ξ_2}.

Proof: Let τ_1 and τ_2 denote the respective topologies of Ξ_1 and Ξ_2. Then S is also a topological space with the relative topology $\tau_S = \{S \bigcap G \mid G \in \tau_1\}$ [Bour71]. Since T is u.s.c. (5) holds. Hence, it follows that:

$$(\forall Q \in \tau_2) \quad \{x \in S \mid T_S(x) \subset Q\} = \{x \in \Xi \mid T(x) \subset Q\} \bigcap S \in \tau_S \tag{6}$$

which proves that T_S is u.s.c. from S into 2^{Ξ_2}. Now if we suppose that T is u.B.s.c., to show that T_S is u.B.s.c. it remains to show that, for an arbitrary x in S, $T_S(x)$ is compact in Ξ_2. This clearly holds since by assumption $T(x)$ is compact subset of Ξ_2 for all x in Ξ_1.

Theorem 3 [Berg59]: Let Ξ_1 and Ξ_2 be two topological spaces and let T be a u.B.s.c. map from Ξ_1 into 2^{Ξ_2}. Then the image TK of a compact set K of Ξ_1 is compact in Ξ_2.

Theorem 4: Let Ξ_1, Ξ_2 and Ξ_3 be topological spaces. Let T_1 be a u.s.c. [respectively u.B.s.c.] set-valued map from Ξ_1 into 2^{Ξ_2} and let T_2 be a u.s.c. [respectively u.B.s.c.] set-valued map from Ξ_2 into 2^{Ξ_3}. Then the composition product $T = T_2 \circ T_1$ is a u.s.c. [respectively u.B.s.c.] set-valued map from Ξ_1 into 2^{Ξ_3}.

Proof: Let Q be an arbitrary open set in Ξ_3. We need to show that the set $A = \{x \in \Xi_1 \mid T(x) \subset Q\}$ is open in Ξ_1. We can write $A = \{x \in \Xi_1 \mid T_1(x) \subset Q'\}$ where: $Q' = \{y \in \Xi_2 \mid T_2(y) \subset Q\}$. But since T_2 is u.s.c. and Q is open in Ξ_3, Q' is open in Ξ_2. Consequently, since T_1 is u.s.c., A is open in Ξ_1 and T is u.s.c. from Ξ_1 into 2^{Ξ_3}. Now, let x be an arbitrary point in Ξ_1. If T_1 is u.B.s.c. then $K = T_1(x)$ is compact in Ξ_2. If T_2 is u.B.s.c. then, by Theorem 3, the set $T(x) = T_2 K$ is compact in Ξ_3, which proves that T is u.B.s.c. from Ξ_1 into 2^{Ξ_3}.

3. Projections in metric spaces:

In this section, unless otherwise stated, $\Xi \equiv (\Xi,d)$ is a metric space.

3.1. Distance to a set:

Let S_1 and S_2 be two nonempty subsets of (Ξ,d). We shall call distance between S_1 and S_2 the number:

$$d(S_1,S_2) = \text{Inf} \left\{ d(x_1,x_2) \mid x_1 \in S_1 \, , \, x_2 \in S_2 \right\} \tag{7}$$

The reader should not be misled by this notation: $d(.,.)$ does not define a metric on $\zeta^{\Xi} \times \zeta^{\Xi}$ [Kell55].

Theorem 5 [Bour74]: Let S_1 and S_2 be respectively a compact and a closed nonempty subset of (Ξ,d) such that $d(S_1,S_2) = 0$. Then $S_1 \bigcap S_2 \neq \emptyset$.

Let S be a nonempty subset of (Ξ,d). The distance from a point of Ξ to S is determined by the functional:

$$\begin{aligned} \phi_S &: \Xi \rightarrow \mathbb{R}_+ \\ & x \mapsto d(x,S) = d(\{x\},S) = \text{Inf} \left\{ d(x,y) \mid y \in S \right\} \end{aligned} \tag{8}$$

Theorem 6 [Bour74]: Let S be a nonempty subset of a metric space (Ξ,d). Then:
(i) ϕ_S is continuous.
(ii) ϕ_S is nonexpansive: $(\forall (x,y) \in \Xi^2) \quad |\phi_S(x) - \phi_S(y)| \leq d(x,y)$.
(iii) $(\forall x \in \Xi) \quad \phi_S(x) = 0 \quad \Longleftrightarrow \quad x \in \overline{S}$.

It is noted that for all x in Ξ: $\phi_S(x) = \phi_{\overline{S}}(x)$.

Theorem 7: Let S and S_1 be respectively a nonempty bounded and a nonempty boundedly compact set in a metric space and let α be a positive real number. Then the set $\{x \in S_1 \mid \phi_S(x) \leq \alpha\}$ is compact.

Proof: Let $A = \{x \in S_1 | \phi_S(x) \leq \alpha\}$. We can write $A = S_1 \bigcap \phi_S^{-1}([0,\alpha])$. Since $[0,\alpha]$ is closed and since, by Theorem 6, ϕ_S is continuous, A is closed. Since S is bounded it is contained in a closed ball centered at some point z in S, say $B[z,r]$. Now let x be an arbitrary point in A and let y be an arbitrary point in $\overline{S}$. Then we can write:
$d(z,x) \leq d(z,y) + d(y,x) \leq r + \phi_{\overline{S}}(x) = r + \phi_S(x) = r'$. Therefore x belong to the closed ball B of center z and radius r', and it follows that $A \subset B \bigcap S_1$. Hence A is a closed subset of $B \bigcap S_1$ which is compact since S_1 is boundedly compact. Thus A is compact.

3.2. Projection maps:

For any point x in (Ξ,d), we shall call a projection of x onto a subset S of (Ξ,d) a point y of S which assumes the greatest lower bound in (8), i.e. a point y in S such that $\phi_S(x) = d(x,y)$. In particular, by a projection of a vector x onto a subset S of a n.v.s.

$(\Xi, ||.||)$, we mean a vector y of S such that:

$$||x - y|| = \text{Inf} \left\{ ||x - z|| \mid z \in S \right\} \tag{9}$$

We strongly stress the fact that a projection may not exist or may not be unique. The set-valued projection map Π_S which assigns to each point x in (Ξ, d) the set of its projections onto a subset S is defined by:

$$\begin{aligned} \Pi_S : \Xi &\rightarrow \zeta^S \\ x &\mapsto \left\{ y \in S \mid \phi_S(x) = d(x,y) \right\} \end{aligned} \tag{10}$$

Theorem 8: Let S be an arbitrary set in a metric space Ξ. Then:

$$(\forall x \in \Xi) \quad \Pi_S(x) = B[x, \phi_S(x)] \bigcap S \tag{11}$$

Proof: Let $B = B[x, \phi_S(x)]$. Then $\Pi_S(x) \subset S \bigcap B$ follows from (8) and (10). Conversely, let $z \in S \bigcap B$. Then z belongs to S and: $d(x,z) \geq \text{Inf} \{ d(x,y) \mid y \in S \} = \phi_S(x)$. But since z also belongs to B: $\phi_S(x) \geq d(x,z)$. Hence: $z \in \Pi_S(x)$.

Theorem 9: Let S be an arbitrary nonempty set in a metric space Ξ and let Π_S denote the projection map onto S. Then if x is an arbitrary point in Ξ:

(i) $\Pi_S(x)$ is bounded.
(ii) $\Pi_S(x)$ is closed if S is closed.
(iii) $\Pi_S(x)$ is compact if S is boundedly compact.

Proof: (i) obvious from (11); (ii) because from (11), it is the intersection of two closed sets; (iii) obvious from (11) and the definition of bounded compactness.

Following I. Singer [Sing74], given a nonempty subset S of Ξ, we can also define a point-valued operator π_S which is a selection of Π_S, i.e. any mapping:

$$\begin{aligned} \pi_S : D &\rightarrow S \\ x &\mapsto y \in \Pi_S(x) \end{aligned} \tag{12}$$

where the domain of π_S is given by: $D = \{x \in \Xi \mid \Pi_S(x) \neq \emptyset\}$. If $D = \Xi$ and $\Pi_S(x)$ reduces to a singleton $\{\pi_S(x)\}$ for all x in D, then π_S will be called the projection operator of Ξ onto S. As usual, the inverse image of a point y of S under π_S is denoted by:

$$(\forall y \in S) \quad \pi_S^{-1}(y) = \left\{ x \in \Xi \mid d(x,y) = \phi_S(x) \right\} \tag{13}$$

3.3. Existence of a projection:

Let S be a nonempty subset of a metric space Ξ. We say that S is an existence set if every point in Ξ has at least one projection onto S, that is: $(\forall x \in \Xi) \quad \Pi_S(x) \neq \emptyset$. It is noted that an existence set is necessarily closed since no point in $\overline{S} - S$ has a projection, which forces $\overline{S} = S$. Therefore if S is an existence set, $\Pi_S(x)$ will be nonempty, closed and

bounded by Theorem 9, and the map Π_S defined in (10) actually maps Ξ into 2^S.

Theorem 10: Let S be a nonempty set in a metric space Ξ. Then S is an existence set if any of the following conditions holds:

(i) S is approximately compact.
(ii) S is boundedly compact.
(iii) S is compact.
(iv) S is closed and all the closed balls are compact in Ξ.

Proof: (i) see [Efim61]; (ii) see [Laur72] or regard it a special case of (i) [Efim61]; (iii) see [Bour74] or regard it as a special case of (ii); (iv) see [Schw80] or regard it as a special case of (ii): if B is an arbitrary closed ball in Ξ, then $S \bigcap B$ is a closed subset of B, and B is compact. Thus if $S \bigcap B$ is not empty it is compact, which makes S boundedly compact.

In the special case of n.v.s.'s, several additional existence criteria can be established:

Theorem 11: Let S be a nonempty set in a n.v.s. Ξ. Then S is an existence set if any of the following conditions holds:

(i) S is approximately weakly compact.
(ii) S is boundedly weakly compact.
(iii) S is weakly compact.
(iv) S is a finite dimensional vector subspace.
(v) S is sequentially weakly closed and Ξ is a uniformly convex Banach space.
(vi) S is closed and Ξ has finite dimension.

Proof: (i) see [Brec68]; (ii) special case of (i); (iii) special case of (ii); (iv) see [Laur72],[Sing74]; (v) because then S is approximately compact [Efim61],[Brec68]; (vi) by Theorem 1, this is a special case of (iv) in Theorem 10 (hence, such an S is boundedly compact).

Theorem 12: In a finite dimensional n.v.s. a nonempty set S is an existence set if and only if it is closed. Then S is boundedly compact.
Proof: An existence set is necessarily closed, as was seen earlier. Sufficiency and bounded compactness of S come from (vi) in Theorem 11.

3.4. Uniqueness of a projection:

A subset S of a metric space (Ξ,d) is called a Chebyshev set if every point of the space has exactly one projection onto S. Thus, if S is a Chebyshev set, the operator π_S defined in (12) may be referred to as a projection operator.

Theorem 13 [Goeb84]: Let S be a nonempty closed and convex set in a uniformly convex Banach space. Then:

(i) S is a Chebyshev set.
(ii) π_S is continuous.

Theorem 14 [Bour81],[Tiss85]: Let S be a nonempty convex and complete subset of a pre-Hilbert space. Then:

(i) S is a Chebyshev set.
(ii) π_S is continuous and nonexpansive: $(\forall (x,y) \in \Xi^2) \quad ||\pi_S(x) - \pi_S(y)|| \leq ||x - y||$.
(iii) Characterization of π_S: $(\forall x \in \Xi)(\forall y \in S) \quad \mathrm{Re}(<x - \pi_S(x), y - \pi_S(x)>) \leq 0$.

In particular, this theorem applies when S is closed and convex and Ξ is a Hilbert space; when S is a Banach subspace and Ξ is a pre-Hilbert space; when S is a closed vector subspace and Ξ is a Hilbert space (in particular S is a finite dimensional vector subspace). The properties of the operator π_S in that latter case are the object of the next theorem.

Theorem 15 [Bour81]: Let S be a closed vector subspace of a Hilbert space Ξ. Then:

(i) $(\forall x \in \Xi)(\exists! y \in S)(\exists! z \perp S) \quad x = y + z, \quad$ furthermore $y = \pi_S(x)$.

(ii) π_S is nonexpansive, linear, and idempotent: $\pi_S^2 = \pi_S$.

(iii) $S \neq \{0\} \implies ||\pi_S|| = 1$.

(iv) π_S is self-adjoint: $(\forall (x,y) \in \Xi^2) \quad <\pi_S(x), y> = <x, \pi_S(y)>$.

Let π_1 and π_2 be respectively the projection operators onto the closed vector subspaces S_1 and S_2 of Ξ and let $V = \{x + y \mid x \in S_1 , y \in S_2\}$. Then:

(v) If $\pi_1 \pi_2 = \pi_2 \pi_1$, then V is closed and we have: $\pi_V = \pi_1 + \pi_2 - \pi_1 \pi_2$.

(vi) $\pi_1 \pi_2 = 0 \iff S_1 \perp S_2$. If so, V is closed and we have: $\pi_V = \pi_1 + \pi_2$.

3.5. Some remarks on Chebyshev sets:

Upon reading the previous section, it is legitimate to ask whether a Chebyshev has to be convex. The answer is negative, for nonconvex Chebyshev sets can be constructed [John87]. However, T. Motzkin has showed that in the Euclidean plane a set is a Chebyshev set if and only if it is closed and convex [Motz35]. Motzkin's result was then generalized to the Euclidean space in [Jess40]. It was then established that in a finite dimensional Banach space Ξ, the class of Chebyshev sets coincide with the class of closed and convex sets if and only if Ξ is strictly convex and smooth [Efim59]. In infinite dimensional spaces, the discussion is more involved and the reader is referred to [Klee61], [Aspl69], [Vlas73], [Sing74], and [Bala82].

It also is worth noting that a Chebyshev set need not be approximately compact. However, a Chebyshev set in a uniformly convex smooth Banach space is convex if and only if it is approximately compact [Efim61].

3.6. Continuity of projections:

Let S be an existence set in a metric space Ξ. In section 3.3 it was noted that the range of the projection map onto S is in fact 2^S. Hence, from this point on, by projection map onto S we mean the set-valued projection map with domain Ξ and range 2^S of (10). When we say that the projection map onto S is u.s.c. [respectively u.B.s.c.] it is understood that we mean u.s.c. [respectively u.B.s.c.] relative to the respective topologies of Ξ and S.

Theorem 16: Let S be an arbitrary nonempty set in a metric space. The projection map onto S is denoted by Π_S and, if S is a Chebyshev set, the projection operator onto S is denoted by π_S. Then:

(i) Π_S is closed if S is an existence set.

(ii) Π_S is u.s.c. if S is approximately compact.

(iii) Π_S is u.B.s.c. if S is boundedly compact.

(iv) π_S is continuous if S is an approximately compact Chebyshev set.

Proof: (i), (ii) and (iv) see [Sing64]; (iii) Π_S is already u.s.c. by (ii) and $\Pi_S(x)$ is compact

for all x from Theorem 9.

3.7. Characterization of a projection:

By a fixed point of a set-valued mapping T, we mean a point x such that $x \in T(x)$. The following characterization is given in [Bros69]. Let S be a nonempty subset of a n.v.s. $(\Xi,||.||)$, y_0 a point in S and x a point in S^c. Then y_0 belongs to $\Pi_S(x)$ if and only if it is a fixed point of the mapping Γ_x where:

$$\begin{aligned} \Gamma_x &: S \rightarrow \zeta^S \\ & y \mapsto S \bigcap B[x, \frac{\phi_S(x) + ||x-y||}{2}] \end{aligned} \tag{14}$$

Moreover, in this case, we have: $\Gamma_x(y_0) \subset \Pi_S(x)$. The problem of characterizing a projection in a n.v.s. is also addressed in [Laur72], where specific cases are studied.

3.8. Projections onto monotonic sequences of sets:

Theorem 17 [Schw80]: Let $\{S_n\}_{n\geq 0}$ be a sequence of nonempty closed and convex sets in a Hilbert space Ξ. Then we have:

$$(\forall x \in \Xi) \quad \lim_{n \rightarrow +\infty} \phi_{S_n}(x) = \phi_S(x) \quad \text{and} \quad \lim_{n \rightarrow +\infty} \pi_{S_n}(x) = \pi_S(x) \tag{15}$$

whenever either one of the following propositions holds:

(i) $\{S_n\}_{n\geq 0}$ is a decreasing sequence with $S = \bigcap_{n\geq 0} S_n \neq \emptyset$,

(ii) $\{S_n\}_{n\geq 0}$ is a increasing sequence with $S = \overline{\bigcup_{n\geq 0} S_n}$.

3.9. A property of the points with multiple projections:

Let S be an existence set in a locally uniformly convex Banach space Ξ and let M denote the set of points with have more than one projection onto S. Then M is a set of the first category, i.e. a countable union of nowhere dense sets [Vlas73]. Therefore, since Ξ is a complete metric space, it may be concluded that the set M^c of points with exactly one projection onto S is everywhere dense: the smallest closed set containing M^c is Ξ itself.

3.10. A few counterexamples:

In this section we have gathered a few cases which illustrate the necessity of the assumptions on which the theorems rest, and the fact that in some respects the results cannot be improved.

- Two disjoint sets S_1 and S_2 in a metric space such that $d(S_1,S_2) = 0$: take S_1 an hyperbola, and S_2 one of its asymptote in the Euclidean plane (compare with Theorem 5).
- A point x with no projection onto a bounded set in a finite dimensional metric vector space: take the Euclidean space $\mathbb{R}$, $S =]-1,1[$, and $x = 2$. Then $\phi_S(x) = 1$ but x has no projection onto S.

- A point x with infinitely many projections onto a closed and convex subset S in a Banach space: take $\mathbb{R}^2$ with the distance d_1 defined in (16), $S=B[0,1]$, and $x=(1,1)$. Then $\Pi_S(x)=\{(\alpha,1-\alpha) \mid 0\leq\alpha\leq 1\}$
- A point with no projection onto a closed set: see [Tiss85].
- A point with no projection onto a closed and bounded set: see [Schw80].
- An existence set in a real Hilbert space which is not approximately compact and a projection mapping onto it which is not u.s.c.: see [Sing64].
- A Chebyshev set in a real pre-Hilbert space which is nonconvex and for which the projection operator is continuous: see [John87].
- A projection operator π_S which is not continuous, where S is a Chebyshev vector subspace in a Banach space: see [Holm68].
- A projection operator π_S which is not nonexpansive, where S is compact and convex in a finite dimensional n.v.s.: see [deFi67].

4. Projections in the Cartesian space:

4.1. Generalities:

Let k be a positive integer. The Cartesian space $\mathbb{R}^k$ is the space of real k-tuples $(x_1, \cdots, x_k)$ and has the structure of a k-dimensional vector space over $\mathbb{R}$, where addition and scalar multiplication are defined in the usual manner. For any positive extended real number p, $(\mathbb{R}^k, d_p)$ possesses the structure of a complete metric space with:

$$(\forall (x,y)\in\mathbb{R}^k\times\mathbb{R}^k) \quad d_p(x,y) = \begin{cases} \sum_{i=1}^{k} |x_i-y_i|^p & \text{if: } 0<p<1 \\ \left(\sum_{i=1}^{k} |x_i-y_i|^p \right)^{1/p} & \text{if: } 1\leq p<+\infty \\ \operatorname{Sup}_{1\leq i\leq k} \left\{ |x_i-y_i| \right\} & \text{if: } p=+\infty \end{cases} \tag{16}$$

For $1\leq p\leq+\infty$, the distance induces a norm and $(\mathbb{R}^k, d_p)$ is a Banach space. In particular, $(\mathbb{R}^k, d_p)$ is a uniformly (and therefore strictly) convex Banach space for $1<p<+\infty$ [Yosi80]. For $p=2$ it becomes a Hilbert space which is referred to as the k-dimensional Euclidean space E^k. It is worth noting that a broader class of Hilbert spaces can be obtained on $\mathbb{R}^k$ with the following distance:

$$(\forall (x,y)\in\mathbb{R}^k\times\mathbb{R}^k) \quad d(x,y) = \sum_{i=1}^{k}\sum_{j=1}^{k} x_i a_{ij} y_j \tag{17}$$

where $\left[a_{ij} \right]$ is a symmetric positive definite $k\times k$ matrix. Figure 1 represents the sphere $\{x\in\mathbb{R}^2 \mid d_p(0,x)=1\}$ for various values of p. It is clear that the notion of proximity strongly depends on the metric. Figure 1 also provides an illustration of the notion of strict convexity: if $1<p<+\infty$, every open segment (as defined in (18)) in the unit ball is disjoint from the unit sphere.

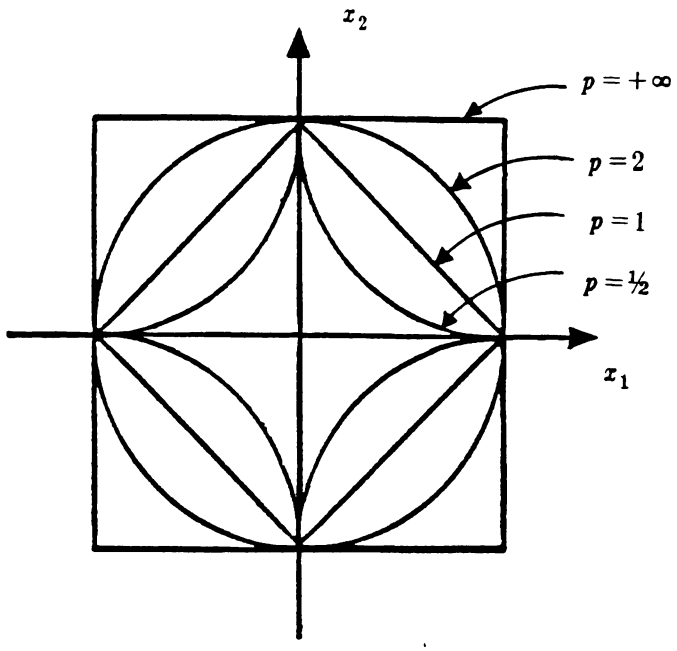


Figure 1. Points at distance 1 from the origin for various p's

4.2. Ordinary and multifurcation points:

Historically, G. Bouligand has been the first to study the properties of points with multiple projections onto a closed subset in the Euclidean space E^k [Boul32]. Upon becoming aware of their importance in Euclidean geometry, he proceeded to categorize the points in the space into what he calls *ordinary points,* i.e. points with a unique projection, and *multifurcation points,* i.e. points with more than one projection (in section 3.9, the set of multifurcation points was denoted by M). A few years later, T. Motzkin gave a characterization of closed and convex subsets of the Euclidean plane (and by straightforward extension, in the Euclidean space) in terms of multifurcation points [Motz35]. Then C. Pauc gave a more detailed geometrical analysis for the case of the Euclidean plane [Pau39a] and subsequently for the Euclidean space [Pau39b]. The next major contribution is due to P. Erdös who approached the problem from a measure theoretic point of view [Erdo45].

4.3. Existence, uniqueness, and continuity of a projection:

If $1 \leq p \leq +\infty$, Theorem 12 states that the class of nonempty closed subsets, the class of boundedly compact sets, and the class of existence sets coincide in $(\mathbb{R}^k, d_p)$. Then the projection map onto a closed subset is u.B.s.c. on account of Theorem 16. For $1 < p < +\infty$, $(\mathbb{R}^k, d_p)$ is a uniformly convex Banach space and therefore, by Theorem 13, any point has a unique projection onto a nonempty closed and convex subset S and the projection

operator π_S is continuous. But in this case the dual of $(\mathbb{R}^k, d_p)$ is $(\mathbb{R}^k, d_q)$, where $p^{-1}+q^{-1}=1$, and it is a strictly convex Banach space as well. Hence $(\mathbb{R}^k, d_p)$ is a strictly convex smooth finite dimensional Banach space and, by a result of section 3.5, the only Chebyshev sets are the closed and convex sets. In addition, π_S is nonexpansive for $p=2$.

4.4. The reach, the convex deficiency, and the skeleton of a set:

The concepts of reach, convex deficiency and skeleton will be introduced here because of their important connection with nonunique projections. S is a nonempty set in the Euclidean space E^k.

The notion of the reach of a set was introduced by H. Federer in [Fede59]. The reach of S is the largest η (possibly $+\infty$) such that if x is a point of the space such that $\phi_S(x)<\eta$, then x is an ordinary point with respect to S. We have:

(i) $\text{reach}(S) > 0 \quad <\!\!=\!\!> \quad S$ is closed

(ii) $\text{reach}(S) = +\infty \quad <\!\!=\!\!> \quad S$ is closed and convex

Loosely speaking, a point is within the reach of a set if it can be projected unambiguously onto it.

Now we suppose that S is closed and we denote by C its convex hull, i.e. the smallest convex set containing S. The convex deficiency of S is defined as [Cala68]: $D = C-S$. A point x in S^c is called a skeletal point of S if $B[x,\phi_S(x)]$ is contained in no other $B[x',\phi_S(x')]$ [Cala68]. The skeleton of S is the set of all its skeletal points and is denoted by K.

4.5. Properties of ordinary and multifurcation points in the Euclidean space:

In this section, we shall provide some interesting properties of the projections onto a nonempty closed subset S of the Euclidean space E^k. The terms projection, ordinary and multifurcation points are relative to the set S. K is the skeleton of S and M the set of its multifurcation points. The Lebesgue measure in $\mathbb{R}^k$ is denoted by λ. We denote by $\text{seg}(x,y)$ the open segment defined by x and y:

$$(\forall (x,y)\in E^k\times E^k) \quad \text{seg}(x,y) = \{\alpha x + (1-\alpha)y \mid 0<\alpha<1\} \tag{18}$$

The following properties are useful and can be inferred from the materials in the preceeding sections.

(i) Let x be an arbitrary point in E^k and y one of its projections. Then any point in $\text{seg}(x,y)$ is an ordinary point: $\text{seg}(x,y)\bigcap M = \emptyset$.

(ii) Let x_1 and x_2 be two multifurcation points and let y_1 and y_2 be one of their respective projections. Then: $\text{seg}(x_1,y_1)\bigcap\text{seg}(x_2,y_2) = \emptyset$.

(iii) The multifurcation points x such that there is no $(n-1)$-dimensional hyperplane containing all the segments $\{\text{seg}(x,y) \mid y\in\Pi_S(x)\}$ are at most countable.

(iv) $M = \emptyset \quad <\!\!=\!\!> \quad S$ is convex.

(v) M is dense in K.

(vi) A point y in S is an interior point of $\pi_S^{-1}(y)$ if and only if it is isolated in S (there exists an open ball B centered at y such that $S\bigcap B=\{y\}$).

(vii) For all y in S, $\pi_S^{-1}(y)$ is closed and convex and contains y.

(viii) For all y in S, $\pi_S^{-1}(y)$ is bounded if and only if y is an interior point of the

convex hull of S.
(ix) π_S is a continuous projection operator on M^c.
(x) Any selection π_S is λ-a.e. differentiable.
(xi) $(\forall r \in \mathbb{R}_+^*) \quad \lambda(\{x \in \mathbb{R}^k \mid \phi_S(x) = r\}) = 0.$
(xii) $\lambda(\bigcup_{x \notin S} \Pi_S(x)) = 0.$
(xiii) $\lambda(M) = 0.$

Proof: (i), (ii) and (iii) see [Boul32]; (iv) and (v) see [Motz35]; (vi), (vii) and (viii) see [Pau39b]; (ix) see [Fede59]; (x) see [Aspl73]; (xi), (xii) and (xiii) see [Erdo45].

The last property is due to P. Erdös and is certainly the most remarkable: in the Euclidean space, the projections onto a closed set are Lebesgue almost everywhere unique. Loosely speaking, M can be covered by a countable number of k-dimensional cubes whose total volume is arbitrarily small. We can also give a probabilistic interpretation of this result: consider a bounded region Ω of the Euclidean space which, for simplicity, we choose to be a closed cube of k-dimensional volume v, however large. Let Σ represents the σ-algebra of Borel subsets of Ω. It is natural to choose $P = 1/v.\lambda$ as a probability measure on (Ω,Σ). Then (Ω,Σ,P) is our probability space. In this context, a point x in Ω has a unique projection with probability one. We could also restate this result by saying that a closed subset of E^k is almost surely a Chebyshev set (the probability of the event "x is a multifurcation point" is zero).

5. Application to set theoretic estimation:

5.1. Introduction:

Most signal processing problems are, in fact, estimation problems: the goal is to estimate an object y (which generally represents a finite collection of parameters) which lie in a solution space Ξ. In many cases, this process takes place in a multi-constrained environment: typically each constraint corresponds to some piece of knowledge about the true solution y which is available a priori.

Set theoretic estimation techniques associate with each of say m known properties of the true solution a set S_i in a solution space Ξ and define as a feasible solution any member in the intersection S of these sets, i.e. an object consistent with all the available a priori information. If the problem is well defined we have: $S = \bigcap_{i=1}^{m} S_i \neq \emptyset$. Hence, the main problem in set theoretic estimation is to find a common point of a finite class of sets. We know that a projection of a given point onto a set, if it exists, will give us a best approximation of x by an element of that set. In most situations, although the task of projecting directly onto the intersection set S is impracticable, it is possible to project onto each set individually. We shall see that under certain hypotheses, one can obtain iteratively a point in S by constructing a sequence a successive projections onto the S_i's.

5.2. The convex case in Hilbert spaces:

For sake of convenience, we shall say that a collection of sets $\{S_1,S_2,...,S_m\}$ satisfies hypothesis H if:

(H) The intersection of the S_i's is nonempty and, for all i in $\{1,2,...,m\}$, S_i is a closed

and convex subset of a Hilbert space Ξ.

Let $\{S_1,S_2,...,S_m\}$ be a collection of sets which satisfies hypothesis H and let S denote their intersection. Then, by Theorem 14, each S_i is a Chebyshev set and we denote by π_i the projection operator onto it. Moreover, we shall denote by π the composition of these individual projection operators, i.e. $\pi = \pi_1 o \pi_2 o ... o \pi_m$. Given a point x_0 in Ξ, one can generate a sequence of points in S_1 by projecting successively onto each set in the following manner:

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = \pi(x_n) = \pi^{n+1}(x_0) \tag{19}$$

The sequence $\{x_n\}_{n \geq 0}$ thus defined is referred to as the sequence of successive projections.

Hereafter we introduce two propositions pertaining to the convergence of the sequence of successive projection to a solution point:

(P_1) For any x_0 in Ξ, the sequence $\{x_n\}_{n \geq 0}$ converges strongly to a point in S.

(P_2) For any x_0 in Ξ, the sequence $\{x_n\}_{n \geq 0}$ converges weakly to a point in S.

Theorem 18 [Halp62]: Let $\{S_1,...,S_m\}$ be a collection of vector subspaces that satisfies H. Let π_S denote the projection operator onto S. Then the sequence $\{\pi^n\}_{n \geq 0}$ converges strongly to π_S and hence P_1 holds.

Theorem 19 [Stil65]: If a collection of sets $\{S_1,...,S_m\}$ satisfies H and if at least one of them is boundedly compact, then P_1 holds.

Theorem 20 [Breg65]: If a collection of sets $\{S_1,...,S_m\}$ satisfies H, then P_2 holds.

Some general remarks regarding the method of successive projections under hypothesis H are in order:

1. The operator π is nonexpansive. This fact is easily established by noting that by Theorem 14 each π_i is nonexpansive.
2. A point is a solution if and only if it is a fixed point of π [Youl82]. In other words: $S = \{x \in \Xi \mid x = \pi(x)\}$.
3. The three above theorems fit into the general problem of the convergence of a sequence of successive approximations $\{T^n(x_0)\}_{n \geq 0}$ to a fixed point of T, where T is a nonexpansive mapping. Such convergence properties are discussed in [Petr73].
4. For $m=2$ Theorem 18 is known as the alternating projection Theorem and was established by Von Neumann in [Vonn49].
5. By straightforward extension of the proof given in [Halp62], Theorem 18 can be generalized to linear varieties (i.e. images under translation of a vector subspaces).
6. Theorem 18 states that for any point x_0 in Ξ, the sequence of successive approximations converges to the projection of x_0 onto the intersection set S. This is not necessarily the case in Theorem 19 and 20.
7. If the dimension of Ξ is finite, Theorem 19 and 20 are equivalent because then a closed set is boundedly compact and the notions of weak and strong convergence coincide.

In order to improve the speed of convergence of the algorithm in Theorem 20, a relaxed version was proposed in [Gubi67] where the projections are extended beyond the boundary of the sets:

Theorem 21 [Gubi67]: Let $\{S_1,...,S_m\}$ be a collection of sets that satisfies H. For all i in $\{1,...,m\}$, let τ_i denote the relaxed projection operator onto S_i, i.e.:

$$(\forall x \in \Xi) \quad \tau_i(x) = x + \lambda_i(\pi_i(x) - x) \quad \text{where: } 0 < \lambda_i < 2 \tag{20}$$

Let $\tau = \tau_1 o \tau_2 o ... o \tau_m$. Then for any x_0 in Ξ, the sequence $\{\tau^n(x_0)\}_{n \geq 0}$ converges weakly to a point in S.
It is noted that the λ_i's are restricted to lie in $]0,2[$ to ensure that each τ_i, and therefore the composition product τ, is nonexpansive. If all the λ_i's are equal to 1, the result is Theorem 20. Conditions for the strong convergence of the method are also established in [Gubi67].

In [Otta88], N. Ottavi presents a unified framework for the problem of obtaining a common point of closed convex sets in a Hilbert space, along with criteria for the strong convergence of a very broad class of projection algorithms. Two salient features characterize these algorithms: at each step a convex linear combination of the projections onto selected sets is formed and an optimal relaxation coefficient is computed. By updating these parameters at each step the speed of convergence is optimized.

The general version of the algorithm is as follows [Otta88]: Let $\{S_\alpha\}_{\alpha \in A}$ be an arbitrary collection of closed and convex sets in a Hilbert space $(\Xi, ||.||)$ whose intersection S is not empty. The projection operator onto S_α is denoted by π_α. Let θ and η be two real numbers in $]0,1[$ and let β be a real number in $[1,+\infty]$. For each x in Ξ, $A(x)$ is a finite nonempty subset of A containing the indices of the sets to be activated and $\bar{x}$ is a weighted average of the corresponding projections:

$$\bar{x} = \sum_{i \in A(x)} \theta_i(x) \pi_i(x) \tag{21}$$

where the weights are determined by:

$$\sum_{i \in A(x)} \theta_i(x) = 1, \quad (\forall i \in A(x)) \quad \theta_i(x) \in [0,1] \quad \text{and:} \quad x \notin S_i \Longrightarrow \theta_i(x) \geq \theta \tag{22}$$

A set-valued map T is defined by:

$$(\forall x \in \Xi) \quad T(x) = \{x + \lambda(x)(\bar{x} - x)\} \tag{23}$$

where $\lambda(x)$ is a relaxation parameter such that:

$$\begin{cases} \lambda(x) \geq \eta \\ x \neq \bar{x} \end{cases} \Longrightarrow \quad \eta \leq \lambda(x) \leq \text{Inf} \left\{ \beta \, , \, \frac{\sum_{i \in A(x)} \theta_i(x) ||x - \pi_i(x)||^2}{||x - \bar{x}||^2} \right\} \tag{24}$$

The successive projection algorithm is then defined by an initial point x_0 in Ξ and the recursion: $(\forall n \in \mathbb{N}) \; x_{n+1} \in T(x_n)$. Sufficient conditions for the strong convergence of $\{x_n\}_{n \geq 0}$ are given in [Otta88]. It is noted that the algorithm of Theorem 21 is a particular case of the above: suppose that $\{S_\alpha\}_{\alpha \in A}$ is a finite collection of sets $\{S_1,...,S_m\}$ and write each n in $\mathbb{N}$ as: $n = km + j$, where k is in $\mathbb{N}$ and j in $\{0,...,m-1\}$. Then proceed by cycles of length m: at each step n activate only the $(m-j)$-th set: $\bar{x}_n = \pi_{m-j}(x_n)$ and fix $\lambda(x_n) = \lambda_{m-j}$ in $]0,2[$. Then if $x = x_0$, the sequence $\{x_{km}\}_{k \geq 0}$ is identical to the sequence $\{\tau^n x\}_{n \geq 0}$ of Theorem 21.

5.3. The general case:

To date, no result has been established toward the problem of obtaining a point in the intersection S of a finite collection of sets $\{S_1,...,S_m\}$ by projection methods outside the case when hypothesis H is satisfied. In this section we present a new method for finding a common point of a finite collection of sets in a metric space. First the method of successive projections is generalized to a collection of existence sets. Then conditions under which a sequence of successive projections would converge to a solution, i.e. a point in the intersection of these sets, are established.

First, we need the following definition: let $\{S_1,S_2,...,S_m\}$ be an ordered collection of existence sets in a metric space. For all i in $\{1,2,...,m-1\}$ we denote by Π_i the restriction to S_{i+1} of the projection map onto S_i, and by Π_m the restriction to S_1 of the projection map onto S_m. The composition map $\Pi=\Pi_1 o \Pi_2 o ... o \Pi_m$ will be called the cyclic projection map onto $\{S_1,S_2,...,S_m\}$. By construction it is a set-valued map from S_1 into 2^{S_1}.

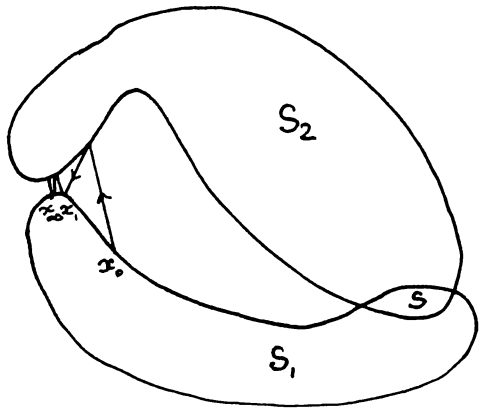
Let Π be cyclic projection map onto an ordered collection of existence sets $\{S_1,S_2,...,S_m\}$ in a metric space Ξ, whose intersection S is not empty, and let x_0 be a point in S_1. One can construct a sequence $\{x_n\}_{n\geq 0}$ of points in S_1 according to the algorithmic rule:

$$(\forall n \in \mathbb{N}) \quad x_{n+1} \in \Pi(x_n) \tag{25}$$

As in (19), the sequence $\{x_n\}_{n\geq 0}$ is called a sequence of successive projections. It is noted that such a sequence exists for $(\forall x \in S_1)\ \Pi(x)\neq\emptyset$. However the sequence may not be unique because in general the Π_i's are multi-valued maps. It is desired to determine those properties of Π and x_0 that will guarantee the convergence of the sequence of successive projections to a point in S.

First of all let us notice that if Π is closed from S_1 into 2^{S_1} the sequence of successive projections $\{x_n\}_{n\geq 0}$ can only converge to a point in $\{x \in S_1 \mid x \in \Pi(x)\}$, i.e. a fixed point of Π. Indeed, suppose that $\{x_n\}_{n\geq 0}$ converges to x. Then $\{x_{n+1}\}_{n\geq 0}$ converges also to x; by (25), since Π is closed, we get that $x \in \Pi(x)$.

It is clear that each point in S is a fixed point of Π. However, the converse statement is false, as illustrated by the following example in the Euclidean plane where the sequence of successive projection terminates at a fixed point of Π which is not in S:



It is seen that, for the set of fixed points of Π to coincide with S, one can impose the following condition:

$$(C) \qquad (\forall x \in \{y \in S_1 - S \mid \phi_S(y) \leq \phi_S(x_0)\})(\forall x' \in \Pi(x)) \quad \phi_S(x') < \phi_S(x)$$

The above condition will be referred to as condition C. Loosely speaking, condition C requires that Π bring strictly closer to S any point of S_1 which is not a solution point and which is at least as close to S as x_0. Under condition C, if the sequence of successive projections converges, it is necessarily to a point in S. It is noted that for all x_0 in S_1, if $\{x_n\}_{n\geq 0}$ is a sequence generated by a set-valued map Π satisfying C according to (19), then the sequence $\{\phi_S(x_n)\}_{n\geq 0}$ is convergent for it is strictly decreasing and bounded from below by zero. It may however not converge to zero. Even if it did, it is not guaranteed that $\{x_n\}_{n\geq 0}$ would converge to a point in S, as illustrated by the following example in the Euclidean plane: let S be the unit square $S=\{(x_1,x_2)\in\mathbb{R}^2 \mid \text{Sup}\{|x_1|,|x_2|\}\leq 1\}$, $x_0=(2,1)$, and let $x_n=(1+2^{-n},(-1)^n)$ for all positive integer n. Then the sequence $\{x_n\}_{n\geq 0}$ admits $(1,1)$ and $(1,-1)$ as cluster values and hence it does not converge.

The following theorem will be needed subsequently.

Theorem 22: The cyclic projection map onto an arbitrary ordered collection of approximately [respectively boundedly] compact sets $\{S_1,...,S_m\}$ in a metric space is a u.s.c. [respectively u.B.s.c.] map from S_1 into 2^{S_1}.

Proof: Let $\Pi=\Pi_1 o...o\Pi_m$ be the cyclic projection map onto $\{S_1,...,S_m\}$. From Theorem 16 and Theorem 2 it is known that if the S_i's are approximately [respectively boundedly] compact, the restricted maps Π_i are u.s.c. [respectively u.B.s.c.] from S_{i+1} into 2^{S_i} if $1\leq i\leq m-1$, and from S_1 into 2^{S_m} if $i=m$. For all i in $\{0,...,m-1\}$ we define a set-valued map T_i from S_1 into $2^{S_{m-i}}$ by $T_i=\Pi_{m-i}o...o\Pi_m$. It is noted that $\Pi=T_{m-1}$. For $i=0$, $T_0=\Pi_m$ is u.s.c. [respectively u.B.s.c.] from S_1 into 2^{S_m}. For any integer i in $\{0,...,m-2\}$ such that T_i is u.s.c. [respectively u.B.s.c.] from S_1 into $2^{S_{m-i}}$, since Π_{m-i-1} is u.s.c. [respectively u.B.s.c.] from S_{m-i} into $2^{S_{m-i-1}}$, the composition product $T_{i+1}=\Pi_{m-i-1}oT_i$ is u.s.c. [respectively u.B.s.c.] from S_1 into $2^{S_{m-i-1}}$, on account of Theorem 4. This completes the proof by induction.

Before we present the main result, let us recall that an ultrametric space is a metric space (Ξ,d) such that [Bour74]:

$$(\forall(x,y,z)\in\Xi^3) \quad d(x,z) \leq \text{Sup}\{d(x,y),d(y,z)\} \tag{26}$$

Theorem 23: Let $\{S_1,...,S_m\}$ be an ordered collection of approximately compact sets in a metric space Ξ whose intersection S is nonempty and bounded, and such that S_1 is boundedly compact. Let Π be the cyclic projection map onto $\{S_1,...,S_m\}$ and suppose that given a point x_0 in S_1 condition C holds. Then the limit of any convergent subsequence of the sequence of successive projections $\{x_n\}_{n\geq 0}$ is a point in S. Moreover, the sequence of successive projections converges itself to a point in S if any of the following conditions holds:

(i) Ξ is an ultrametric space.

(ii) $\sum_{n\geq 0} \phi_S(x_n) < +\infty$.

Proof: Since condition C holds, the sequence of successive projections $\{x_n\}_{n\geq 0}$ lies in the set $A=\{x\in S_1 \mid \phi_S(x)\leq\phi_S(x_0)\}$, which is compact on account of Theorem 7. Therefore we can extract a subsequence $\{x_{n_k}\}_{k\geq 0}$ converging to a point y in A. Since y is a cluster value of $\{x_n\}_{n\geq 0}$, $\phi_S(y)$ is a cluster value of $\{\phi_S(x_n)\}_{n\geq 0}$ by continuity of ϕ_S [Bour71]. For all nonnegative integer k, let y_k denote the successor of x_{n_k} in the main sequence $\{x_n\}_{n\geq 0}$, i.e.: $(\forall k\in\mathbb{N})\ \ y_k=x_{n_k+1}$. Then $\{y_k\}_{k\geq 0}$ also lies in A and it thereby possesses a subsequence $\{y_{k_j}\}_{j\geq 0}$ converging to a point y' in A. But by construction $\phi_S(y')$ is also a cluster value of $\{\phi_S(x_n)\}_{n\geq 0}$. This sequence was seen to be convergent and hence it admits a unique cluster value, which yields: $\phi_S(y')=\phi_S(y)$. For all nonnegative integer j, let z_j denote the predecessor of y_{k_j} in the main sequence $\{x_n\}_{n\geq 0}$. Then $\{z_j\}_{j\geq 0}$ is a subsequence of $\{x_{n_k}\}_{k\geq 0}$ and must converge to y. On the other hand, $\{y_{k_j}\}_{j\geq 0}$ converges to y' and moreover $(\forall j\in\mathbb{N})\ \ y_{k_j}\in\Pi(z_j)$. By Theorem 22, Π is u.s.c. and a fortiori closed, which forces: $y'\in\Pi(y)$. Now suppose that y is not in S. Then condition C yields: $\phi_S(y')<\phi_S(y)$, which contradicts a previous equality statement. Whence y must belong to S and the first assertion is proved.

We have just shown that $\{x_n\}_{n\geq 0}$ has a cluster value in S. Hence to prove that $\{x_n\}_{n\geq 0}$ converges to a point in S, it is enough to show that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence (for if a Cauchy sequence admits a cluster value y, it converges to y [Bour71]). We have: $\Pi=\Pi_1\circ\ldots\circ\Pi_m$. Let n be fixed in $\mathbb{N}$ and let $y_0=x_n$. For all j in $\{0,\ldots,m-1\}$, we denote by y_{j+1} the projection of y_j onto S_{m-j} which has been selected in the process of obtaining x_{n+1}. We have $x_{n+1}=y_m$. It is noted that S is boundedly compact as a closed subset of S_1 and hence, by Theorem 10, there exists a point z in S which is a projection of y_0 onto S. Since z belongs to S, it also belongs to all the S_{m-j}'s, and hence:

$$(\forall j\in\{0,\ldots,m-1\})\quad d(y_j,y_{j+1}) = \mathrm{Inf}\,\{d(y_j,y) \mid y\in S_{m-j}\} \leq d(y_j,z) \tag{27}$$

Let $d=d(y_0,z)$. Let us prove that:

$$(\forall j\in\{0,\ldots,m\})\quad d(y_j,z)\leq 2^j d \tag{28}$$

The statement is clearly true for $j=0$. For any integer j in $\{0,\ldots,m-1\}$ for which (28) holds, (27) yields:

$$d(y_{j+1},z)\leq d(y_j,y_{j+1})+d(y_j,z)\leq 2d(y_j,z)\leq 2^{j+1}d \tag{29}$$

which completes the proof by induction. By using (27) and (28), we get:

$$d(x_n,x_{n+1})\leq\sum_{j=0}^{m-1}d(y_j,y_{j+1})\leq\sum_{j=0}^{m-1}d(y_j,z)\leq\sum_{j=0}^{m-1}2^j d=(2^m-1)d \tag{30}$$

It is easy to see that $d=\phi_S(x_n)$. Therefore:

$$(\forall n\in\mathbb{N})\quad 0\leq d(x_n,x_{n+1})\leq(2^m-1)\phi_S(x_n) \tag{31}$$

But $\{x_n\}_{n\geq 0}$ admits a cluster value in S and therefore the limit of $\{\phi_S(x_n)\}_{n\geq 0}$ is in fact zero. Therefore, by (31), $d(x_n,x_{n+1})$ converges to zero as n goes to infinity. In an ultrametric space, this condition guarantees that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence [Bour74], which proves (i). To prove (ii), let p and q be any two nonnegative integers such that $p<q$. We have:

$$0\leq d(x_p,x_q)\leq\sum_{n=p}^{q-1}d(x_n,x_{n+1})\leq\sum_{n=p}^{+\infty}d(x_n,x_{n+1})\leq(2^m-1)\sum_{n=p}^{+\infty}\phi_S(x_n) \tag{32}$$

Since the rightmost expression in (32) is nothing but the tail of a convergent series, it must go to zero as p goes to infinity. Consequently $d(x_p,x_q)$ goes to zero as p and q go to infinity and $\{x_n\}_{n\geq 0}$ is a Cauchy sequence.

Corollary: Let $\{S_1,...,S_m\}$ be an ordered collection of closed sets in a finite dimensional normed vector space Ξ whose intersection S is nonempty and bounded. Let Π be the cyclic projection map onto $\{S_1,...,S_m\}$ and suppose that given a point x_0 in S_1 condition C holds. Then the strong limit of any convergent subsequence of the sequence of successive projections $\{x_n\}_{n\geq 0}$ is a point in S. Moreover, the sequence of successive projections converges strongly itself to a point in S if any of the following conditions holds:

(i) Ξ is an ultrametric space.

(ii) $\sum_{n\geq 0} \phi_S(x_n) < +\infty$.

Proof: By Theorem 12, Theorem 23 applies.

Comment: In a practical situation, given a collection of sets $\{S_1,...,S_m\}$ and an initial point x_0 which satisfy the conditions of Theorem 23, a sequence is generated according to the algorithm:

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \text{Stop if: } (\forall i \in \{1,...,m\}) \;\; x_n \in S_i \\ x_{n+1} \in \Pi(x_n) \qquad \text{else} \end{cases} \tag{33}$$

In words, the iterations are stopped if the algorithm produces a solution in a finite number steps. If not, the algorithm gives a solution in a limiting sense; namely, a solution is any cluster value of $\{x_n\}_{n\geq 0}$ or its limit itself if either (i) or (ii) holds.

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